Reference

Reliability Growth and Repairable System Analysis

ReliaSoft

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Reliability Growth Analysis Overview

IN THIS CHAPTER

When conducting distribution analysis (and life data analysis), the events that are observed are assumed to be statistically independent and identically distributed (IID). A sequence or collection of random variables is IID if:

- Each has the same probability distribution as any of the others.
- All are mutually independent, which implies that knowing whether or not one occurred makes it neither more nor less probable that the other occurred.

In life data analysis, the unit/component placed on test is assumed to be *as-good-as-new*. However, this is not the case when dealing with repairable systems that have more than one life. They are able to have multiple lives as they fail, are repaired and then put back into service. The age just after the repair is basically the same as it was just before the failure. This is called *as-bad-as-old*. For reliability growth and repairable systems analysis, the events that are observed are part of a *stochastic process*. A stochastic process is defined as a sequence of inter-dependent random events. Therefore, the events are dependent and are not identically distributed.

- The time-to-failure of a product after a redesign is dependent on how good or bad the redesign action was.

- The time-to-failure of the product after the redesign may follow a distribution that is different than the times-to-failure distribution before the redesign.

There is a dependency between the failures that occur on a repairable system. Events that occur first will affect future failures. Given this dependency, applying a Weibull distribution, for example, is not valid since life data analysis assumes that the events are IID. Reliability growth and repairable systems analysis provide methodologies for analyzing data/events that are associated with systems that are part of a stochastic process.

What is Reliability Growth?

In general, the first prototypes produced during the development of a new complex system will contain design, manufacturing and/or engineering deficiencies. Because of these deficiencies, the initial reliability of the prototypes may be below the system's reliability goal or requirement. In order to identify and correct these deficiencies, the prototypes are often subjected to a rigorous testing program. During testing, problem areas are identified and appropriate corrective actions (or redesigns) are taken. Reliability growth is defined as the positive improvement in a reliability metric (or parameter) of a product (component, subsystem or system) over a period of time due to changes in the product's design and/or the manufacturing process. A reliability growth program is a well-structured process of finding reliability problems by testing, incorporating corrective actions and monitoring the increase of the product's reliability throughout the test phases. The term "growth" is used since it is assumed that the reliability of the product will increase over time as design changes and fixes are implemented. However, in practice, no growth or negative growth may occur.

Reliability goals are generally associated with a reliability growth program. A program may have more than one reliability goal. For example, there may be a reliability goal associated with failures resulting in unscheduled maintenance actions and a separate goal associated with those failures causing a mission abort or catastrophic failure. Other reliability goals may be associated with failure modes that are safety related. The monitoring of the increase of the product's reliability through successive phases in a reliability growth testing program is an important aspect of attaining these goals. Reliability growth analysis (RGA) concerns itself with the quantification and assessment of parameters (or metrics) relating to the product's reliability growth over time. Reliability growth management addresses the attainment of the reliability objectives through planning and controlling of the reliability growth process.

Reliability growth testing can take place at the system, major subsystem or lower unit level. For a comprehensive program, the testing may employ two general approaches: integrated and dedicated. Most development programs have considerable testing that takes place for reasons other than reliability. *Integrated* reliability growth utilizes this existing testing to uncover reliability problems and incorporate corrective actions. *Dedicated* reliability growth testing is a test program focused on uncovering reliability problems, incorporating corrective actions and typically, the achievement of a reliability goal. With lower level testing, the primary focus is to improve the reliability of a unit of the system, such as an engine, water pump, etc. Lower level testing, which may be dedicated or integrated, may take place, for example, during the early part of the design phase. Later, the system and subsystem prototypes may be subjected to dedicated reliability growth testing, integrated reliability growth testing or both. Modern applications of reliability growth extend these methods to early design and to in-service customer use. Reliability growth management concerns itself with the planning and management of an item's reliability growth as a function of time and resources.

Reliability growth occurs from corrective and/or preventive actions based on experience gained from failures and from analysis of the equipment, design, production and operation processes. The reliability growth "Test-Analyze-Fix" concept in design is applied by uncovering weaknesses during the testing stages and performing appropriate corrective actions before full-scale production. A corrective action takes place at the problem and root cause level. Therefore, a *failure mode* is a problem and root cause. Reliability growth addresses failure modes. For example, a problem such as a seal leak may have more than one cause. Each problem and cause constitutes a separate failure mode and, in some cases, requires separate corrective actions. Consequently, there may be several failure modes and design corrections corresponding to a seal leak problem. The formal procedures and manuals associated with the maintenance and support of the product are part of the system design and may require improvement. Reliability growth is due to permanent improvements in the reliability of a product that result from changes in the product design and/or the manufacturing process. Rework, repair and temporary fixes do not constitute reliability growth.

Screening addresses the reliability of an individual unit and not the inherent reliability of the design. If the population of devices is heterogeneous then the high failure rate items are naturally screened out through operational use or testing. Such screening can improve the mixture of a heterogeneous population, generating an apparent growth phenomenon when in fact the devices themselves are not improving. This is not considered reliability growth. Screening is a form of rework. Reliability growth is concerned with permanent corrective actions focused on prevention of problems.

Learning by operator and maintenance personnel also plays an important role in the improvement scenario. Through continued use of the equipment, operator and maintenance personnel become more familiar with it. This is called *natural learning*. Natural learning is a continuous process that improves reliability as fewer mistakes are made in operation and maintenance, since the equipment is being used more effectively. The learning rate will be increasing in early stages and then level off when familiarity is achieved. Natural learning can generate lessons learned and may be accompanied by revisions of technical manuals or even specialized training for improved operation and maintenance. Reliability improvement due to written and institutionalized formal procedures and manuals that are a permanent implementation to the system design is part of the reliability growth process. Natural learning is an individual characteristic and is not reliability growth.

The concept of reliability growth is not just theoretical or absolute. Reliability growth is related to factors such as the management strategy toward taking corrective actions, effectiveness of the fixes, reliability requirements, the initial reliability level, reliability funding and competitive factors. For example, one management team may take corrective actions for 90% of the failures seen during testing, while another management team with the same design and test information may take corrective actions on only 65% of the failures seen during testing. Different management strategies may attain different reliability values with the same basic design. The effectiveness of the corrective actions is also relative when compared to the initial reliability at the beginning of testing. If corrective actions give a 400% improvement in reliability for equipment that initially had one tenth of the reliability goal, this is not as significant as a 50% improvement in reliability if the system initially had one half the reliability goal.

Why Reliability Growth?

It is typical in the development of a new technology or complex system to have reliability goals. Each goal will generally be associated with a failure definition. The attainment of the various reliability goals usually involves implementing a reliability program and performing reliability tasks. These tasks will vary from program to program. A reference of common reliability tasks is MIL-STD-785B. It is widely used and readily available. The following table presents the tasks included in MIL-STD 785B.

MIL-STD-785B reliability tasks

	Design and Evaluation		Program Surveillance and Control		Development and Production Testing
101	Reliability Program Plan	201	Reliability modeling	301	Environmental Stress Screening (ESS)
102	Monitor/Control of Subcontractors and Suppliers	202	Reliability Allocations	302	Reliability Development/Growth Test (RDGT) Program

103	Program Reviews	203	Reliability Predictions	303	Reliability Qualification Test (RGT) Program
104	Failure Reporting, Analysis and Corrective Action System (FRACAS)	204	Failure Modes, Effects and Criticality Analysis (FMECA)	304	Production Reliability Acceptance Test (PRAT) Program
105	Failure Review Board (FRB)	205	Sneak Circuit Analysis (SCA)		
		206	Electronic Parts/Circuit Tolerance Analysis		
		207	Parts Program		
		208	Reliability Critical Items		
		209	Effects of Functional Testing, Storage, Handling, Packaging, Transportation and Maintenance		

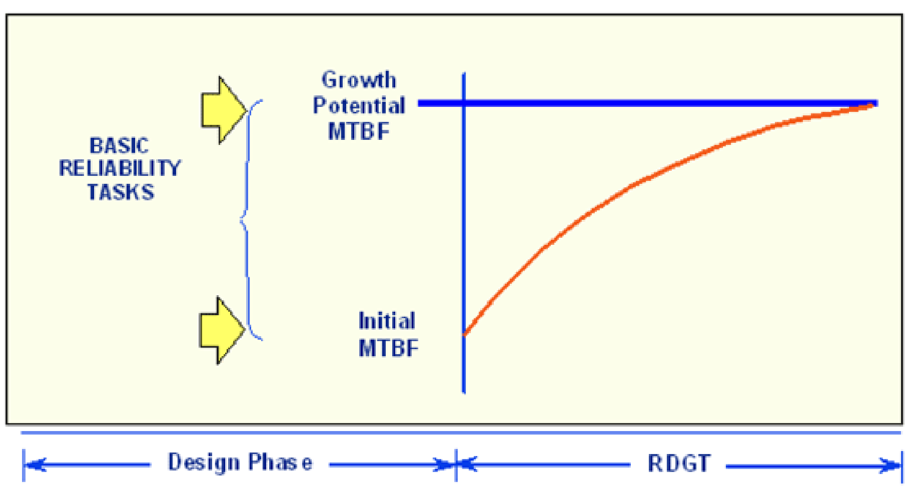
The Program Surveillance and Control tasks (101-105) and Design and Evaluation tasks (201-209) can be combined into a group called *basic reliability tasks*. These are basic tasks in the sense that many of these tasks are included in a comprehensive reliability program. Of the MIL-STD-785B Development & Production Testing tasks (301-304) only the RDGT reliability growth testing task is specifically directed toward finding and correcting reliability deficiencies.

For discussion purposes, consider the reliability metric mean time between failures (MTBF). This term is used for continuous systems, as well as one shot systems. For one shot systems this metric is the mean trial or shot between failures and is equal to $\frac{1}{failure\ probability}$.

The MTBF of the prototypes immediately after the basic reliability tasks are completed is called the *initial MTBF*. This is a key basic reliability task output parameter. If the system is tested after the completion of the basic reliability tasks then the initial MTBF is the mean time between failures as demonstrated from actual data. The initial MTBF is the reliability achieved by the basic reliability tasks and would be the system MTBF if the reliability program were stopped after the basic reliability tasks had been completed.

The initial MTBF after the completion of the basic reliability tasks will generally be lower than the goal. If this is the case then a reliability growth program is appropriate. Formal reliability growth testing is usually conducted after the basic reliability tasks have been completed. For a system subjected to RDGT, the initial MTBF is the system reliability at the beginning of the test. The objective of the testing is to find problems, implement corrective actions and increase the initial reliability. During RDGT, failures are observed and an underlying failure mode is associated with each failure. A failure mode is defined by a problem and a cause. When a new failure mode is observed during testing, management makes a decision not to correct or to correct the failure mode in accordance with the management strategy. Failure modes that are not corrected are called *A modes* and failure modes that receive a corrective action are called *B modes*. If the corrective action is effective for a B mode, then the failure intensity for the failure mode will decrease. The effectiveness of the corrective actions is part of the overall management strategy. If the RDGT testing and corrective action process are conducted long enough, the system MTBF will grow to a mature MTBF value in which further corrective actions are very infrequent. This mature MTBF value is called the *growth potential*. It is a direct function of the design and management strategy. The system growth potential MTBF is the MTBF that would be attained at the end of the basic reliability tasks if all the problem failure modes were uncovered in early design and corrected in accordance with the management strategy.

In summary, the initial MTBF is the value actually achieved by the basic reliability tasks. The growth potential is the MTBF that can be attained if the test is conducted long enough with the current management strategy. See the figure below.



Elements of a Reliability Growth Program

In a formal reliability growth program, one or more reliability goals are set and should be achieved during the development testing program with the necessary allocation or reallocation of resources. Therefore, planning and evaluating are essential factors in a growth process program. A comprehensive reliability growth program needs well-structured planning of the assessment techniques. A reliability growth program differs from a conventional reliability program in that there is a more objectively developed growth standard against which assessment techniques are compared. A comparison between the assessment and the planned value provides a good estimate of whether or not the program is progressing as scheduled. If the program does not progress as planned, then new strategies should be considered. For example, a reexamination of the problem areas may result in changing the management strategy so that more problem failure modes that surface during the testing actually receive a corrective action instead of a repair. Several important factors for an effective reliability growth program are:

- Management: the decisions made regarding the management strategy to correct problems or not correct problems and the effectiveness of the corrective actions
- Testing: provides opportunities to identify the weaknesses and failure modes in the design and manufacturing process
- Failure mode root cause identification: funding, personnel and procedures are provided to analyze, isolate and identify the cause of failures
- Corrective action effectiveness: design resources to implement corrective actions that are effective and support attainment of the reliability goals
- Valid reliability assessments: using valid statistical methodologies to analyze test data in order to assess reliability

The management strategy may be driven by budget and schedule but it is defined by the actual decisions of management in correcting reliability problems. If the reliability of a failure mode is known through analysis or testing, then management makes the decision either not to fix (no corrective action) or to fix (implement a corrective action) that failure mode. Generally, if the reliability of the failure mode meets the expectations of management, then no corrective actions would be expected. If the reliability of the failure mode is below expectations, the management strategy would generally call for the implementation of a corrective action.

Another part of the management strategy is the effectiveness of the corrective actions. A corrective action typically does not eliminate a failure mode from occurring again. It simply reduces its rate of occurrence. A corrective action, or fix, for a problem failure mode typically removes a certain amount of the mode's failure intensity, but a certain amount will remain in the system. The fraction decrease in the problem mode failure intensity due to the corrective action is called the *effectiveness factor* (EF). The EF will vary from failure mode to failure mode but a typical average for government and industry systems has been reported to be about 0.70. With an EF equal to 0.70, a corrective action for a failure mode removes about 70% of the failure intensity, but 30% remains in the system.

Corrective action implementation raises the following question: "What if some of the fixes cannot be incorporated during testing?" It is possible that only some fixes can be incorporated into the product during testing. However, others may be delayed until the end of the test since it may be too expensive to stop and then restart the test, or the equipment may be too complex for performing a complete teardown. Implementing delayed fixes usually results in a distinct jump in the reliability of the system at the end of the test phase. For corrective actions implemented during testing, the additional follow-on testing provides feedback on how effective the corrective actions are and provides opportunity to uncover additional problems that can be corrected.

Evaluation of the delayed corrective actions is provided by projected reliability values. The *demonstrated reliability* is based on the actual current system performance and estimates the system reliability due to corrective actions incorporated during testing. The *projected reliability* is based on the impact of the delayed fixes that will be incorporated at the end of the test or between test phases.

When does a reliability growth program take place in the development process? Actually, there is more than one answer to this question. The modern approach to reliability realizes that typical reliability tasks often do not yield a system that has attained the reliability goals or attained the cost-effective reliability potential in the system. Therefore, reliability growth may start very early in a program, utilizing Integrated Reliability Growth Testing (IRGT). This approach recognizes that reliability problems often surface early in engineering tests. The focus of these engineering tests is typically on performance and not reliability. IRGT simply piggybacks reliability failure reporting, in an informal fashion, on all engineering tests. When a potential reliability problem is observed, reliability engineering is notified and appropriate design action is taken. IRGT will usually be implemented at the same time as the basic reliability tasks. In addition to IRGT, reliability growth may take place during early prototype testing, during dedicated system testing, during production testing, and from feedback through any manufacturing or quality testing or inspections. The formal dedicated testing or RDGT will typically take place after the basic reliability tasks have been completed.

Note that when testing and assessing against a product's specifications, the test environment must be consistent with the specified environmental conditions under which the product specifications are defined. In addition, when testing subsystems it is important to realize that interaction failure modes may not be generated until the subsystems are integrated into the total system.

Why Are Reliability Growth Models Needed?

In order to effectively manage a reliability growth program and attain the reliability goals, it is imperative that valid reliability assessments of the system be available. Assessments of interest generally include estimating the current reliability of the system configuration under test and estimating the projected increase in reliability if proposed corrective actions are incorporated into the system. These and other metrics give management information on what actions to take in order to attain the reliability goals. Reliability growth assessments are made in a dynamic environment where the reliability is changing due to corrective actions. The objective of most reliability growth models is to account for this changing situation in order to estimate the current and future reliability and other metrics of interest. The decision for choosing a particular growth model is typically based on how well it is expected to provide useful information to management and engineering. Reliability growth can be quantified by looking at various metrics of interest such as the increase in the MTBF, the decrease in the failure intensity or the increase in the mission success probability, which are generally mathematically related and can be derived from each other. All key estimates used in reliability growth management such as demonstrated reliability, projected reliability and estimates of the growth potential can generally be expressed in terms of the MTBF, failure intensity or mission reliability. Changes in these values, typically as a function of test time, are collectively called *reliability growth trends* and are usually presented as reliability growth curves. These curves are often constructed based on certain mathematical and statistical models called *reliability growth models*. The ability to accurately estimate the demonstrated reliability and calculate projections to some point in the future can help determine the following:

- Whether the stated reliability requirements will be achieved
- The associated time for meeting such requirements
- The associated costs of meeting such requirements
- The correlation of reliability changes with reliability activities

In addition, demonstrated reliability and projections assessments aid in:

- Establishing warranties
- Planning for maintenance resources and logistic activities
- Life-cycle-cost analysis

Terminology

Some basic terms that relate to reliability growth and repairable systems analysis are presented below. Additional information on terminology in Weibull++ can be found in the Reliability Growth Analysis Glossary.

Failure Rate vs. Failure Intensity

Failure rate and failure intensity are very similar concepts. The term *failure intensity* typically refers to a process such as a reliability growth program. The system age when a system is first put into service is time 0. Under the non-homogeneous Poisson process (NHPP), the first failure is governed by a distribution $F(x)$ with failure rate of $r(x)$. Each succeeding failure is governed by the intensity function $u(t)$ of the process. Let t be the age of the system and let Δt be a very small value. The probability that a system of age t fails between t and $t+\Delta t$ is given by the intensity function $u(t)\Delta t$. Notice that this probability is not conditioned on not having any system failures up to time t, as is the case for a failure rate. The failure intensity $u(t)$ for the NHPP has the same functional form as the failure rate governing the first system failure. Therefore, $u(t)=r(t)$, where $r(t)$ is the failure rate for the distribution function of the first system failure. If the first system failure follows the Weibull distribution, the failure rate is:

$$r(x)=\lambda\beta x^{\beta-1}$$

Under minimal repair, the system intensity function is:

$$u(t)=\lambda\beta t^{\beta-1}$$

This is the power law model. It can be viewed as an extension of the Weibull distribution. The Weibull distribution governs the first system failure and the power law model governs each succeeding system failure. Additional information on the power law model is available in Repairable Systems Analysis.

Instantaneous vs. Cumulative

In Weibull++, metrics such as MTBF and failure intensity can be calculated as instantaneous or cumulative. *Cumulative MTBF* is the average time-between-failure from the beginning of the test (i.e., t = 0) up to time t. The *instantaneous MTBF* is the average time-between-failure in a given interval, dt. Consider a grouped data set, where 4 failures are found in the interval 0-100 hours and 2 failures are found in the second interval 100-180 hours. The cumulative MTBF at 180 hours of test time is equal to 180/6 = 30 hours. However, the instantaneous MTBF is equal to 80/2 = 40 hours. Another analogy for this relates to the miles per gallon (mpg) that your car gets. In many cars, there are readouts that indicates the current mpg as you are driving. This is the instantaneous value. The average mpg on the current tank of gas is the cumulative value.

Relative to the Crow-AMSAA (NHPP) model, when beta is equal to one, the system's MTBF is not changing over time; therefore, the cumulative MTBF equals the instantaneous MTBF. If beta is greater than one, then the system's MTBF is decreasing over time and the cumulative MTBF is greater than the instantaneous MTBF. If beta is less than one, then the system's MTBF is increasing over time and the cumulative MTBF is less than the instantaneous MTBF.

Now the concern is, how do you know whether you should estimate the instantaneous or cumulative value of a metric (e.g., MTBF or failure intensity)? In general, system requirements are usually represented as instantaneous values. After all, you want to know where the system is *now* in terms of its MTBF. The cumulative MTBF can be used to observe trends in the data.

Time Terminated vs. Failure Terminated

When a reliability growth test is conducted, it must be determined how much operation time the system will accumulate during the test (i.e., the point at which the test will reach its end). There are two possible options under which a reliability growth test may be terminated (or truncated): time terminated or failure terminated.

- A time terminated test is stopped after a certain amount of test time.
- A failure terminated test is stopped at a time which corresponds to a failure (e.g., the time of the tenth failure).

It is determined a priori whether a test is time terminated or failure terminated. The test data does not determine this.

Reliability Growth Analysis Data Types

IN THIS CHAPTER

Reliability growth analysis can be conducted using different data types. This chapter explores and examines the possible data schemes, and outlines the available models for each data type. The data types for developmental testing (traditional reliability growth analysis) will be discussed first. Then we will discuss the data types that support the use of reliability growth analysis models for analyzing fielded systems (either for repairable systems analysis or fleet data analysis).

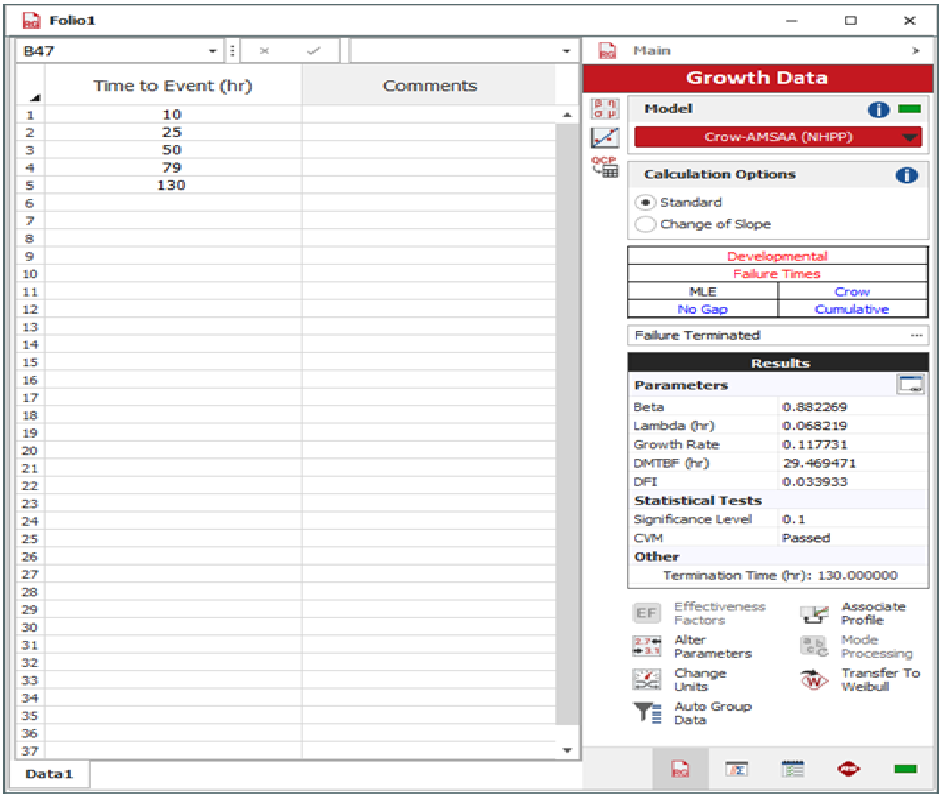
Developmental Testing Data Types

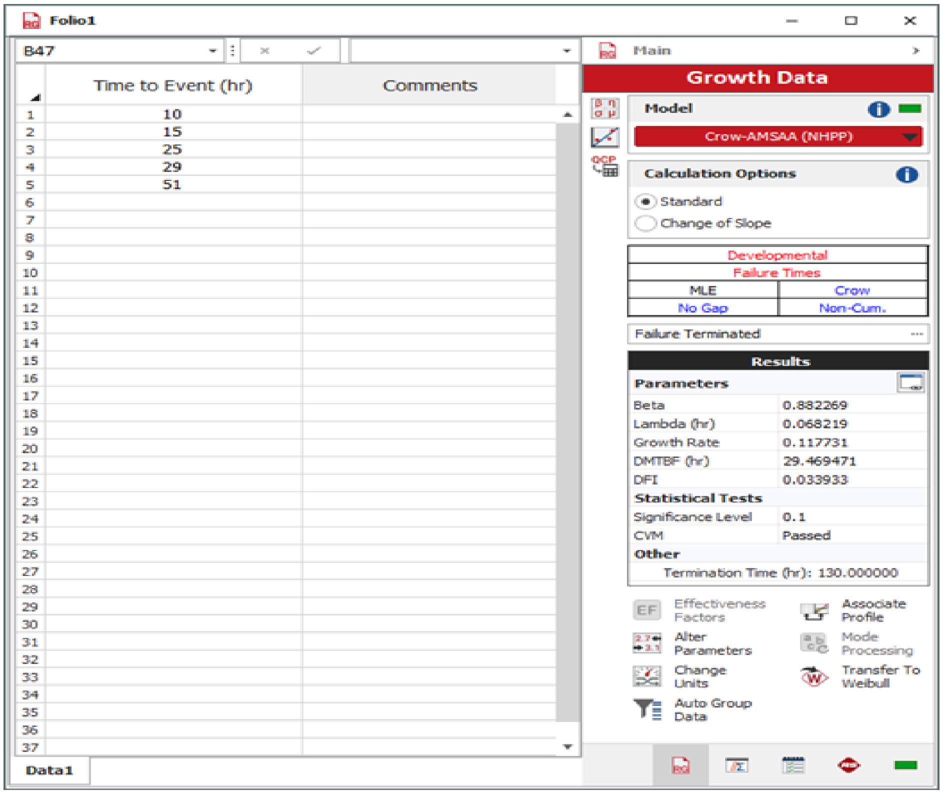
Time-to-Failure Data

Time-to-failure (continuous) data is the most commonly observed type of reliability growth data. It involves recording the times-to-failure for the unit(s) under test. Time-to-failure data can be applied to a single unit or system or to multiple units or systems. There are multiple data entry schemes for this data type and each is presented next.

Failure Times Data

This data type is used for tests where the actual system failure times are tracked and recorded. The data can be entered in a cumulative format (where each row shows the total amount of test time) or non-cumulative format (where each row shows the incremental test time), as shown next.





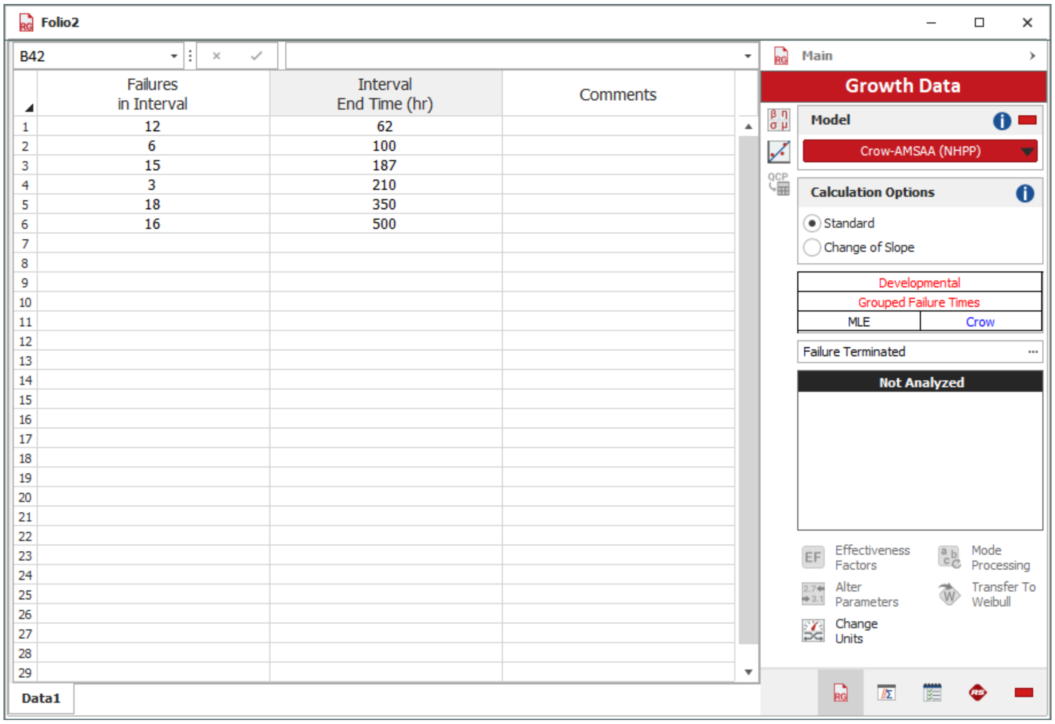
Grouped Failure Times

This data type is used for tests where the exact failure times are unknown and only the number of failures within a time interval are recorded (e.g., inspection data). For a single system, multiple failures can occur before the operator stops the test. In this case, X number of failures are found after Y hours of test time. Failures X_1 , X_2 , X_3 , etc. could have occurred at any time period up to the termination time, thus exact times for each failure are not available. This is commonly called *interval* or *grouped* data. Grouped data can be helpful to merge data from multiple locations.

When multiple units are tested, the units are inspected at predetermined time intervals and the number of failed units is recorded. When entering the time at which the failures occurred for grouped data, the time is equal to the total accumulated test time for all of the units being tested. The number of failed units is simply equal to the number of failures that occurred during the current interval. The next figure shows an example of data entry for grouped data.

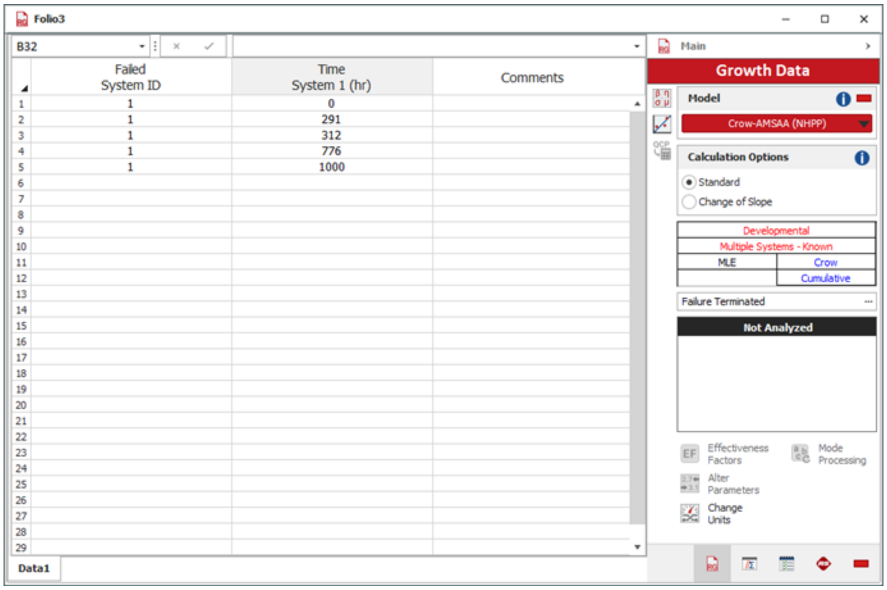
Assumptions:

- Intervals do not have to be of equal length.
- All units within each interval should be the same configuration. However, the number of units does not have to be the same.



Multiple Systems (Known Operating Times)

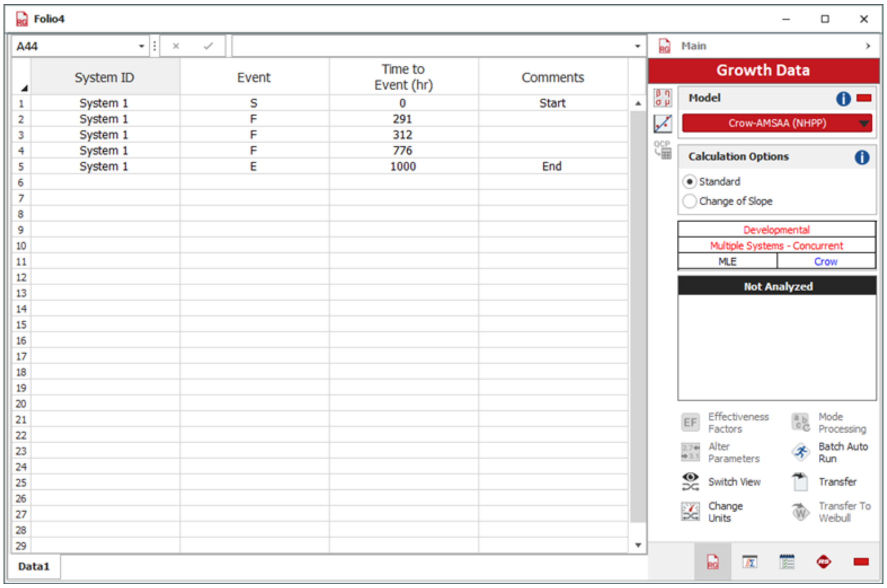
This data type is used for tests where a number of systems are tested, and if a failure occurs in any system, a corrective action is taken on the failed unit and any design changes are incorporated into all test systems. Once the corrective actions have been implemented, the test is resumed. In this data type, the systems can accumulate usage at different rates. In addition, the systems do not have to start the test at the same time. The basic idea of Multiple Systems (Known Operating Times) is that when one system fails, the usage on the other systems on test must also be known. This is a flexible data type, but it is also the most demanding given the information required on all systems. The time-to-failure for the failed system, along with the current operating times of all other systems, is recorded. The data can be cumulative or non-cumulative. Consider the following data sheet where the **Failed Unit ID** column indicates which unit failed. For example, if you enter **2** into the "Failed Unit ID" column, this indicates that the unit in the **Time Unit 2** column is the one that has failed. For the units that did not fail, you must enter the operating time at the time of the other unit's failure.



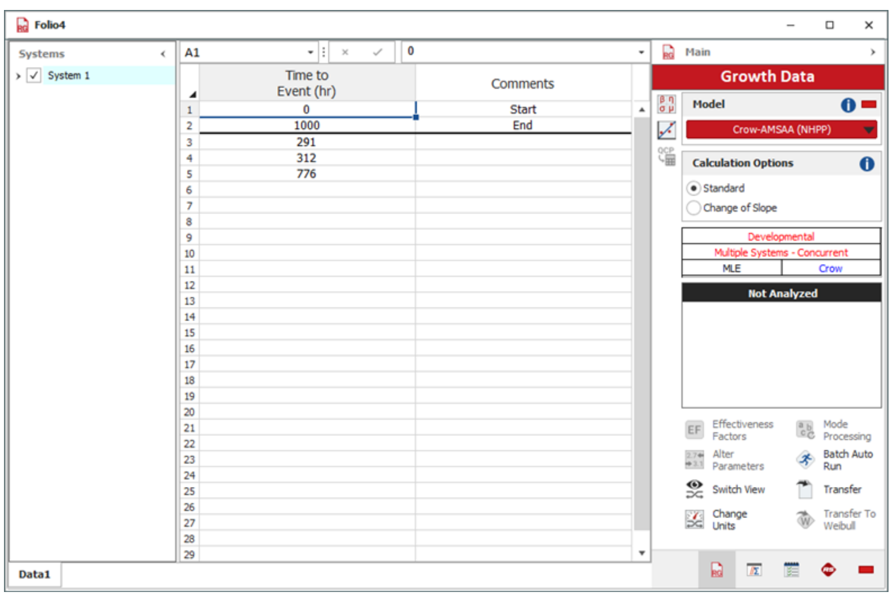
In this example, two units are undergoing testing and the units do not accumulate age at the same rate. At 10 hours into the test, unit 1 fails and corrective action is taken on both units 1 and 2. By this time, both units have accumulated 10 hours of operation. At 17 hours, unit 2 fails, and corrective action is again implemented on both units; however, unit 1 has accumulated 5 hours of operating time and unit 2 has accumulated 7 hours since the last event. The rest of the data can be interpreted in a similar manner.

Multiple Systems (Concurrent Operating Times)

This data type is used for tests where a number of systems are tested. The start time, failure time(s) and end time for each system are recorded. This data type assumes uniform time accumulation and that the systems are tested simultaneously. When a corrective action is implemented on a failed system, the fix is also applied to the non-failed systems. This results in all systems on test having the same configuration throughout the test. As an example, consider the data of two systems shown in the following figure. In this case, the folio is shown in Normal view.

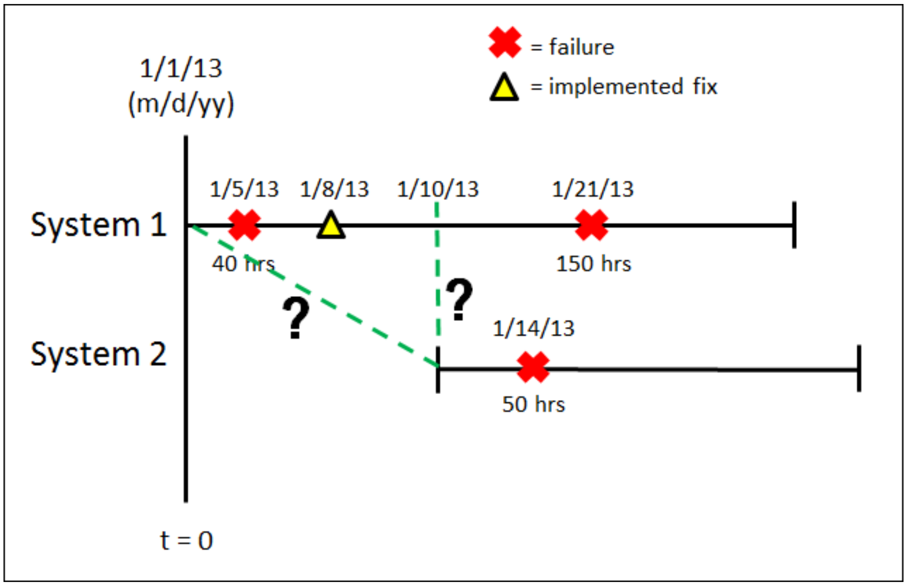


System 1 begins testing at time equals 0 (with a start event, S). Note that when entering data within the Normal view, each system must be initiated with a start event. The failures encountered by System1 are corrected at 281, 312 and 776 (with failure events, F). Testing stops at 1000 hours (with an end event, E). System 2 begins testing at time equals 0 and failures are encountered and corrected at 40, 222 and 436. Testing stops at 500 hours. The next figure shows the folio and the same data set in the Advanced Systems view.



Multiple Systems with Dates

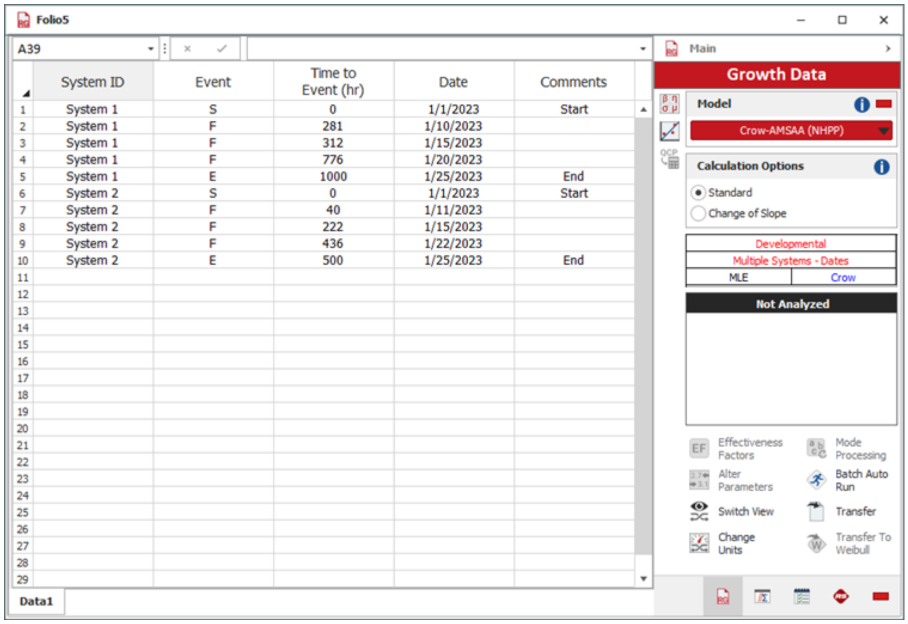
This data type is similar to Multiple Systems (Concurrent Operating Times) except that a date is now associated with each failure time, as well as for the start and end time of each system. This assumes noncontinuous usage, and the software computes equivalent (average) usage rates. To determine when dates are important and whether or not they should be included in your analysis, consider the picture below (not drawn to scale).



Two systems are placed in a reliability growth test. System 1 starts testing on January 1, 2013 and System 2 does not start testing until January 10, 2013. When testing begins on System 2, what is its configuration? Does the configuration of System 2 on January 10 match System 1 on January 10 (there was a fix implemented on System 1 on January 8) or does it match System 1's configuration on January 1? If the configuration of System 2 matches System 1 on January 1, then dates are not important. This means the previous data type, Multiple Systems (Concurrent Operating Times) can be used, and the timeline for System 2 can be shifted to the left. However, if the configuration of System 2 matches System 1 on January 10, then dates are important and the Multiple Systems with Dates data type should be used.

The goal is to sum up the accumulated hours across all systems with the same configuration. If System 2 on January 10 matches System 1 on January 10, then it would not be correct to shift System 2 to the left to t = 0. If this were done, the first failure on System 1 on January 5 would correspond to 80 hours: 40 hours for System 1 plus 40 hours for System 2 on the equivalent system timeline. However, with dates this same failure now corresponds to 40 hours on the equivalent system timeline since System 2 is not operating yet. For the failure on January 14 on System 2, the total test time on this date must also account for the test time accumulated on System 1. However, there is not an event on System 1 on January 14. Therefore, Weibull++ uses linear interpolation to estimate the operating time on System 1 on January 14.

The next figure shows an example of Multiple Systems with Dates in Weibull++.

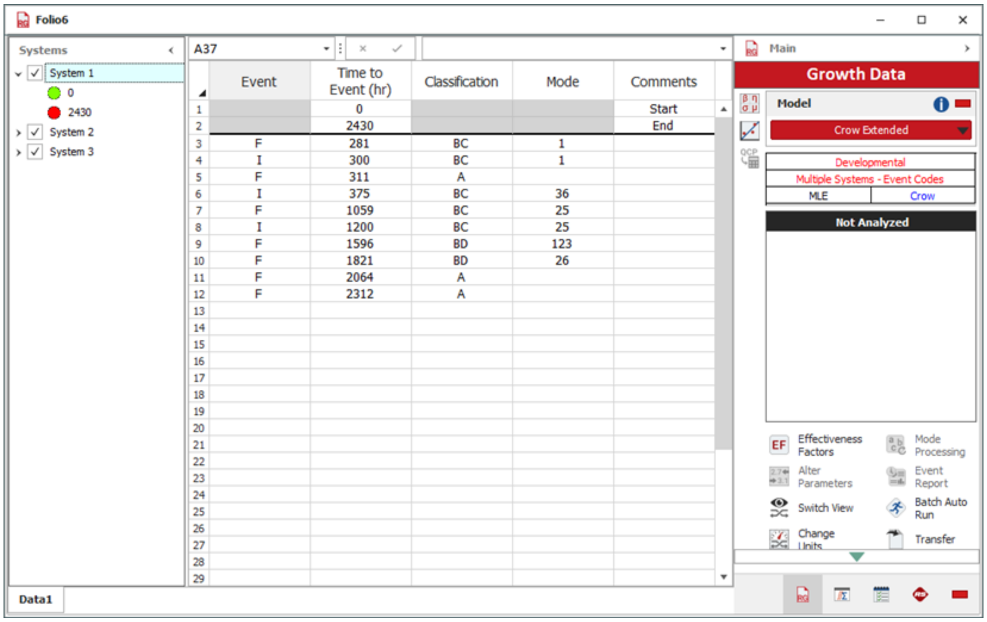


Multiple Systems with Event Codes

The Multiple Systems with Event Codes data type is basically the same as the Multiple Systems (Concurrent Operating Times) data type. These data types are used to analyze the failure data from a reliability growth test in which a number of systems are tested concurrently and the implemented fixes are tracked during the test phase. All of the systems under test are assumed to have the same system hours at any given time. However, the Multiple Systems with Event Codes data type does have two differences:

- The Crow Extended model is always the underlying model.
- Event codes, similar to the ones used for the Crow Extended - Continuous Evaluation model, are part of the data entry.

Even though event codes are added to the data entry, the underlying assumptions associated with the Crow Extended model have not changed. Additional information on Multiple Systems with Event Codes is presented on the Crow Extended page. The next figure shows an example of this type of data.



Models for Time-to-Failure (Continuous) Data

The following models can be used to analyze time-to-failure (continuous) data sets. Models and examples using the different data types are discussed in later chapters.

- Duane
- Crow-AMSAA (NHPP)
- Crow Extended

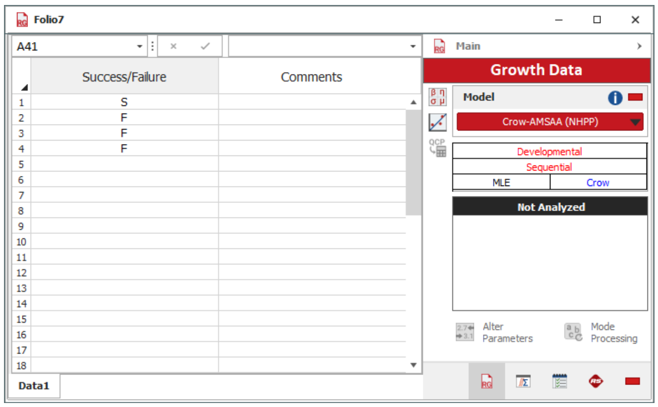
Discrete Data

Discrete data are also referred to as *success/failure* or *attribute data*. These data sets consist of data from a test where a unit is tested with only two possible outcomes: success or failure. An example of this is a missile that gets fired once and it either succeeds or fails. The data types available for analyzing discrete data with the Weibull++ software are:

- Sequential
- Sequential with Mode
- Grouped per Configuration
- Mixed Data

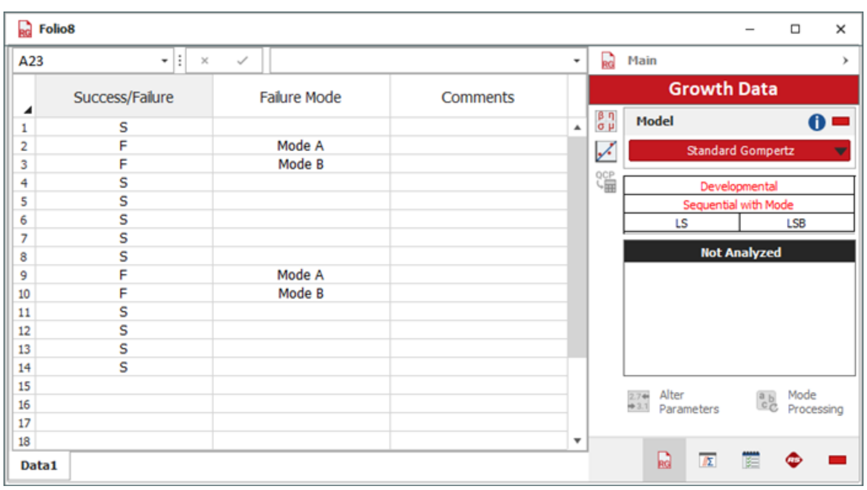
Sequential Data

For sequential data, an item is tested with only two possible outcomes: success or failure. This could be a one-shot item such as a missile or an entire system that either succeeds or fails. The item is then inspected after each trial and redesigned/repaired before the next trial. The following figure shows an example of this data type, where the row number in the data sheet represents the sequence of the trials. In this data set, trial #1 succeeded, trial #2 failed, and so on.



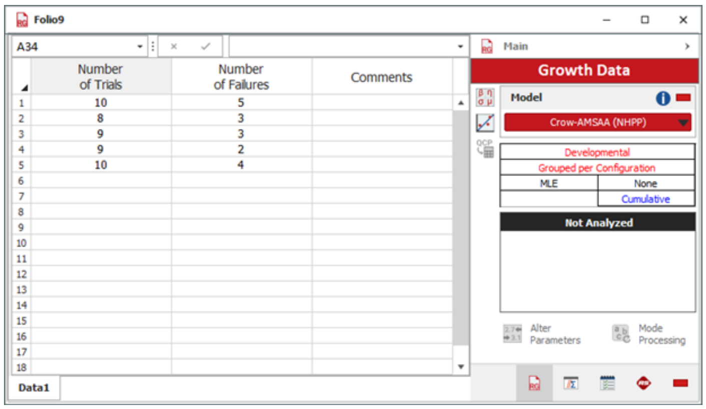
Sequential with Mode Data

Often after failure analysis you know the reason for failure during a particular trial. If this is the case, the reason for each failure can also be used in the analysis. This data entry is identical to the sequential data with the exception that a failure code, mode or ID is added after each failure so that the analysis can take into account different failure modes. The next figure shows an example of this type of data.



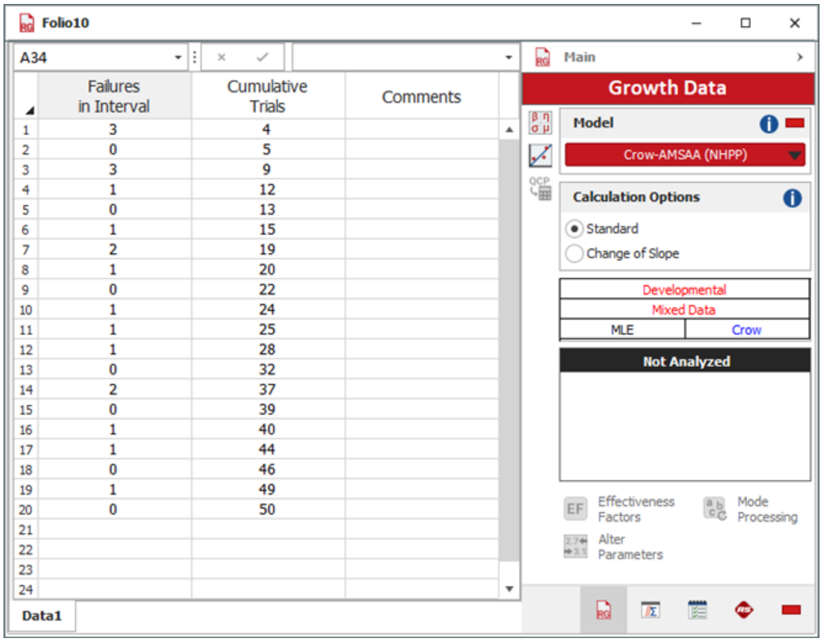
Grouped per Configuration Data

This data type is used when multiple items, instead of a single item, are tested and the number of units that fail are recorded for each configuration. The row numbers that appear on the left side of the worksheet, as shown in the following figure, represent the unique configurations. For example, row 1 indicates configuration 1 in which 10 missiles were fired and 5 failed, row 2 indicates configuration 2 in which 8 missiles were fired and 3 failed, etc. The data can be cumulative or non-cumulative.



Mixed Data

The Mixed data type is basically a combination of Grouped per Configuration and Sequential data. It allows for testing to be conducted in groups of configurations, individual trial by trial or a combination of individual trials and configurations where more than one trial is tested. The following figure shows an example of this data type. For example the first row of this data sheet shows that 3 failures occurred in the first 4 trials, the second row shows that there were no failures in the next trial, while the third row shows that 3 failures occurred in the next 4 trials.



Models for Discrete Data

The following models can be used to analyze discrete data. Models and examples using the different data types are discussed in later chapters.

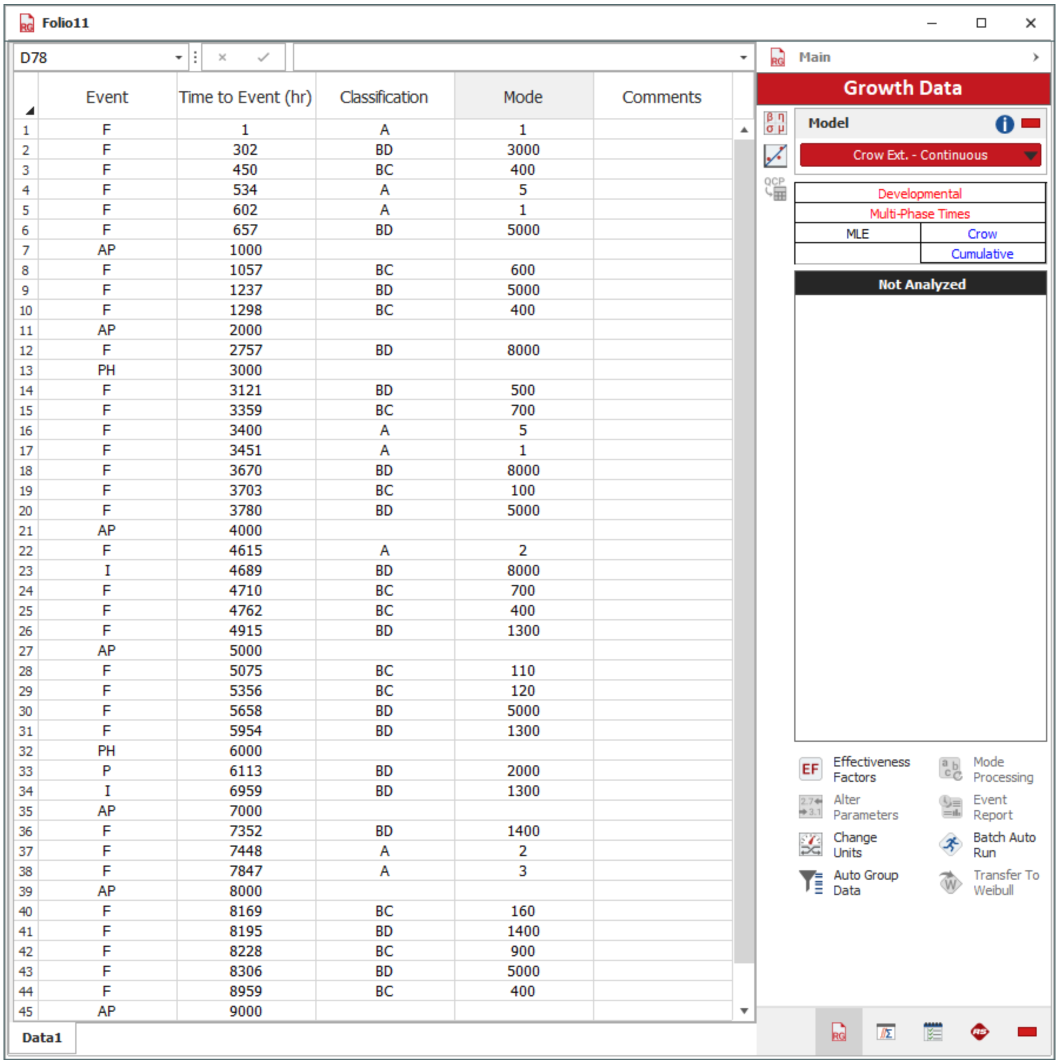
- Duane
- Crow-AMSAA (NHPP)
- Crow Extended
- Lloyd-Lipow
- Gompertz and Modified Gompertz
- Logistic

Multi-Phase Data

Reliability data can be analyzed across multiple phases. This is useful when an overall reliability growth program is planned and involves multiple test phases.

(Multi-Phase) Failure Times Data

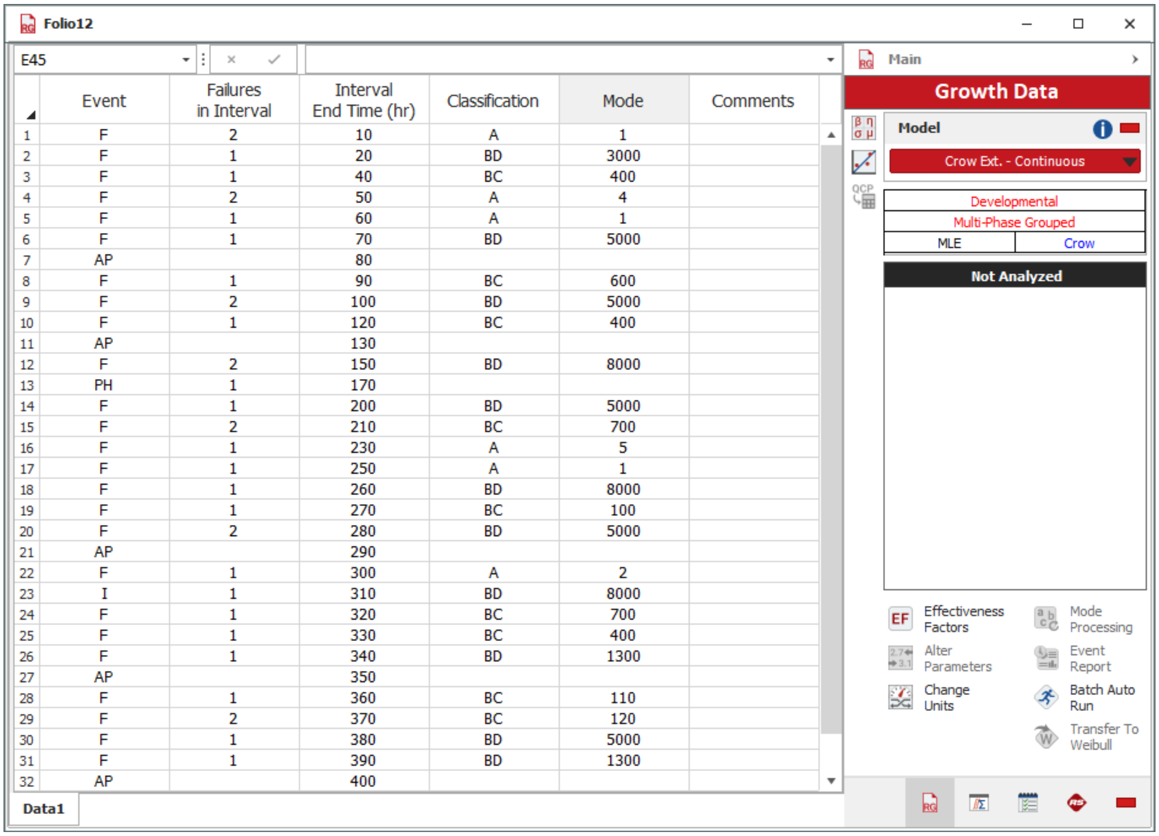
This data type can be used for tests that span multiple phases and the exact failure times are recorded. The figure below shows an example of multi-phase failure times data, where the different events signify failures (F), test phases (PH) or analysis points (AP).



	Event	Time to Event (hr)	Classification	Mode	Comments
1	F	1	A	1	
2	F	302	BD	3000	
3	F	450	BC	400	
4	F	534	A	5	
5	F	602	A	1	
6	F	657	BD	5000	
7	AP	1000			
8	F	1057	BC	600	
9	F	1237	BD	5000	
10	F	1298	BC	400	
11	AP	2000			
12	F	2757	BD	8000	
13	PH	3000			
14	F	3121	BD	500	
15	F	3359	BC	700	
16	F	3400	A	5	
17	F	3451	A	1	
18	F	3670	BD	8000	
19	F	3703	BC	100	
20	F	3780	BD	5000	
21	AP	4000			
22	F	4615	A	2	
23	I	4689	BD	8000	
24	F	4710	BC	700	
25	F	4762	BC	400	
26	F	4915	BD	1300	
27	AP	5000			
28	F	5075	BC	110	
29	F	5356	BC	120	
30	F	5658	BD	5000	
31	F	5954	BD	1300	
32	PH	6000			
33	P	6113	BD	2000	
34	I	6959	BD	1300	
35	AP	7000			
36	F	7352	BD	1400	
37	F	7448	A	2	
38	F	7847	A	3	
39	AP	8000			
40	F	8169	BC	160	
41	F	8195	BD	1400	
42	F	8228	BC	900	
43	F	8306	BD	5000	
44	F	8959	BC	400	
45	AP	9000			

(Multi-Phase) Grouped Failure Times Data

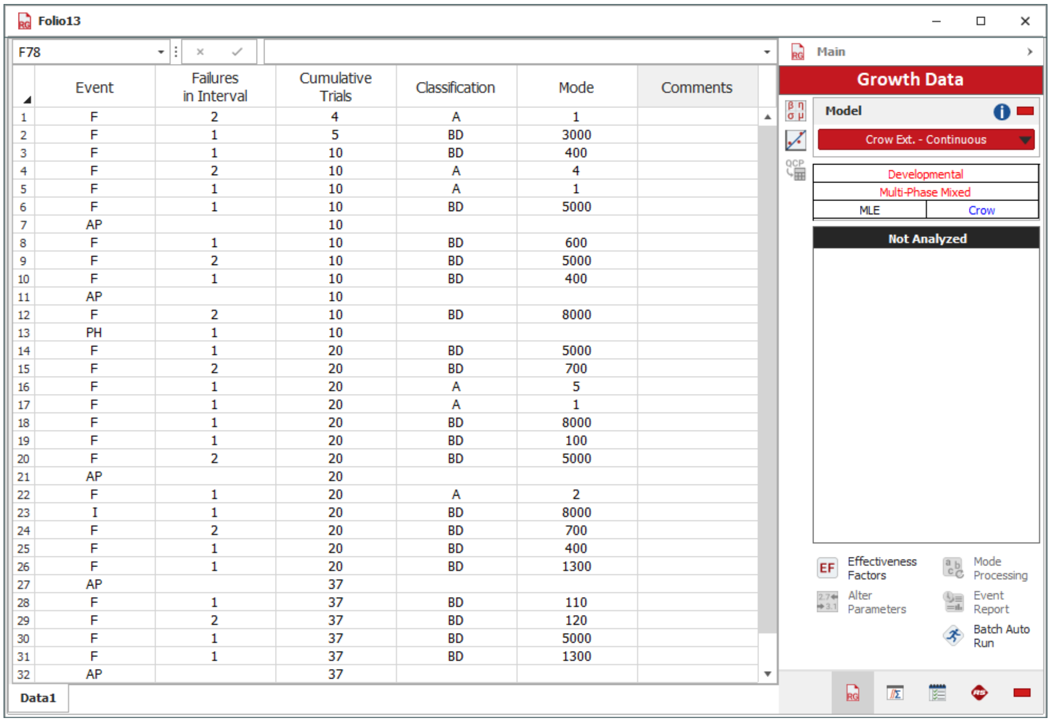
This data type can be used for tests that span multiple phases and the exact failure times are unknown. Only the number of failures within a time interval are recorded, as shown in the figure below.



	Event	Failures in Interval	Interval End Time (hr)	Classification	Mode	Comments
1	F	2	10	A	1	
2	F	1	20	BD	3000	
3	F	1	40	BC	400	
4	F	2	50	A	4	
5	F	1	60	A	1	
6	F	1	70	BD	5000	
7	AP		80			
8	F	1	90	BC	600	
9	F	2	100	BD	5000	
10	F	1	120	BC	400	
11	AP		130			
12	F	2	150	BD	8000	
13	PH	1	170			
14	F	1	200	BD	5000	
15	F	2	210	BC	700	
16	F	1	230	A	5	
17	F	1	250	A	1	
18	F	1	260	BD	8000	
19	F	1	270	BC	100	
20	F	2	280	BD	5000	
21	AP		290			
22	F	1	300	A	2	
23	I	1	310	BD	8000	
24	F	1	320	BC	700	
25	F	1	330	BC	400	
26	F	1	340	BD	1300	
27	AP		350			
28	F	1	360	BC	110	
29	F	2	370	BC	120	
30	F	1	380	BD	5000	
31	F	1	390	BD	1300	
32	AP		400			

(Multi-Phase) Mixed Data

This data type can be used for tests that span multiple phases and it allows for configuration in groups, individual trial by trial, or a mixed combination of individual trials and configurations of more than one trial. An example of this data type can be seen in the figure below.



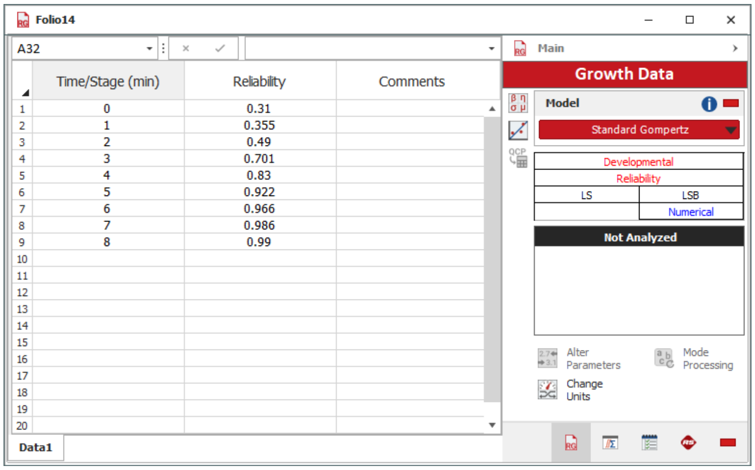
	Event	Failures in Interval	Cumulative Trials	Classification	Mode	Comments
1	F	2	4	A	1	
2	F	1	5	BD	3000	
3	F	1	10	BD	400	
4	F	2	10	A	4	
5	F	1	10	A	1	
6	F	1	10	BD	5000	
7	AP		10			
8	F	1	10	BD	600	
9	F	2	10	BD	5000	
10	F	1	10	BD	400	
11	AP		10			
12	F	2	10	BD	8000	
13	PH	1	10			
14	F	1	20	BD	5000	
15	F	2	20	BD	700	
16	F	1	20	A	5	
17	F	1	20	A	1	
18	F	1	20	BD	8000	
19	F	1	20	BD	100	
20	F	2	20	BD	5000	
21	AP		20			
22	F	1	20	A	2	
23	I	1	20	BD	8000	
24	F	2	20	BD	700	
25	F	1	20	BD	400	
26	F	1	20	BD	1300	
27	AP		37			
28	F	1	37	BD	110	
29	F	2	37	BD	120	
30	F	1	37	BD	5000	
31	F	1	37	BD	1300	
32	AP		37			

Models for Multi-Phase Data

The Crow Extended - Continuous Evaluation model is used to analyze data across multiple phases.

Reliability Data

Reliability data consists of entering the reliability of the equipment at different times or stages. An example is shown in the figure below. In this case, the process is monitored at pre-defined time intervals and the reliability is recorded. The reliability can be computed by a simple ratio of the number of units still functioning vs. the number of units that entered the test stage or by using life data analysis and related methods (e.g., Weibull analysis).



Models for Reliability Data

The following models can be used to analyze reliability data sets. Models and examples using different data types are discussed in later chapters.

- Lloyd-Lipow
- Gompertz and Modified Gompertz
- Logistic

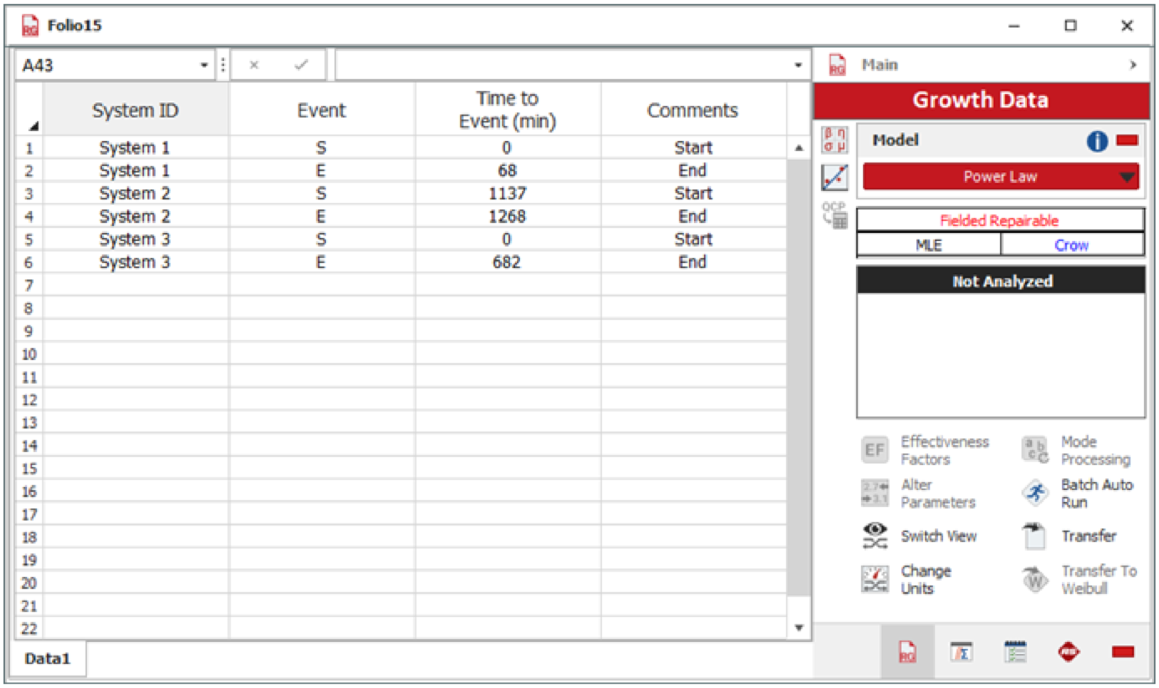
Fielded Systems

Fielded systems are systems that are used by customers in the field and for which failure information is not derived from an in-house test. This type of data is analogous to warranty data. The data types available for fielded systems data entry are:

- Repairable Systems
- Fleet

Repairable Systems

Repairable Systems data is identical in format to the Multiple Systems (Concurrent Operating Times) data. It also can be entered in the Normal or Advanced Systems view. The following figure illustrates a sample data set. In repairable systems, the purpose of the analysis is not to evaluate reliability growth but rather to obtain reliability estimates for the system, including expected number of failures, reliability at a given time, and so forth.



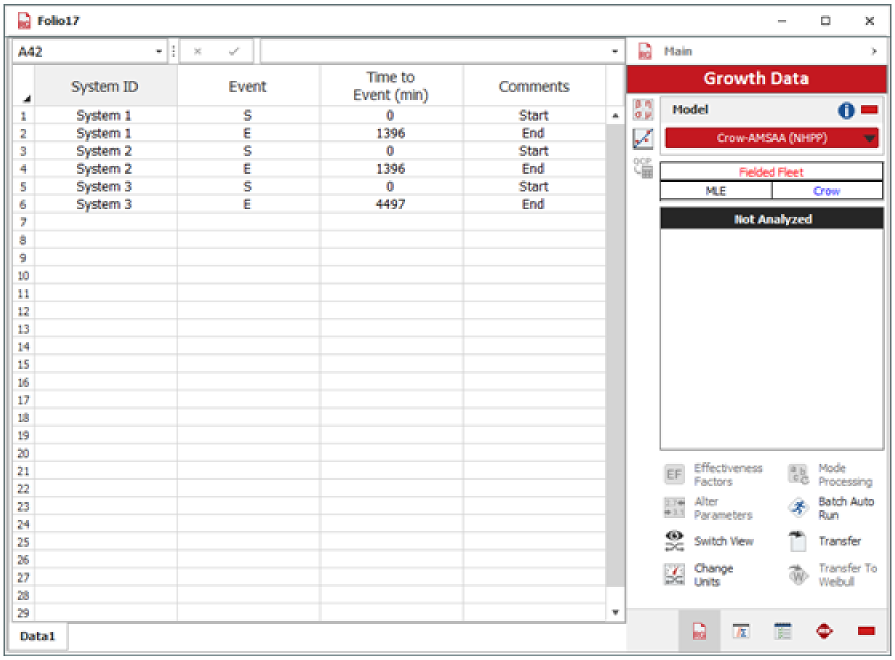
Models for Repairable Systems Data

The following models can be used to analyze repairable systems data. Models and examples using different data types are discussed in Repairable Systems Analysis.

- Power Law
- Crow Extended (also called Operational Testing)

Fleet Data

This data type is used to analyze the entire population (fleet). The data entry for this data type is similar to the data entry for repairable systems; however, the overall data analysis is again different. In repairable systems, the reliability of a single system can be tracked and quantified, whereas in a fleet analysis, data from the entire fleet as a whole is analyzed. The figure below presents an example of data entered for fleet analysis.



Models for Fleet Data

The following models can be used to analyze fleet data. Models and examples using different data types are discussed in later chapters.

- Crow-AMSAA (NHPP)
- Crow Extended

Developmental Testing

IN THIS CHAPTER

Reliability growth analysis is the process of collecting, modeling, analyzing and interpreting data from the reliability growth development test program (development testing). In addition, reliability growth models can be applied for data collected from the field (fielded systems). Fielded systems analysis also includes the ability to analyze data of complex repairable systems. Depending on the metric(s) of interest and the data collection method, different models can be utilized (or developed) to analyze the growth processes.

Example

As an example of such a model development, consider the simple case of a wine glass designed to withstand a fall of three feet onto a level cement surface.

The success/failure result of such a drop is determined by whether or not the glass breaks.

Furthermore, assume that:

- You will continue to drop the glass, looking at the results and then adjusting the design after each failure until you are sure that the glass will not break.
- Any redesign effort is either completely successful or it does not change the inherent reliability (R). In other words, the reliability is either 1 or R , $0 < R < 1$.
- When testing the product, if a success is encountered on any given trial, no corrective action or redesign is implemented.
- If the trial fails, then you will redesign the product.
- When the product is redesigned, assume that the probability of fixing the product permanently before the next trial is α . In other words, the glass may or may not have been fixed.
- Let $P_n(0)$ and $P_n(1)$ be the probabilities that the glass is unreliable and reliable, respectively, just before the n^{th} trial, and that the glass is in the unreliable state just before the first trial, $P_1(0)$.

Now given the above assumptions, the question of how the glass could be in the unreliable state just before trial n can be answered in two mutually exclusive ways:

The first possibility is the probability of a successful trial, $(1-p)$, where p is the probability of failure in trial $n-1$, while being in the unreliable state, $P_{n-1}(0)$, before the $n-1$ trial, or:

$$(1-p)P_{n-1}(0)$$

Secondly, the glass could have failed the trial, with probability p , when in the unreliable state, $P_{n-1}(0)$, and having failed the trial, an unsuccessful attempt was made to fix, with probability $(1-\alpha)$, or:

$$p(1-\alpha)P_{n-1}(0)$$

Therefore, the sum of these two probabilities, or possible events, gives the probability of being unreliable just before trial n :

$$P_n(0) = (1-p)P_{n-1}(0) + p(1-\alpha)P_{n-1}(0)$$

or:

$$P_n(0) = (1-p\alpha)P_{n-1}(0)$$

By induction, since $P_1(0) = 1$:

$$P_n(0) = (1-p\alpha)^{n-1}$$

To determine the probability of being in the reliable state just before trial n, the above equation is subtracted from 1, therefore:

$$P_n(1) = 1-(1-p\alpha)^{n-1}$$

Define the reliability R_n of the glass as the probability of not failing at trial n. The probability of not failing at trial n is the sum of being reliable just before trial n, $(1-(1-p\alpha)^{n-1})$, and being unreliable just before trial n but not failing $((1-p\alpha)^{n-1}(1-p))$, thus:

$$R_n = \left(1-(1-p\alpha)^{n-1}\right) + \left((1-p)(1-p\alpha)^{n-1}\right)$$

or:

$$R_n = 1-(1-p\alpha)^{n-1}\cdot p$$

Now instead of $P_1(0) = 1$, assume that the glass has some initial reliability or that the probability that the glass is in the unreliable state at $n = 1$, $P_1(0) = \beta$, then:

$$R_n = 1-\beta p(1-p\alpha)^{n-1}$$

When $\beta < 1$, the reliability at the n^{th} trial is larger than when it was certain that the device was unreliable at trial $n = 1$. A trend of reliability growth is observed in the above equation.

Let $A = \beta p$ and $C = ln\left(\frac{1}{1-p\alpha}\right) > 0$, then:

$$R_n = 1-Ae^{-C(n-1)}$$

This equation is now a model that can be utilized to obtain the reliability (or probability that the glass will not break) after the n^{th} trial.

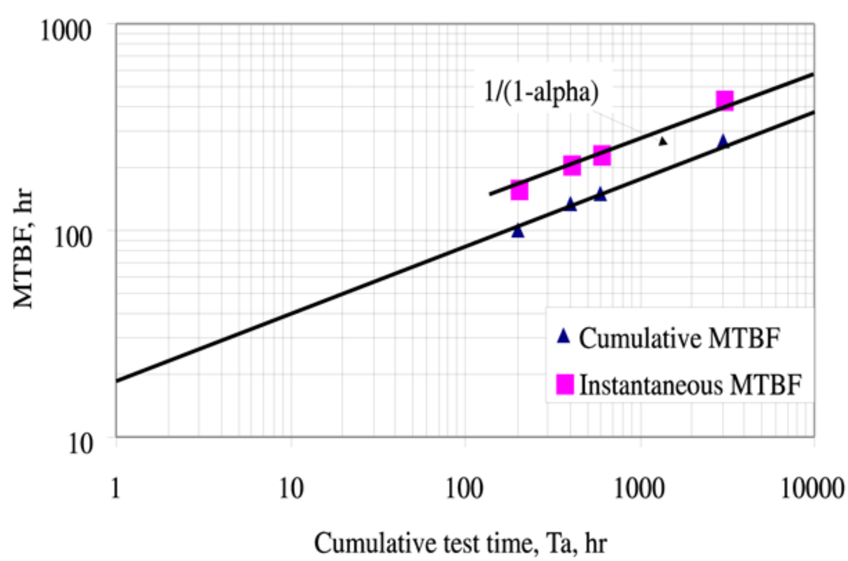
Applicable Models

The following chapters contain additional information on each of the models that are available for developmental testing:

- Duane
- Crow-AMSAA (NHPP)
- Crow Extended - Continuous Evaluation
- Lloyd-Lipow
- Gompertz Models
- Logistic

Duane Model

In 1962, J. T. Duane published a report in which he presented failure data of different systems during their development programs [8]. While analyzing the data, it was observed that the cumulative MTBF versus cumulative operating time followed a straight line when plotted on log-log paper, as shown next.



Based on that observation, Duane developed his model as follows. If $N(T)$ is the number of failures by time T, the observed mean (average) time between failures, $MTBF_c$, at time T is:

$$MTBF_c = \frac{T}{N(T)}$$

The equation of the line can be expressed as:

$$y = mx + c$$

Setting:

$$\begin{aligned} y =& \ln(MTBF_c) \\ x =& \ln(T) \\ m =& \alpha \\ c =& \ln b \end{aligned}$$

yields:

$$\ln(MTBF_c) = \alpha \ln(T) + \ln b$$

Then equating $MTBF_c$ to its expected value, and assuming an exact linear relationship, gives:

$$E(MTBF_c) = bT^{\alpha}$$

or:

$$MTBF_c = bT^{\alpha}$$

And, if you assume a constant failure intensity, then the cumulative failure intensity, λ_c , is:

$$E(\lambda_c) = \frac{1}{b}T^{-\alpha}$$

or:

$$\hat{\lambda}_c = \frac{1}{b}T^{-\alpha}$$

Also, the expected number of failures up to time T is:

$$\begin{aligned} E(N(T)) =& \hat{\lambda}_c \cdot T \\ =& \frac{1}{b}T^{1-\alpha} \end{aligned}$$

where:

- $\hat{\lambda}_c$ is the average estimate of the cumulative failure intensity, failures/hour.
- T is the total accumulated unit hours of test and/or development time.

- $1/b$ is the cumulative failure intensity at $T=1$ or at the beginning of the test, or the earliest time at which the first $\hat{\lambda}$ is predicted, or the $\hat{\lambda}$ for the equipment at the start of the design and development process.

- α is the improvement rate in the $\hat{\lambda}, 0 \leq \alpha \leq 1$.

The corresponding $MTBF_c$, or $\hat{m}_c$, is equal to:

$$\hat{m}_c = bT^{\alpha}$$

where $b =$ cumulative MTBF at $T=1$ or at the beginning of the test, or the earliest time at which the first $\hat{m}$ can be determined, or the $\hat{m}$ predicted at the start of the design and development process ($b > 0$).

The cumulative MTBF, $\hat{m}_c$, and $\hat{\lambda}_c$ tell whether m is increasing or λ is decreasing with time, utilizing all data up to that time. You may want to know, however, the instantaneous $\hat{m}_i$ or $\hat{\lambda}_i$ to see what you are doing at a specific instant or after a specific test and development time. The instantaneous failure intensity, λ_i , is:

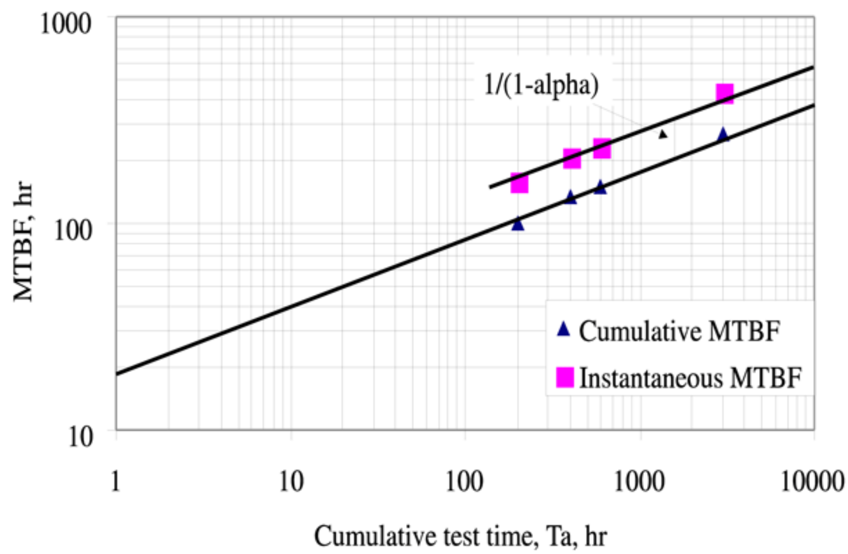
$$\begin{aligned}\lambda_i =& \frac{d(E(N(T)))}{dT} \\ =& \frac{1}{b}(1-\alpha)T^{-\alpha} \\ =& (1-\alpha)\lambda_c\end{aligned}$$

Similarly, using the equation for the expected number of failures up to time T , this procedure yields:

$$\begin{aligned}m_i =& \frac{1}{1-\alpha} bT^{\alpha} \\ =& \frac{1}{1-\alpha}\hat{m}_c, : \ \alpha \neq 1\end{aligned}$$

where $\alpha = 1$ implies infinite MTBF growth.

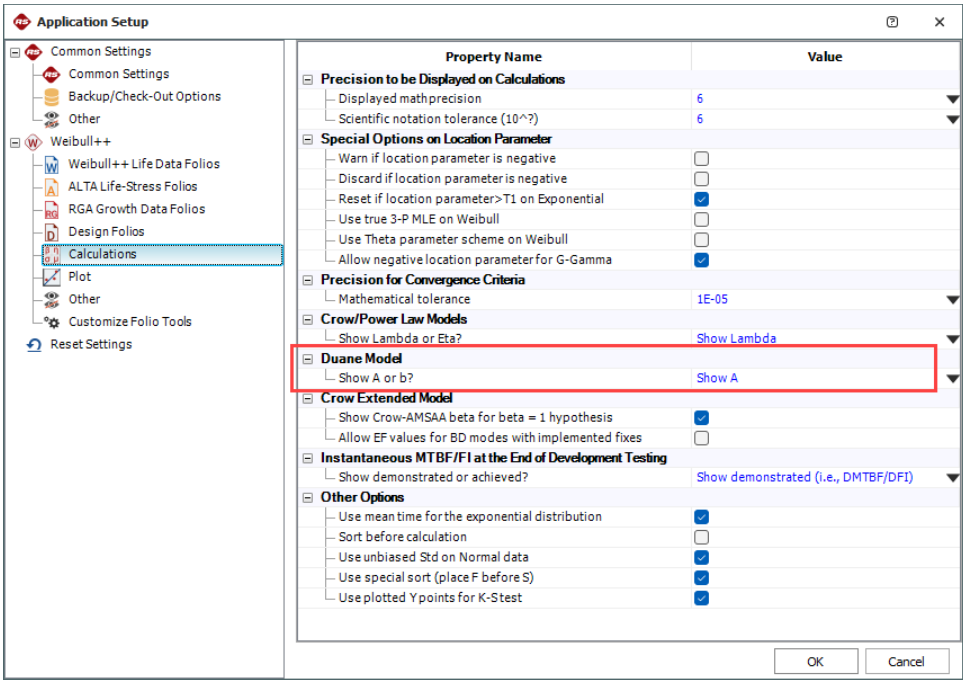
As shown in these derivations, the instantaneous failure intensity improvement line is obtained by shifting the cumulative failure intensity line down, parallel to itself, by a distance of $(1-\alpha)$. Similarly, the current or instantaneous MTBF growth line is obtained by shifting the cumulative MTBF line up, parallel to itself, by a distance of $\frac{1}{1-\alpha}$, as illustrated in the figure below.



Parameter Estimation

The Duane model is a two parameter model. Therefore, to use this model as a basis for predicting the reliability growth that could be expected in an equipment development program, procedures must be defined for estimating these parameters as a function of equipment characteristics. Note that, while these parameters can be estimated for a given data set using curve-fitting methods, there exists no underlying theory for the Duane model that could provide a basis for *a priori* estimation.

One of the parameters of the Duane model is α. The second parameter can be represented as A or b where $A = \frac{1}{b}$. There is an option within the Application Setup that allows you to determine whether to display A or b for the Duane model. All formulation within this reference uses the parameter b.



Graphical Method

The Duane model for the cumulative failure intensity is:

$$\hat{\lambda}_c = \frac{1}{b} T^{-\alpha}$$

This equation may be linearized by taking the natural log of both sides:

$$\ln\left(\hat{\lambda}_c\right) = \ln\left(\frac{1}{b}\right) - \alpha \ln(T)$$

Consequently, plotting $\hat{\lambda}$ versus T on log-log paper will result in a straight line with a negative slope, such that:

- $\ln\left(\frac{1}{b}\right)$ is the y-intercept at $T = 1$
- $\frac{1}{b}$ is the cumulative failure intensity at $T = 1$
- α is the slope of the straight line on the log-log plot

Similarly, the corresponding MTBF of the cumulative failure intensity can also be linearized by taking the natural log of both sides:

$$\hat{m}_c = bT^{\alpha}$$
$$\ln \hat{m}_c = \ln b + \alpha \ln T$$

Plotting $\hat{m}$ versus T on log-log paper will result in a straight line with a positive slope such that:

- $\ln b$ is the y-intercept at $T = 1$
- b is the cumulative mean time between failure at $T = 1$
- α is the slope of the straight line on the log-log plot

Two ways of determining these curves are as follows:

1. Predict the $\hat{\lambda}_0$ and $\hat{m} = \frac{1}{\hat{\lambda}_0}$ of the system from its reliability block diagram and available component failure intensities. Plot this value on log-log plotting paper at $T = 1$. From past experience and from past data for similar equipment, find values of α_1 , the slope of the improvement lines for $\hat{\lambda}$ or $\hat{m}$. Modify this α as necessary. If a better design effort is expected and a more intensive research, test and development or TAAF program is to be implemented, then a 15% improvement in the growth rate may be attainable. Consequently, the available value for slope α , and α_1 , should be adjusted by this amount. The value to be used will then be $\alpha = 1.15\alpha_1$. A line is then drawn through point $\hat{\lambda}_0$ and $T = 1$ with the just determined slope α , keeping in mind that α is negative for the $\hat{\lambda}$ curve. This line should be extended to the design, development and test time scheduled to be expended to see if the failure intensity goal will indeed be achieved on schedule. It is also possible to find that the design, development and test time to achieve the goal may be earlier than the delivery date or later. If earlier, then either the reliability program effort can be judiciously and appropriately trimmed; or if it is an incentive contract, full advantage is taken of the fact that the failure intensity goal can be exceeded with the associated increased profits to the company. A similar approach may be used for the MTBF growth model, where $\hat{m}_0 = \frac{1}{\hat{\lambda}_0}$ is plotted at $T = 1$, and a line is drawn through the point $\hat{m}_0$ and $T = 1$ with slope α to obtain the MTBF growth line. If α values are not available, consult the table below, which gives actual α values for various types of equipment. These have been obtained from literature or by MTBF growth tests. It may be seen from the following table that α values range between 0.24 and 0.65. The lower values reflect slow early growth and the higher values reflect fast early growth.

Sample Slope Values for Various Equipment

Equipment		Slope(α)
Computer system	Actual	0.24

	Easy to find failures were eliminated	0.26
	All known failure causes were eliminated	0.36
Mainframe computer		0.50
Aerospace electronics	All malfunctions	0.57
	Relevant failures only	0.65
Attack radar		0.60
Rocket engine		0.46
Afterburning turbojet		0.35
Complex hydromechanical system		0.60
Aircraft generator		0.38
Modern dry turbojet		0.48

2. During the design, development and test phase and at specific milestones, the $\hat{\hat{\lambda}} = \frac{1}{\hat{m}}$ is calculated from the total failures and T values. These values of $\hat{\hat{\lambda}}$ or $\hat{m}$ are plotted above the corresponding T values on log-log paper. A straight line is drawn favoring these points to minimize the distance between the points and the line, thus establishing the improvement or growth model and its parameters graphically. If needed, linear regression analysis techniques can be used to determine these parameters.

GRAPHICAL METHOD EXAMPLE

A complex system's reliability growth is being monitored and the data set is given in the table below.

Cumulative Test Hours and the Corresponding Observed Failures for the Complex System

Point Number	Cumulative Test Time (hours)	Cumulative Failures	Cumulative MTBF(hours)	Instantaneous MTBF (hours)
1	200	2	100.0	100
2	400	3	133.0	200
3	600	4	150.0	200

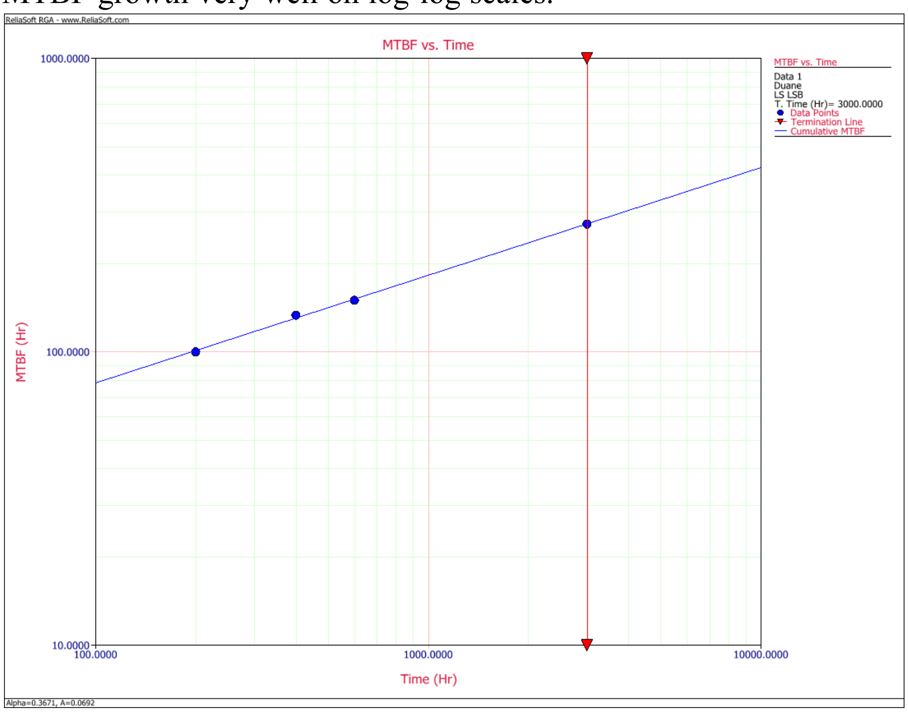
4	3,000	11	273.0	342.8

Do the following:

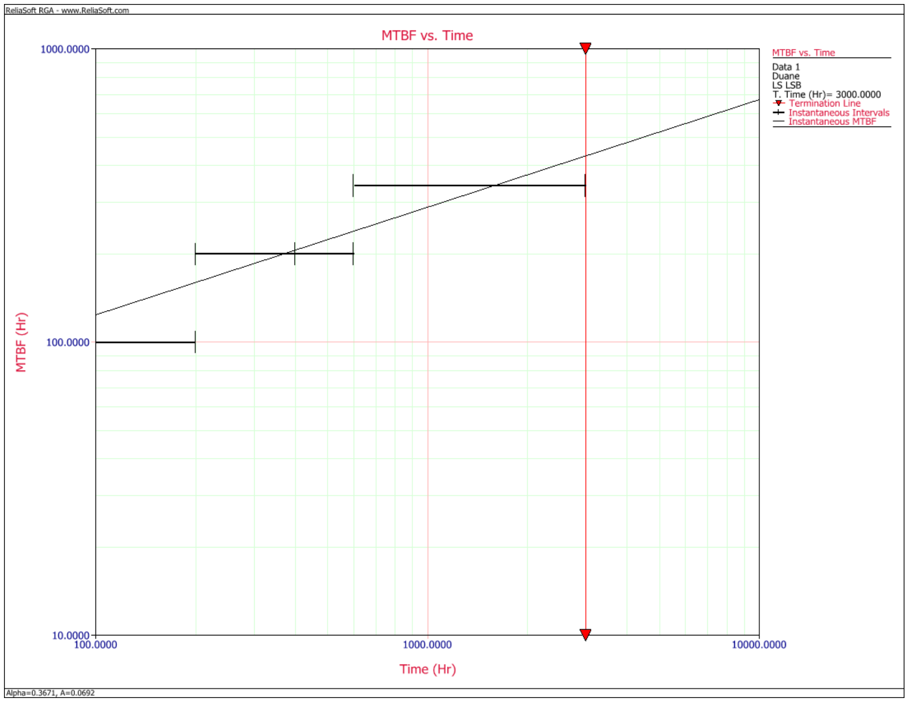
1. Plot the cumulative MTBF growth curve.
2. Write the equation of this growth curve.
3. Write the equation of the instantaneous MTBF growth model.
4. Plot the instantaneous MTBF growth curve.

Solution

1. Given the data in the second and third columns of the above table, the cumulative MTBF, $\hat{m}_c$, values are calculated in the fourth column. The information in the second and fourth columns are then plotted. The first figure below shows the cumulative MTBF while the second figure below shows the instantaneous MTBF. It can be seen that a straight line represents the MTBF growth very well on log-log scales.

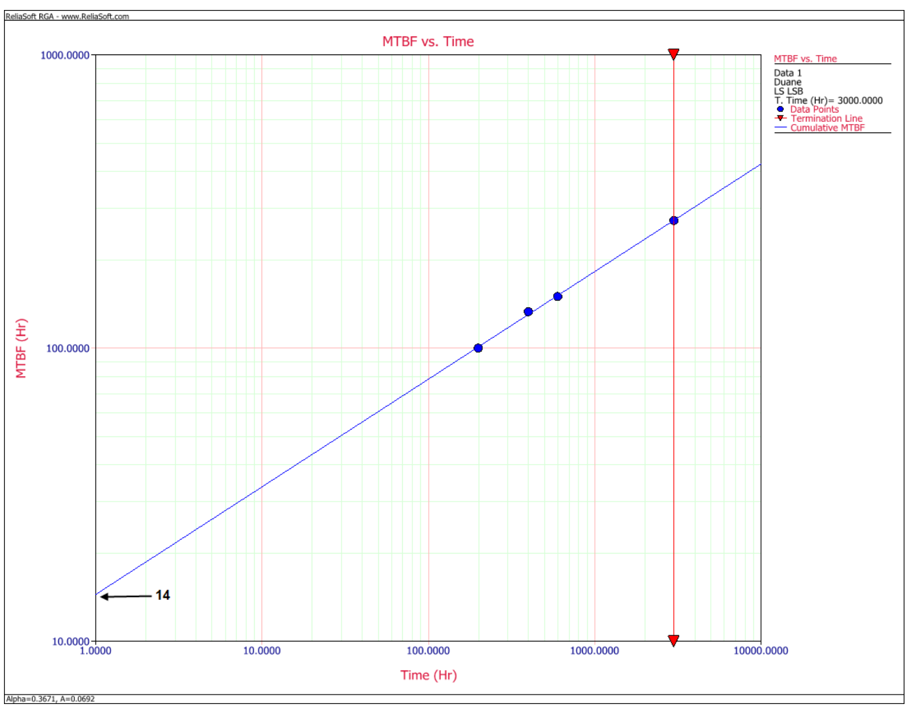


Cumulative MTBF plot



Instantaneous MTBF plot

By changing the x-axis scaling, you are able to extend the line to $T = 1$. You can get the value of b from the graph by positioning the cursor at the point where the line meets the y-axis. Then read the value of the y-coordinate position at the bottom left corner. In this case, b is approximately 14 hours. The next figure illustrates this.



Cumulative MTBF plot for $b \approx 14$ at $T = 1$

Another way of determining b is to calculate α by using two points on the fitted straight line and substituting the corresponding $\hat{m}_c$ and T values into:

$$\alpha = \frac{\ln(\hat{m}_{c_2}) - \ln(\hat{m}_{c_1})}{\ln(T_2) - \ln(T_1)}$$

Then substitute this α and choose a set of values for $\hat{m}_{c_1}$ and T_1 into the cumulative MTBF equation, $\hat{m}_c = bT^{\alpha}$, and solve for b. The slope of the line, α, may also be found from the linearized form of the cumulative MTBF equation, or :

$$\alpha = \frac{\ln(\hat{m}_c) - \ln(b)}{\ln(T) - \ln(1)}$$

Using the cumulative MTBF plot for the example, at $T_1 = 200$ hours, $\hat{m}_{c_1} = 100$ hours, and $T_2 = 3,500$ hours, $\hat{m}_{c_2} = 300$ hours. From the cumulative MTBF plot for $b = 14$ hours when $T = 1$, substituting the first set of values, $b = 14$ hours and $\ln 1 = 0$, into the equation yields:

$$\alpha_1 = \frac{\ln(100) - \ln(14)}{\ln(200) - \ln(1)}$$
$$= 0.3711$$

2. Substituting the second set of values, $b = 14$ hours and $\ln 1 = 0,$ into the equation yields:

$$\alpha_2 = \frac{\ln(300) - \ln(14)}{\ln(3,500) - \ln(1)}$$
$$= 0.3755$$

Averaging these two α values yields a better estimate of $\hat{\alpha} = 0.3733$.

3. Now the equation for the cumulative MTBF growth curve is:

$$\hat{m}_c = 14 \cdot T^{\,0.3733}$$

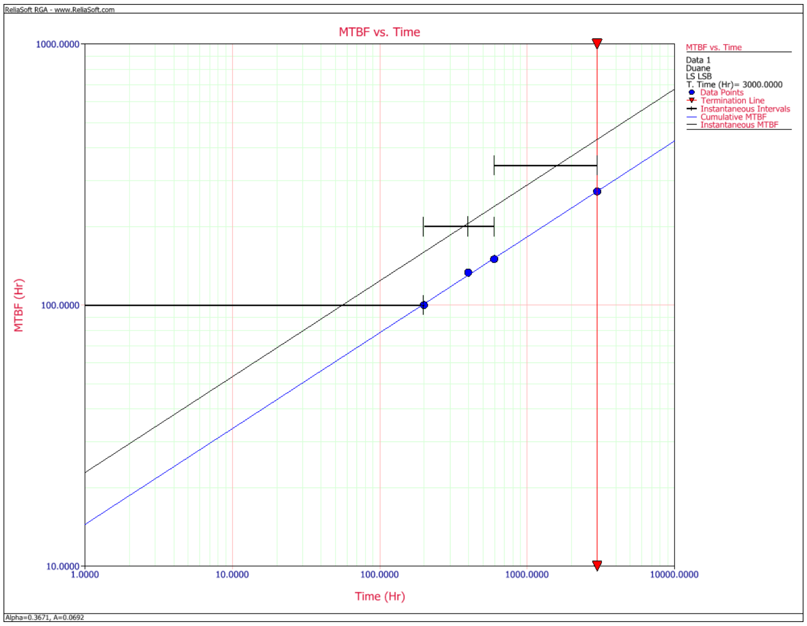
4. Using the following equation for the instantaneous MTBF, or

$$m_i = \frac{1}{1-\alpha}\hat{m}_c, : \ \alpha \neq 1$$

The equation for the instantaneous MTBF growth curve is:

$$\hat{m}_i = \frac{1}{1-0.3733} \cdot 14T^{0.3733}$$

The above equation is plotted in the instantaneous MTBF plot shown in the example and in the cumulative and instantaneous MTBF vs. Time plot. In the following figure, you can see that a parallel shift upward of the cumulative MTBF, $\hat{m}_c$, line by a distance of $\frac{1}{1-\alpha}$ gives the instantaneous MTBF, or the $\hat{m}_i$, line.



Cumulative and Instantaneous MTBF vs. Time plot

Least Squares (Linear Regression)

The parameters can also be estimated using a mathematical approach. To do this, linearize the MTBF of the cumulative failure intensity by taking the natural log of both sides, and then apply least squares analysis. For example:

$$\begin{aligned}\hat{m}_c &= bT^{\alpha} \\ \ln \hat{m}_c &= \ln b + \alpha \ln T\end{aligned}$$

For simplicity in the calculations, let:

$$\begin{aligned}\ln(m_{ci}) &= Y_i \\ \ln(b) &= a \\ \alpha &= c \\ \ln(T_i) &= X_i\end{aligned}$$

Therefore, the equation becomes:

$$Y_i = \hat{a} + \hat{c}X_i$$

Assume that a set of data pairs (X_1, Y_1) , (X_2, Y_2) ,..., (X_N, Y_N) were obtained and plotted. Then according to the Least Squares Principle, which minimizes the vertical distance between the data points and the straight line fitted to the data, the best fitting straight line to this data set is the straight line $Y = \hat{a} + \hat{c}X$ such that:

$$\sum_{i=1}^{N}(\hat{a}+\hat{c}X_i-Y_i)^2=\underset{(a,c)}{min}\sum_{i=1}^{N}(a+cX_i-Y_i)^2$$

where $\hat{a}$ and $\hat{c}$ are the least squares estimates of a and c. To obtain $\hat{a}$ and $\hat{c}$, let:

$$F=\sum_{i=1}^{N}(a+cX_i-Y_i)^2$$

Differentiating F with respect to a and c yields:

$$\frac{\partial F}{\partial a}=2\sum_{i=1}^{N}(a+cX_i-Y_i)$$

and:

$$\frac{\partial F}{\partial c}=2\sum_{i=1}^{N}(a+cX_i-Y_i)X_i$$

Set those two equations equal to zero:

$$\sum_{i=1}^{N}(a+cX_i-Y_i)=\sum_{i=1}^{N}(\hat{Y}_i-Y_i)=-\sum_{i=1}^{N}(Y_i-\hat{Y}_i)=0$$

and:

$$\sum_{i=1}^{N}(a+cX_i-Y_i)X_i=\sum_{i=1}^{N}(\hat{Y}_i-Y_i)X_i=-\sum_{i=1}^{N}(Y_i-\hat{Y}_i)X_i=0$$

Solve the equations simultaneously:

$$\begin{aligned}\hat{a}=&\frac{\sum\limits_{i=1}^{N}Y_i}{N}-\hat{c}\frac{\sum\limits_{i=1}^{N}X_i}{N}\\=&\bar{Y}-\hat{c}\bar{X}\end{aligned}$$

and:

$$\hat{c}=\frac{\sum^{N}{}_{i=1}X_iY_i-\frac{\left(\sum^{N}{}_{i=1}X_i\sum^{N}{}_{i=1}Y_i\right)}{N}}{\sum^{N}{}_{i=1}X_i^2-\frac{\left(\sum^{N}{}_{i=1}X_i\right)^2}{N}}$$

Now substituting back $\ln(m_{ci})=Y_i, \ln(b)=a,$ $\alpha=c$ and $\ln(T_i)=X_i,$ we have:

$$\hat{b} = e^{\frac{1}{n}[\sum^n{}_{i=1} \ln(m_{ci}) - \alpha \sum^n{}_{i=1} \ln(T_i)]}$$

where:

$$\hat{\alpha} = \frac{\sum^n{}_{i=1} \ln(T_i)\ln(m_{ci}) - \frac{\sum^n{}_{i=1} \ln(T_i) \sum^n{}_{i=1} \ln(m_{ci})}{n}}{\sum^n{}_{i=1} [\ln(T_i)]^2 - \frac{(\sum^n{}_{i=1} \ln(T_i))^2}{n}}$$

EXAMPLE 1

Using the same data set from the graphical approach example, estimate the parameters of the MTBF model using least squares.

Solution

From the data table:

$$\sum_{i=1}^{n} \ln(T_i) = \quad 25.693$$

$$\sum_{i=1}^{n} \ln(T_i)\ln(m_{ci}) = \quad 130.66$$

$$\sum_{i=1}^{n} \ln(m_{ci}) = \quad 20.116$$

$$\sum_{i=1}^{n} [\ln(T_i)]^2 = \quad 168.99$$

Obtain the value of $\hat{\alpha}$ from the least squares analysis, or:

$$\begin{aligned}\hat{\alpha} &= \frac{\sum^n{}_{i=1} \ln(T_i)\ln(m_{ci}) - \frac{\sum^n{}_{i=1} \ln(T_i) \sum^n{}_{i=1} \ln(m_{ci})}{n}}{\sum^n{}_{i=1} [\ln(T_i)]^2 - \frac{(\sum^n{}_{i=1} \ln(T_i))^2}{n}} \\ &= \frac{130.66 - \frac{25.693 \cdot 20.116}{4}}{168.99 - \frac{25.693^2}{4}} \\ &= 0.3671\end{aligned}$$

Obtain the value $\hat{b}$ from the least squares analysis, or:

$$\begin{aligned}\hat{b} &= e^{\frac{1}{n}[\sum^n{}_{i=1} \ln(m_{ci}) - \alpha \sum^n{}_{i=1} \ln(T_i)]} \\ &= e^{\frac{1}{4}(20.116 - 0.3671 \cdot 25.693)} \\ &= 14.456\end{aligned}$$

Therefore, the cumulative MTBF becomes:

$$\hat{m}_c = bT^{\alpha}$$
$$= 14.456 \cdot T^{0.3671}$$

The equation for the instantaneous MTBF growth curve is:

$$\hat{m}_i = \frac{1}{1-\alpha}\hat{m}_c,: \quad \alpha \neq 1$$
$$= \frac{1}{1-0.3671}(14.456)T^{0.3671}$$

EXAMPLE 2

For the data given in columns 1 and 2 of the following table, estimate the Duane parameters using least squares.

Failure Times Data

(1)Failure Number	(2)Failure Time (hours)	(3)$\ln T_i$	(4)$\ln T_i{}^2$	(5)m_c	(6)$\ln m_c$	(7) $\ln m_c \cdot \ln T_i$
1	9.2	2.219	4.925	9.200	2.219	4.925
2	25	3.219	10.361	12.500	2.526	8.130
3	61.5	4.119	16.966	20.500	3.020	12.441
4	260	5.561	30.921	65.000	4.174	23.212
5	300	5.704	32.533	60.000	4.094	23.353
6	710	6.565	43.103	118.333	4.774	31.339
7	916	6.820	46.513	130.857	4.874	33.241
8	1010	6.918	47.855	126.250	4.838	33.470
9	1220	7.107	50.504	135.556	4.909	34.889
10	2530	7.836	61.402	253.000	5.533	43.359
11	3350	8.117	65.881	304.545	5.719	46.418
12	4200	8.343	69.603	350.000	5.858	48.872
13	4410	8.392	70.419	339.231	5.827	48.895
14	4990	8.515	72.508	356.429	5.876	50.036

15	5570	8.625	74.393	371.333	5.917	51.036
16	8310	9.025	81.455	519.375	6.253	56.431
17	8530	9.051	81.927	501.765	6.218	56.282
18	9200	9.127	83.301	511.111	6.237	56.921
19	10500	9.259	85.731	552.632	6.315	58.469
20	12100	9.401	88.378	605.000	6.405	60.215
21	13400	9.503	90.307	638.095	6.458	61.375
22	14600	9.589	91.945	663.636	6.498	62.305
23	22000	9.999	99.976	956.522	6.863	68.625
	Sum =	173.013	1400.908	7600.870	121.406	974.242

Solution

To estimate the parameters using least squares, the values in columns 3, 4, 5, 6 and 7 are calculated. The cumulative MTBF, m_c , is calculated by dividing the failure time by the failure number. The value of $\hat{\alpha}$ is:

$$\begin{aligned}\hat{\alpha} &= \frac{\sum^n{}_{i=1} \ln(T_i)\ln(m_{ci}) - \frac{\sum^n{}_{i=1} \ln(T_i)\sum^n{}_{i=1} \ln(m_{ci})}{n}}{\sum^n{}_{i=1} [\ln(T_i)]^2 - \frac{(\sum^n{}_{i=1} \ln(T_i))^2}{n}} \\ &= \frac{974.242 - \frac{173.013 \cdot 121.406}{23}}{1400.908 - \frac{(173.013)^2}{23}} \\ &= 0.6133\end{aligned}$$

The estimator of b is estimated to be:

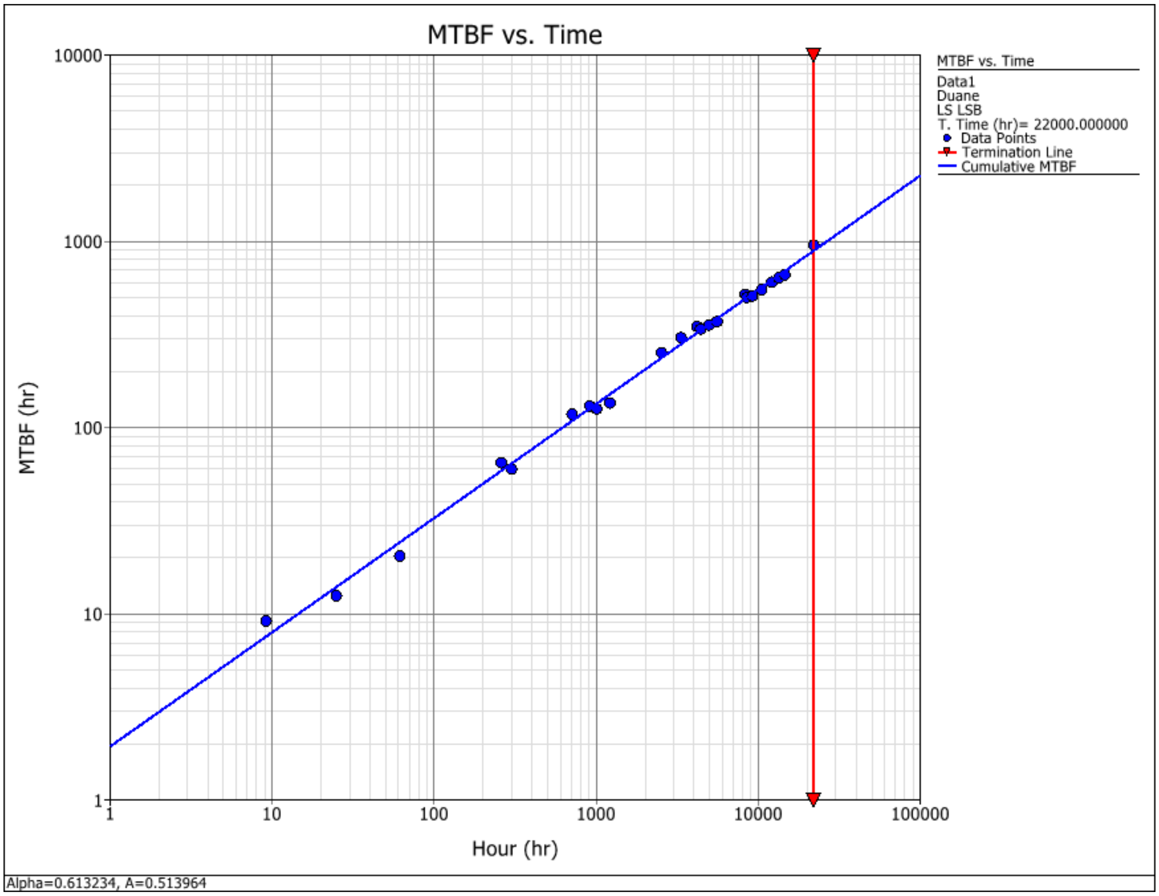
$$\begin{aligned}\hat{b} &= e^{\frac{1}{n}[\sum^n{}_{i=1} \ln(m_{ci}) - \alpha\sum^n{}_{i=1} \ln(T_i)]} \\ &= e^{\frac{1}{23}(121.406 - 0.6133 \cdot 173.013)} \\ &= 1.9453\end{aligned}$$

Therefore, the cumulative MTBF becomes:

$$\begin{aligned}\hat{m}_c &= bT^{\alpha} \\ &= 1.9453 \cdot T^{0.613}\end{aligned}$$

Using the equation for the instantaneous MTBF growth curve,

$$\hat{m}_i = \frac{1}{1-\alpha}\hat{m}_c,: \quad \alpha \neq 1$$
$$= \frac{1}{1-0.613}(1.945)T^{0.613}$$



EXAMPLE 3

For the data given in the following table, estimate the Duane parameters using least squares.

Multiple Systems (Known Operating Times) Data}

Run Number	Failed Unit	Test Time 1	Test Time 2	Cumulative Time
1	1	0.2	2.0	2.2
2	2	1.7	2.9	4.6
3	2	4.5	5.2	9.7
4	2	5.8	9.1	14.9

5	2	17.3	9.2	26.5
6	2	29.3	24.1	53.4
7	1	36.5	61.1	97.6
8	2	46.3	69.6	115.9
9	1	63.6	78.1	141.7
10	2	64.4	85.4	149.8
11	1	74.3	93.6	167.9
12	1	106.6	103	209.6
13	2	195.2	117	312.2
14	2	235.1	134.3	369.4
15	1	248.7	150.2	398.9
16	2	256.8	164.6	421.4
17	2	261.1	174.3	435.4
18	2	299.4	193.2	492.6
19	1	305.3	234.2	539.5
20	1	326.9	257.3	584.2
21	1	339.2	290.2	629.4
22	1	366.1	293.1	659.2
23	2	466.4	316.4	782.8
24	1	504	373.2	877.2
25	1	510	375.1	885.1
26	2	543.2	386.1	929.3
27	2	635.4	453.3	1088.7
28	1	641.2	485.8	1127
29	2	755.8	573.6	1329.4

Solution

The solution to this example follows the same procedure as the previous example. Therefore, from the table shown above:

$$\sum_{i=1}^{29} \ln(T_i) = 154.151$$

$$\sum_{i=1}^{29} \ln(T_i)^2 = 902.592$$

$$\sum_{i=1}^{29} \ln(m_c) = 82.884$$

$$\sum_{i=1}^{29} \ln(T_i) \cdot \ln(m_c) = 483.154$$

For least squares, the value of α is:

$$\begin{aligned}
\hat{\alpha} &= \frac{\sum_{i=1}^{n} \ln(T_i)\ln(m_{ci}) - \frac{\sum_{i=1}^{n} \ln(T_i)\sum_{i=1}^{n} \ln(m_{ci})}{n}}{\sum_{i=1}^{n} [\ln(T_i)]^2 - \frac{\left(\sum_{i=1}^{n} \ln(T_i)\right)^2}{n}} \\
&= \frac{483.154 - \frac{154.151 \cdot 82.884}{29}}{902.592 - \frac{(154.151)^2}{29}} \\
&= 0.5115
\end{aligned}$$

The value of the estimator b is:

$$\begin{aligned}
\hat{b} &= e^{\frac{1}{n}\left[\sum_{i=1}^{n} \ln(m_{ci}) - \alpha \sum_{i=1}^{n} \ln(T_i)\right]} \\
&= e^{\frac{1}{29}(82.884 - 0.5115 \cdot 154.151)} \\
&= 1.1495
\end{aligned}$$

Therefore, the cumulative MTBF is:

$$\hat{m}_c = bT^{\alpha} \qquad = 1.1495 \cdot T^{0.5115}$$

Using the equation for the instantaneous MTBF growth,

$$\begin{aligned}
\hat{m}_i &= \frac{1}{1-\alpha}\hat{m}_c ; \quad \alpha \neq 1 \\
&= \frac{1}{1-0.5115}(1.1495)T^{0.5115}
\end{aligned}$$

Maximum Likelihood Estimators

L. H. Crow [17] noted that the Duane model could be stochastically represented as a Weibull process, allowing for statistical procedures to be used in the application of this model in reliability growth. This statistical extension became what is known as the *Crow-AMSAA (NHPP)* model. The Crow-AMSAA model, which is described in the Crow-AMSAA (NHPP) chapter, provides a complete Maximum Likelihood Estimation (MLE) solution to the Duane model.

Confidence Bounds

Least squares confidence bounds can be computed for both the model parameters and metrics of interest for the Duane model.

Parameter Bounds

Apply least squares analysis on the Duane model:

$$\ln(\hat{m}_c) = \ln(b) + \alpha\ln(t)$$

The unbiased estimator of σ^2 can be obtained from:

$$\sigma^2 = Var\left[\ln m_c(t)\right] = \frac{SSE}{(n-2)}$$

where:

$$SSE = \sum_{i=1}^{n}\left[\ln\hat{m}_c(t_i) - \ln m_c(t_i)\right]^2$$

Thus, the confidence bounds on α and b are:

$$CB_\alpha = \hat{\alpha} \pm t_{n-2,\alpha/2}SE(\hat{\alpha})$$

$$CB_b = \hat{b}e^{\pm t_{n-2,\alpha/2}SE\left[\ln(\hat{b})\right]}$$

where $t_{n-2,\alpha/2}$ denotes the percentage point of the t distribution with $n-2$ degrees of freedom such that $P\{t_{n-2} \geq t_{\alpha/2,n-2}\} = \alpha/2$ and:

$$SE(\hat{\alpha}) = \frac{\sigma}{\sqrt{S_{xx}}}$$

$$SE\left[\ln(\hat{b})\right] = \sigma\cdot\sqrt{\frac{\sum^{n}{}_{i=1}\left(\ln t_i\right)^2}{n\cdot S_{xx}}}$$

$$S_{xx} = \left[\sum_{i=1}^{n}(\ln t_i)^2\right] - \frac{1}{n}\left(\sum_{i=1}^{n}\ln(t_i)\right)^2$$

Other Bounds

Confidence bounds also can be obtained on the cumulative MTBF and the cumulative failure intensity:

$$CB_{m_c} = \hat{m}_c(t)e^{\pm z_\alpha\sqrt{Var[\ln(\hat{m}_c)]}}$$

$$Var\left[\ln(\hat{m}_c)\right] = \frac{\sum_{i=1}^{n}(\ln\hat{m}_c(t_i) - \ln m_c(t_i))^2}{n-2}\cdot\left(\frac{1}{n} + \frac{\left(\ln(t) - \frac{\sum_{i=1}^{n}\ln(t_i)}{n}\right)^2}{\sum_{i=1}^{n}\left(\ln(t_i) - \frac{\sum_{i=1}^{n}\ln(t_i)}{n}\right)^2}\right)$$

$$[\lambda_c(t)]_L = \frac{1}{[m_c(t)]_u}$$
$$[\lambda_c(t)]_U = \frac{1}{[m_c(t)]_l}$$

When n is large, the approximate $100(1-\alpha)\%$ confidence bounds for instantaneous MTBF are given by:

$$m_i(t)_L = \frac{[m_c(t)]_L}{\hat{\beta}}$$
$$m_i(t)_U = \frac{[m_c(t)]_U}{\hat{\beta}}$$

and

$$\lambda_i(t) = \frac{1}{m_i(t)}$$

therefore, the confidence bounds on the instantaneous failure intensity are:

$$[\lambda_i(t)]_L = \frac{1}{[m_i(t)]_U}$$
$$[\lambda_c(t)]_U = \frac{1}{[m_i(t)]_L}$$

Duane Confidence Bounds Example

Using the values of $\hat{b}$ and $\hat{\alpha}$ estimated from the least squares analysis in Least Squares Example 2:

$$\hat{b} = 1.9453$$
$$\hat{\alpha} = 0.6133$$

Calculate the 90% confidence bounds for the following:

1. The parameters α and b.
2. The cumulative and instantaneous failure intensity.
3. The cumulative and instantaneous MTBF.

Solution

1. Use the values of $\hat{b}$ and $\hat{\alpha}$ estimated from the least squares analysis. Then:

$$S_{xx} = \left[\sum_{i=1}^{n} (\ln t_i)^2\right] - \frac{1}{n}\left(\sum_{i=1}^{n} \ln(t_i)\right)^2$$
$$= 1400.9084 - 1301.4545$$
$$= 99.4539$$

$$SE(\hat{\alpha}) = \frac{\sigma}{\sqrt{S_{xx}}}$$
$$= \frac{0.08428}{9.9727}$$
$$= 0.008452$$

$$SE(\ln \hat{b}) = \sigma \cdot \sqrt{\frac{\sum^{n}{}_{i=1} (\ln T_i)^2}{n \cdot S_{xx}}}$$
$$= 0.065960$$

Thus, the 90% confidence bounds on parameter α are:

$$CB_\alpha = \hat{\alpha} \pm t_{n-2,\alpha/2} SE(\hat{\alpha})$$

$$\alpha_L = 0.602050$$
$$\alpha_U = 0.624417$$

And 90% confidence bounds on parameter b are:

$$CB_b = \hat{b} e^{\pm t_{n-2,\alpha/2} SE[\ln(\hat{b})]}$$

$$b_L = 1.7831$$
$$b_U = 2.1231$$

2. The cumulative failure intensity is:

$$\lambda_c = \frac{1}{1.9453} \cdot 22000^{-0.6133} = 0.00111689$$

And the instantaneous failure intensity is equal to:

$$\lambda_i = \frac{1}{1.9453} \cdot (1 - 0.6133) \cdot 22000^{-0.6133} = 0.00043198$$

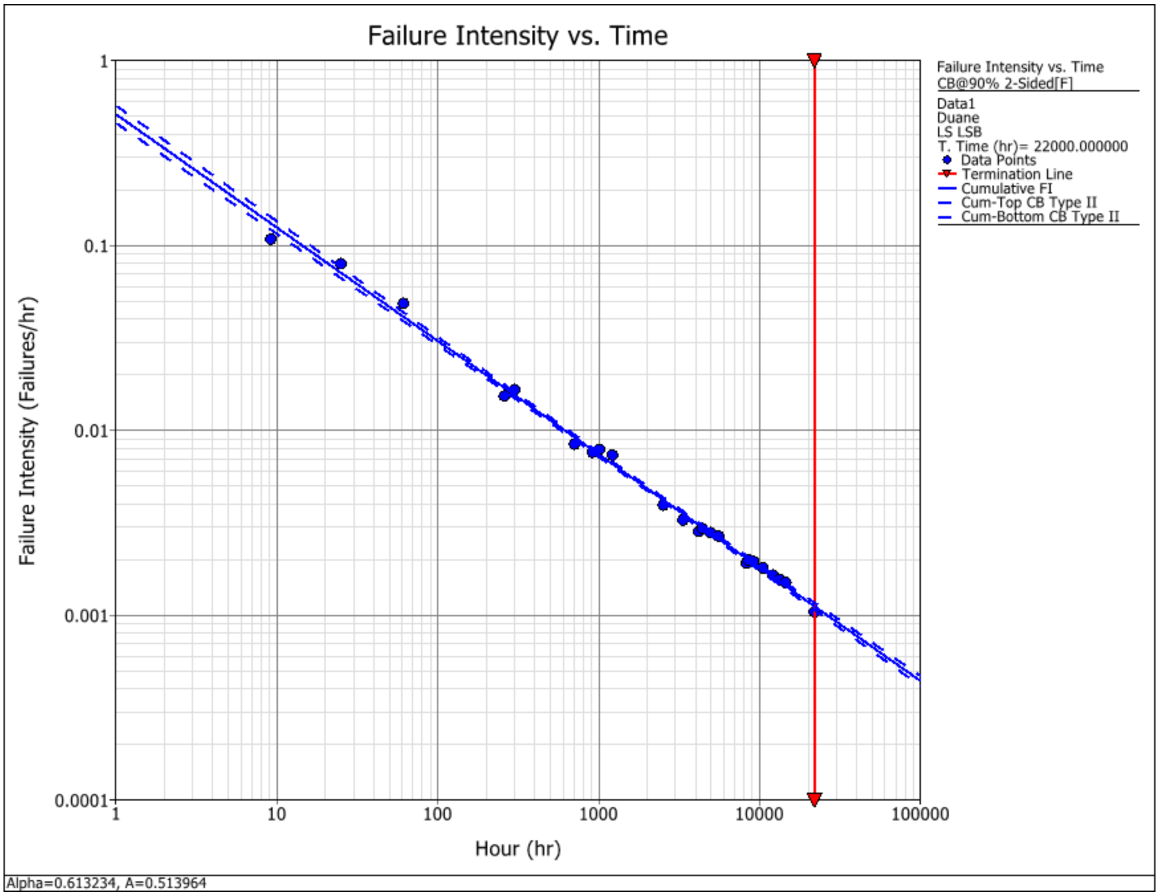
So, at the 90% confidence level and for $T = 22,000$ hours, the confidence bounds on cumulative failure intensity are:

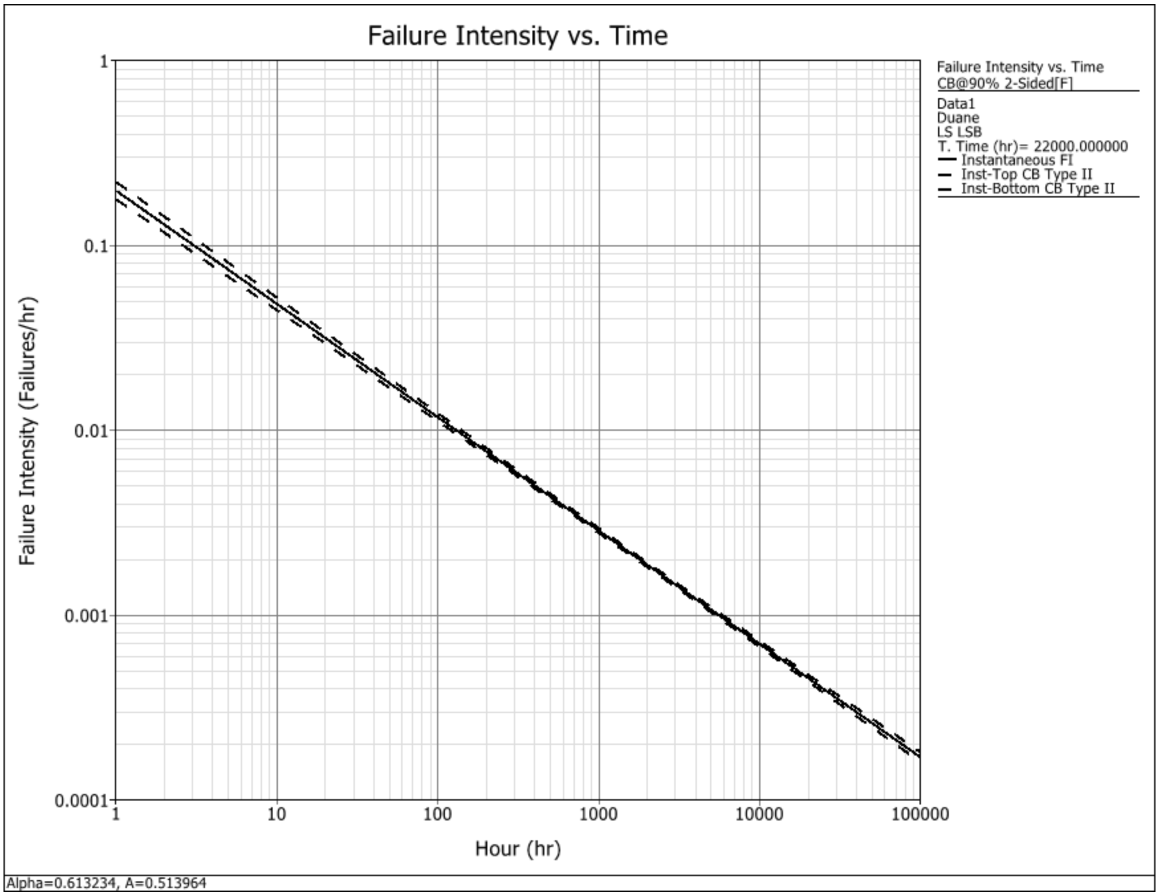
$$[\lambda_c(t)]_L = 0.00106780$$
$$[\lambda_c(t)]_U = 0.00116825$$

For the instantaneous failure intensity:

$$[\lambda_i(t)]_L = 0.00041299$$
$$[\lambda_c(t)]_U = 0.00045184$$

The following figures show the graphs of the cumulative and instantaneous failure intensity. Both are plotted with confidence bounds.


Failure Intensity vs. Time
Failure Intensity (Failures/hr)
Hour (hr)
1
0.1
0.01
0.001
0.0001
1
10
100
1000
10000
100000
Failure Intensity vs. Time
CB@90% 2-Sided[F]
Data1
Duane
LS LSB
T. Time (hr)= 22000.000000
Data Points
Termination Line
Cumulative FI
Cum-Top CB Type II
Cum-Bottom CB Type II
Alpha=0.613234, A=0.513964



3. The cumulative MTBF is:

$$\begin{aligned} m_c(T) &= 1.9453 \cdot 22000^{0.6133} \\ &= 895.3395 \end{aligned}$$

And the instantaneous MTBF is:

$$\begin{aligned} m_i(T) &= \frac{1.9453}{1 - 0.6133} \cdot 22000^{0.6133} \\ &= 2314.9369 \end{aligned}$$

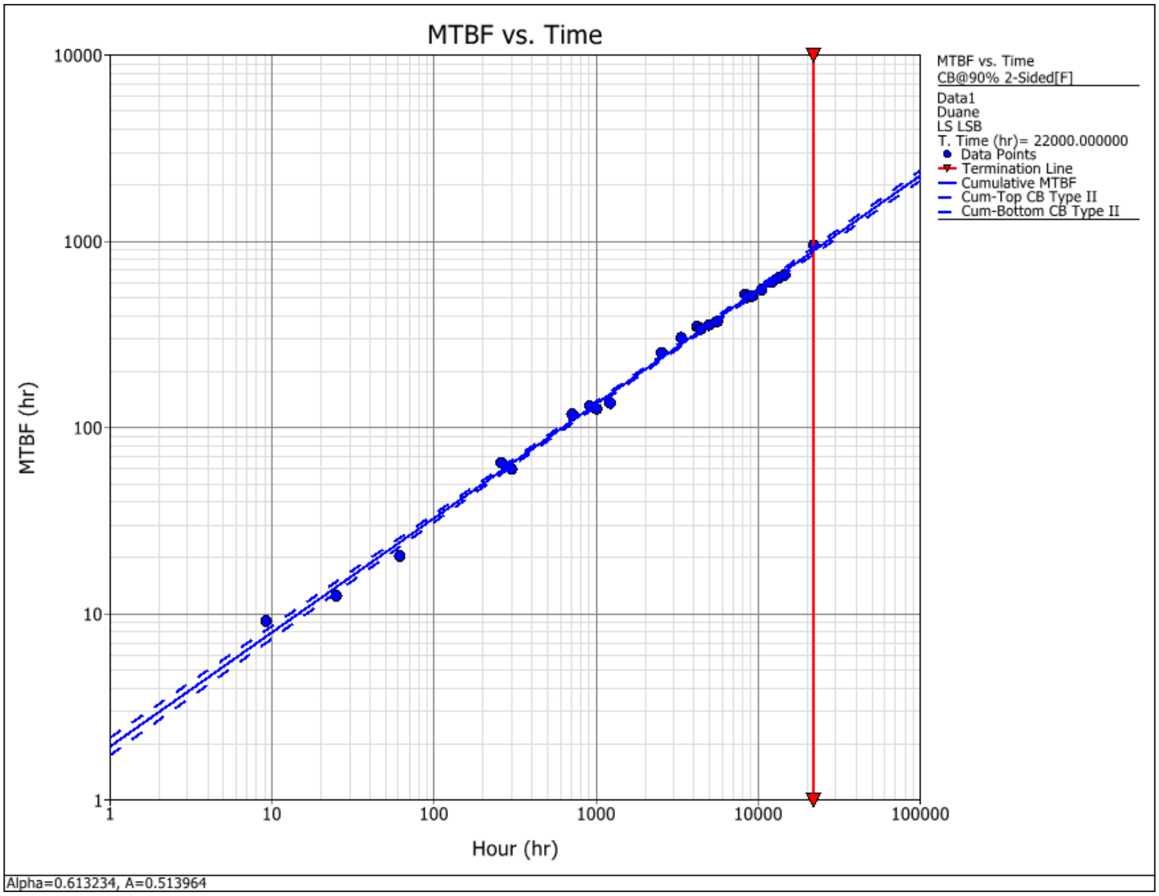
So, at 90% confidence level and for $T = 22,000$ hours, the confidence bounds on the cumulative MTBF are:

$$\begin{aligned} m_c(t)_l &= 855.9815 \\ m_c(t)_u &= 936.5071 \end{aligned}$$

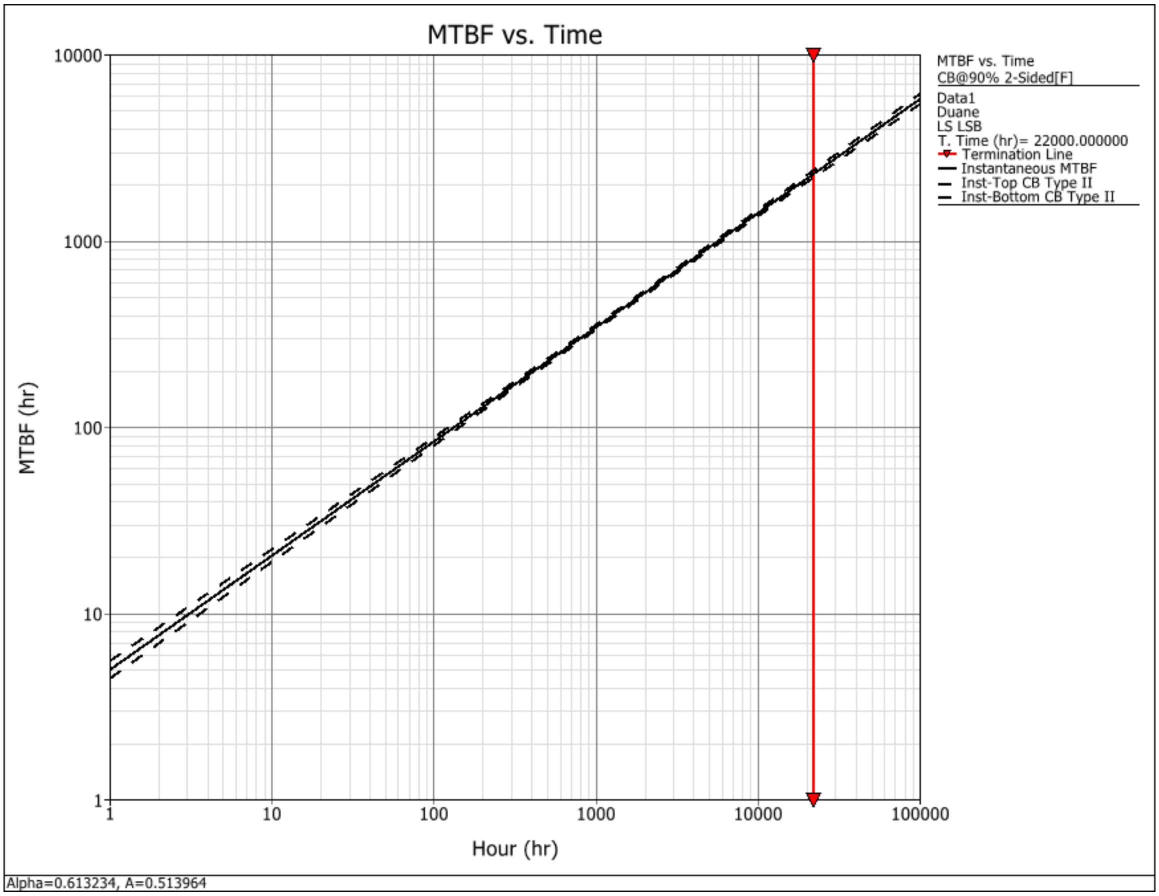
The confidence bounds for the instantaneous MTBF are:

$$\begin{aligned} m_i(t)_l &= 2213.1753 \\ m_i(t)_u &= 2421.3776 \end{aligned}$$

The figure below displays the cumulative MTBF.



The next figure displays the instantaneous MTBF. Both are plotted with confidence bounds.



More Examples

Estimating the Time Required to Meet an MTBF Goal

A prototype of a system was tested with design changes incorporated during the test. A total of 12 failures occurred. The data set is given in the following table.

Developmental Test Data

Failure Number	Cumulative Test Time (hours)
1	80
2	175
3	265
4	400
5	590

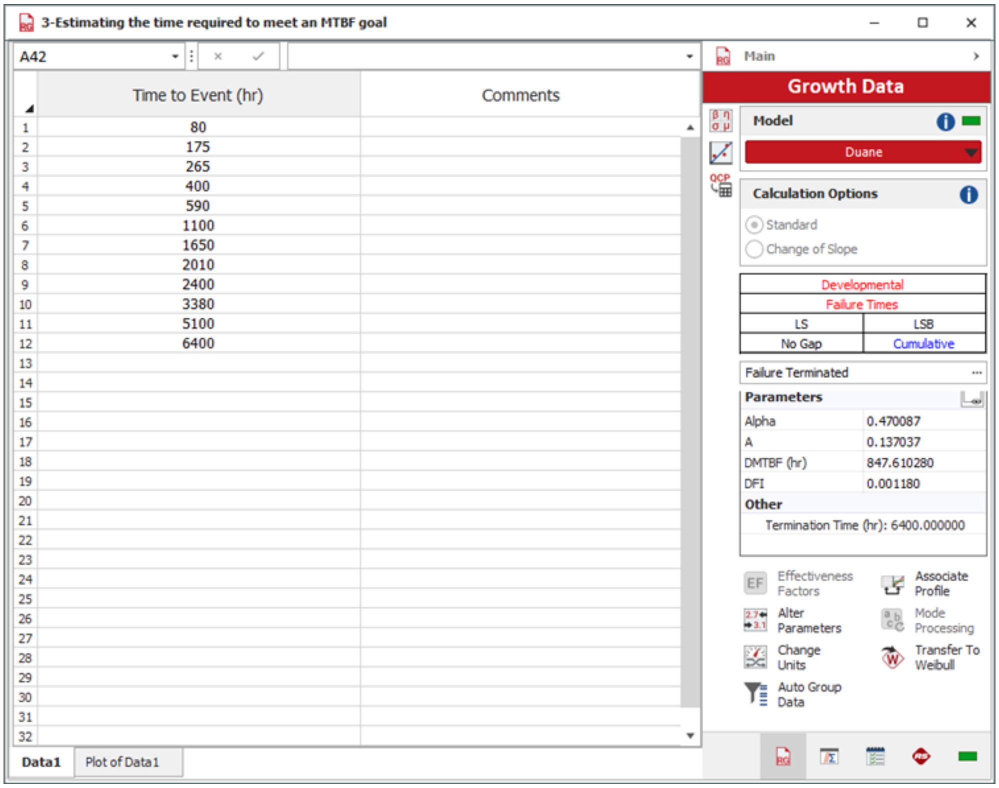
6	1100
7	1650
8	2010
9	2400
10	3380
11	5100
12	6400

Do the following:

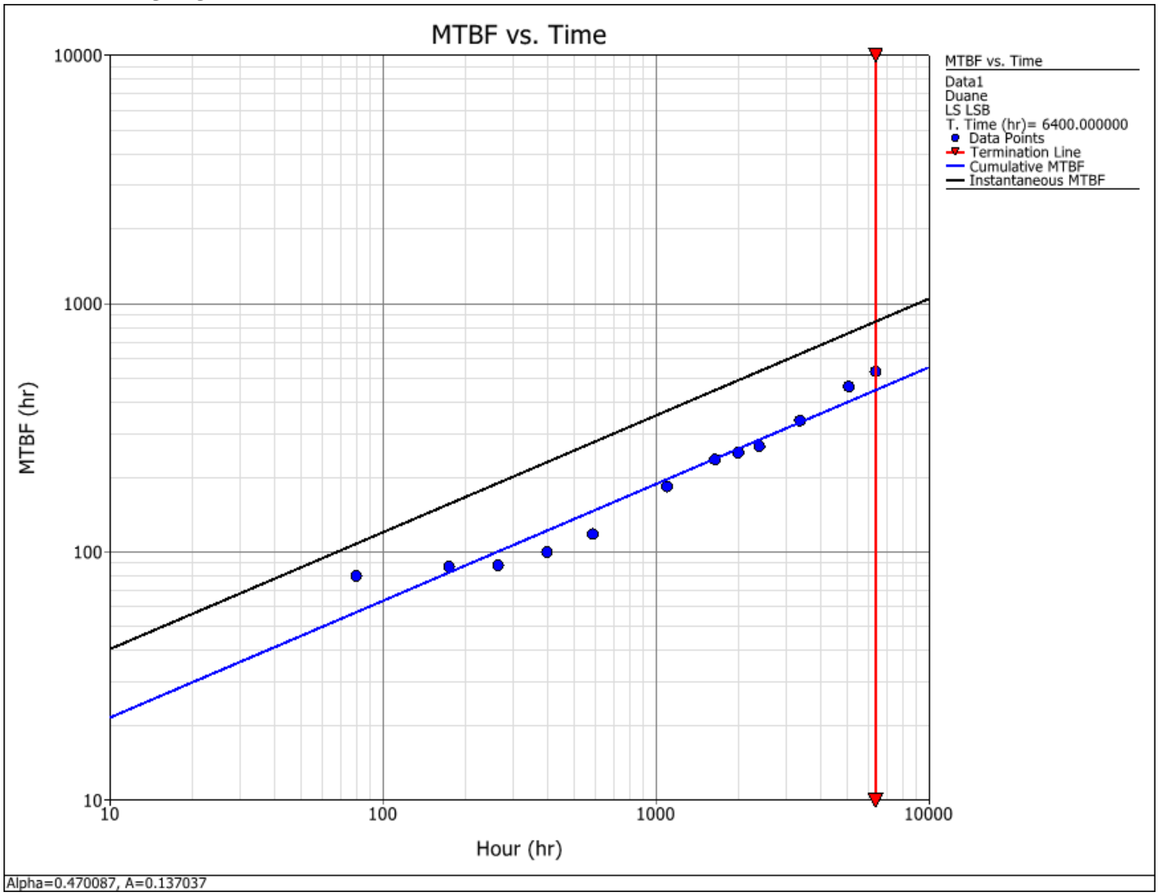
1. Estimate the Duane parameters.
2. Plot the cumulative and instantaneous MTBF curves.
3. How many cumulative test and development hours are required to meet an *instantaneous* MTBF goal of 500 hours?
4. How many cumulative test and development hours are required to meet a *cumulative* MTBF goal of 500 hours?

Solution

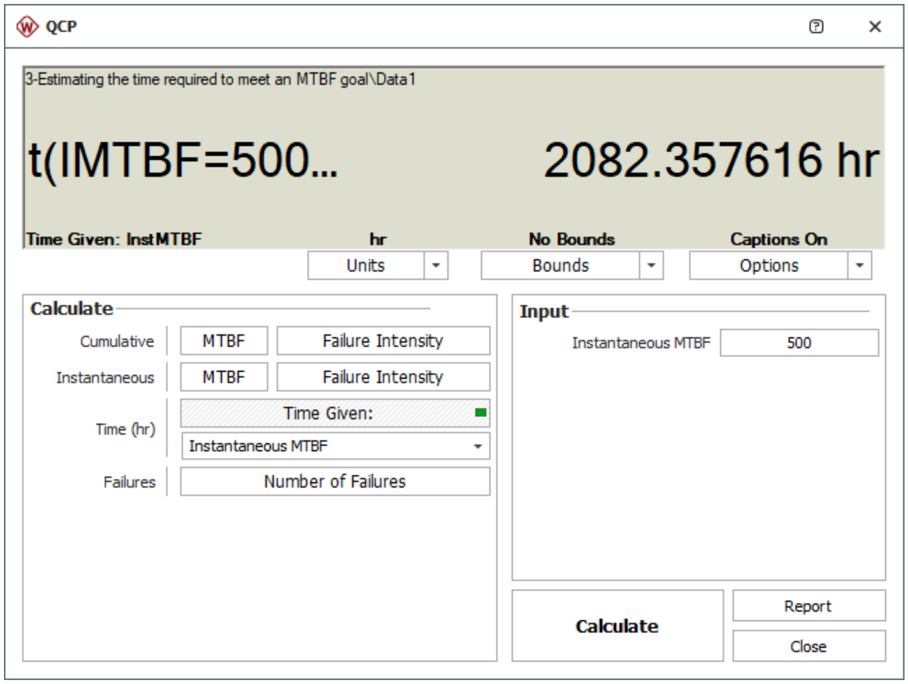
1. The next figure shows the data entered into Weibull++ along with the estimated Duane parameters.


3-Estimating the time required to meet an MTBF goal
A42
Time to Event (hr)
Comments
80
175
265
400
590
1100
1650
2010
2400
3380
5100
6400
Main
Growth Data
Model
Duane
Calculation Options
Standard
Change of Slope
Developmental
Failure Times
LS
LSB
No Gap
Cumulative
Failure Terminated
Parameters
Alpha 0.470087
A 0.137037
DMTBF (hr) 847.610280
DFI 0.001180
Other
Termination Time (hr): 6400.000000
Effectiveness Factors
Associate Profile
Alter Parameters
Mode Processing
Change Units
Transfer To Weibull
Auto Group Data
Data1
Plot of Data1

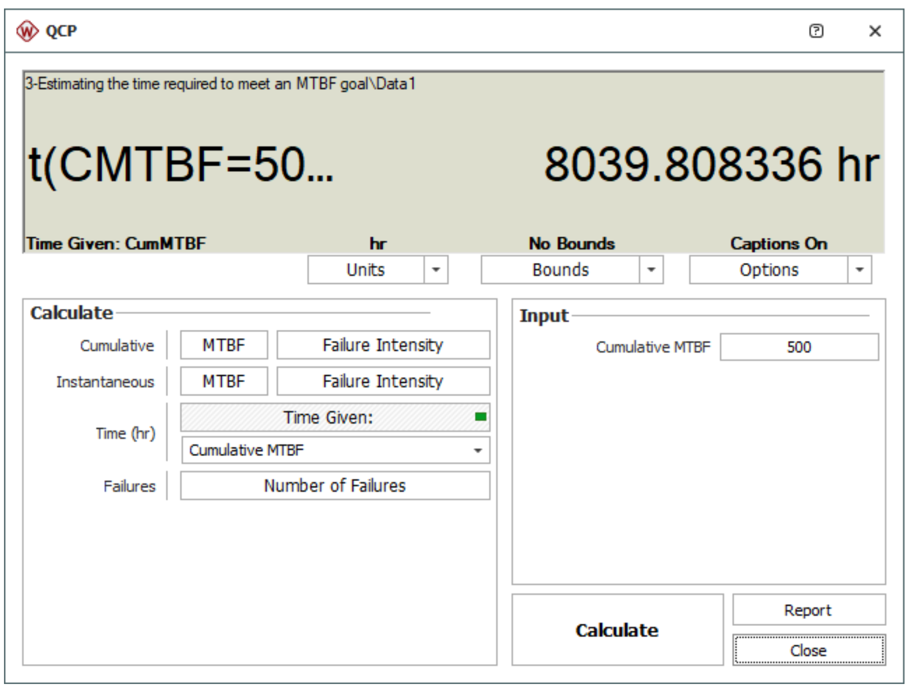
2. The following figure shows the cumulative and instantaneous MTBF curves.



3. The next figure shows the cumulative test and development hours needed for an instantaneous MTBF goal of 500 hours.



4. The figure below shows the cumulative test and development hours needed for a cumulative MTBF goal of 500 hours.



Multiple Systems - Known Operating Times Example

Two identical systems were tested. Any design changes made to improve the reliability of these systems were incorporated into both systems when any system failed. A total of 29 failures occurred. The data set is given in the table below.

Do the following:

1. Estimate the Duane parameters.
2. Assume both units are tested for an additional 100 hours each. How many failures do you expect in that period?
3. If testing/development were halted at this point, what would the reliability equation for this system be?

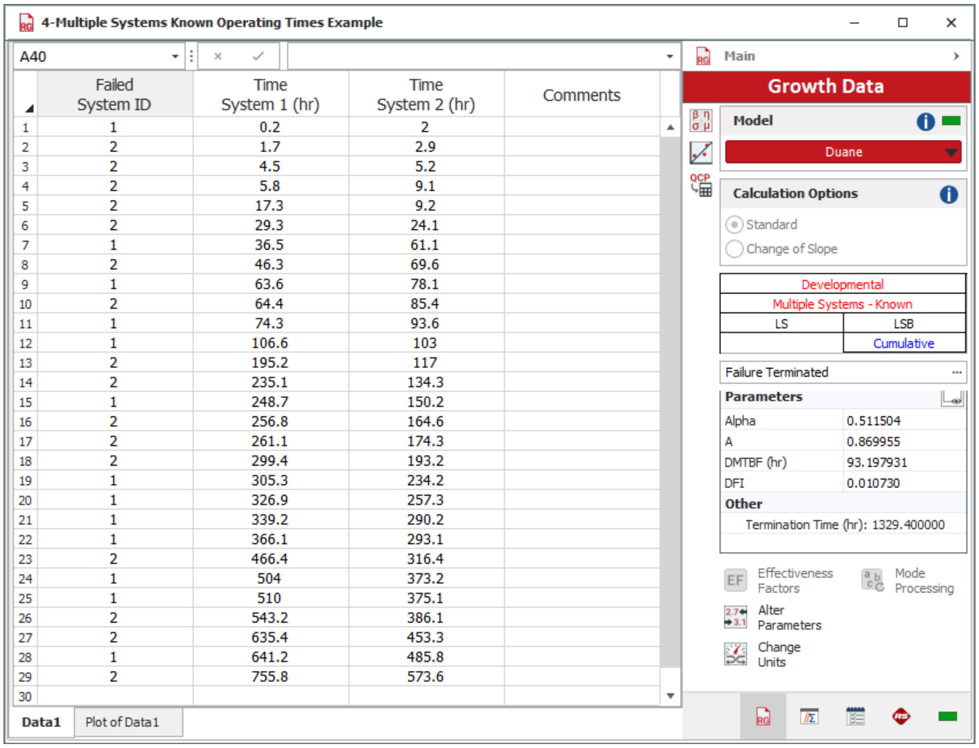
Developmental Test Data

Failure Number	Failed Unit	Test Time Unit 1 (hours)	Test Time Unit 2 (hours)
1	1	0.2	2.0
2	2	1.7	2.9
3	2	4.5	5.2
4	2	5.8	9.1
5	2	17.3	9.2
6	2	29.3	24.1
7	1	36.5	61.1
8	2	46.3	69.6
9	1	63.6	78.1
10	2	64.4	85.4
11	1	74.3	93.6
12	1	106.6	103
13	2	195.2	117
14	2	235.1	134.3
15	1	248.7	150.2
16	2	256.8	164.6

17	2	261.1	174.3
18	2	299.4	193.2
19	1	305.3	234.2
20	1	326.9	257.3
21	1	339.2	290.2
22	1	366.1	293.1
23	2	466.4	316.4
24	1	504	373.2
25	1	510	375.1
26	2	543.2	386.1
27	2	635.4	453.3
28	1	641.2	485.8
29	2	755.8	573.6

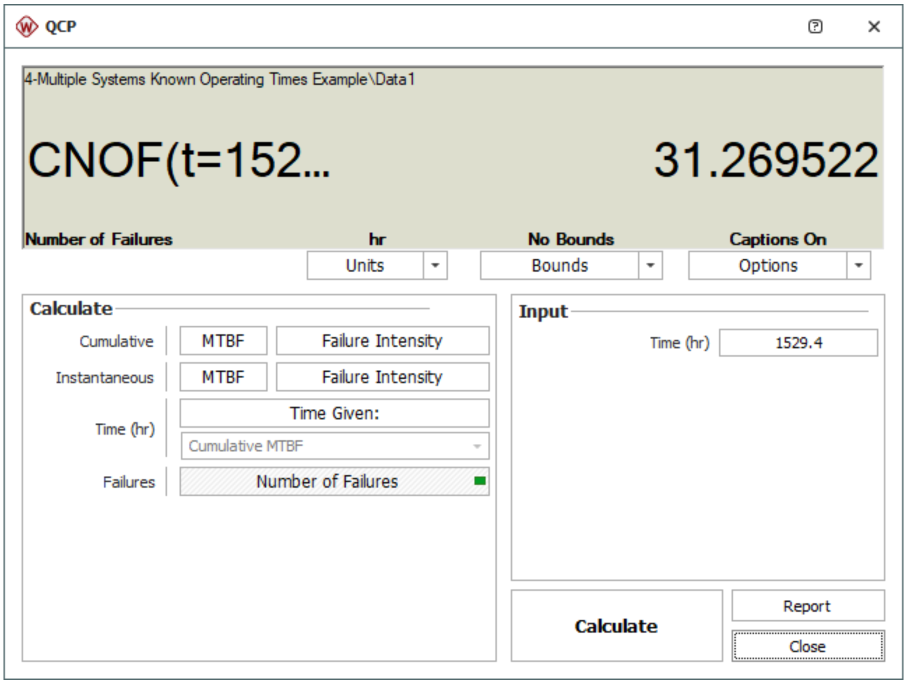
Solution

1. The figure below shows the data entered into Weibull++ along with the estimated Duane parameters.

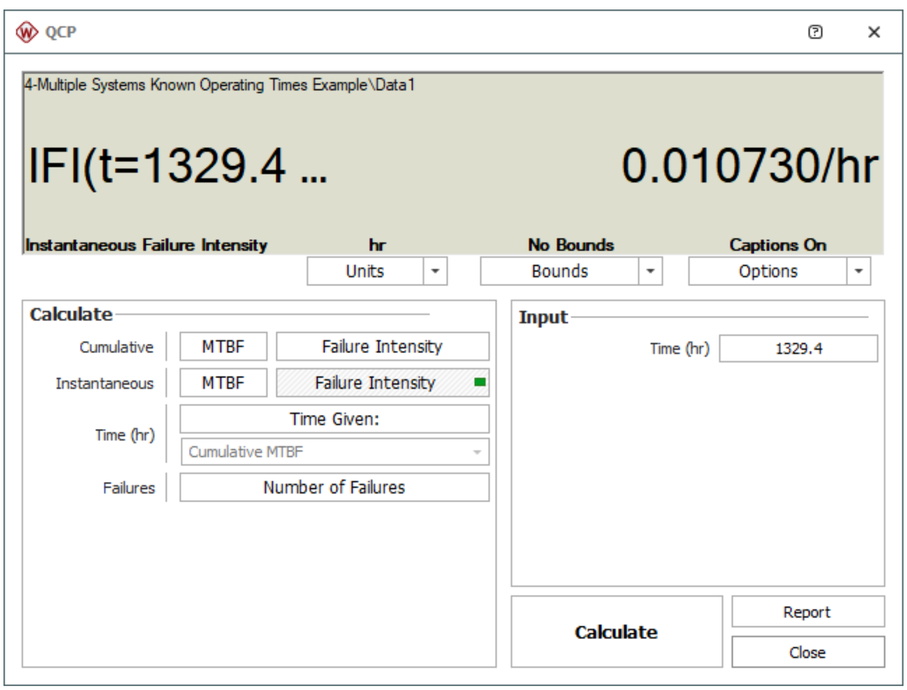


	Failed System ID	Time System 1 (hr)	Time System 2 (hr)	Comments
1	1	0.2	2	
2	2	1.7	2.9	
3	2	4.5	5.2	
4	2	5.8	9.1	
5	2	17.3	9.2	
6	2	29.3	24.1	
7	1	36.5	61.1	
8	2	46.3	69.6	
9	1	63.6	78.1	
10	2	64.4	85.4	
11	1	74.3	93.6	
12	1	106.6	103	
13	2	195.2	117	
14	2	235.1	134.3	
15	1	248.7	150.2	
16	2	256.8	164.6	
17	2	261.1	174.3	
18	2	299.4	193.2	
19	1	305.3	234.2	
20	1	326.9	257.3	
21	1	339.2	290.2	
22	1	366.1	293.1	
23	2	466.4	316.4	
24	1	504	373.2	
25	1	510	375.1	
26	2	543.2	386.1	
27	2	635.4	453.3	
28	1	641.2	485.8	
29	2	755.8	573.6	
30				

2. The current accumulated test time for both units is 1329.4 hours. If the process were to continue for an additional combined time of 200 hours, the expected cumulative number of failures at $T = 1529.4$ is 31.2695, as shown in the figure below. At T = 1329.4, the expected number of failures is 29.2004. Therefore, the expected number of failures that would be observed over the additional 200 hours is $31.2695 - 29.2004 = 2.0691 \approx 2$.



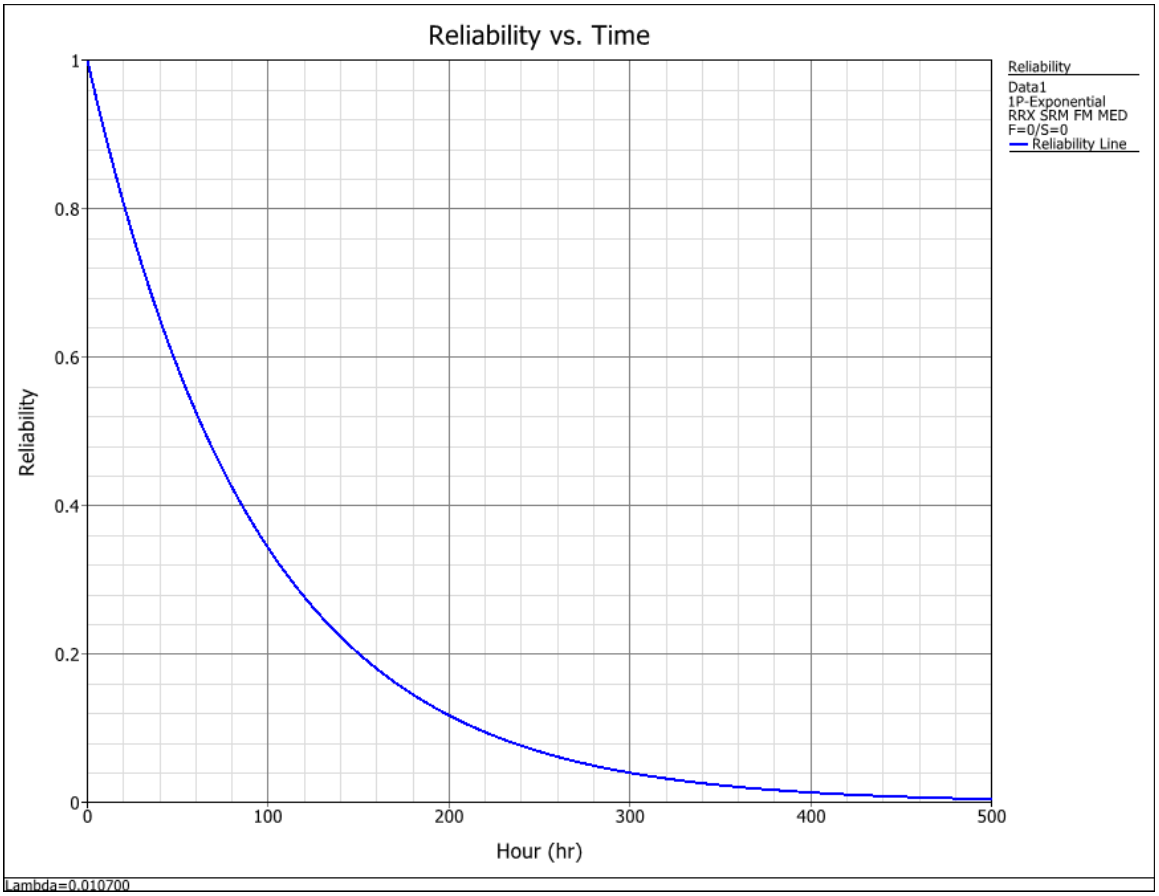
3. If testing/development were halted at this point, the system failure intensity would be equal to the instantaneous failure intensity at that time, or $\lambda = 0.0107$ failures/hour. See the following figure.



An exponential distribution can be assumed since the value of the failure intensity at that instant in time is known. Therefore:

$$R(t) = e^{-\lambda t}$$
$$= e^{-(0.0107)t}$$

Weibull++ can be utilized to provide a Reliability vs. Time plot which is shown below.



Using the Duane Model for Success/Failure Data

Given the sequential success/failure data in the table below, do the following:

1. Estimate the Duane parameters.
2. What is the instantaneous Reliability at the end of the test?
3. How many additional test runs with a one-sided 90% confidence level are required to meet an instantaneous Reliability goal of 80%?

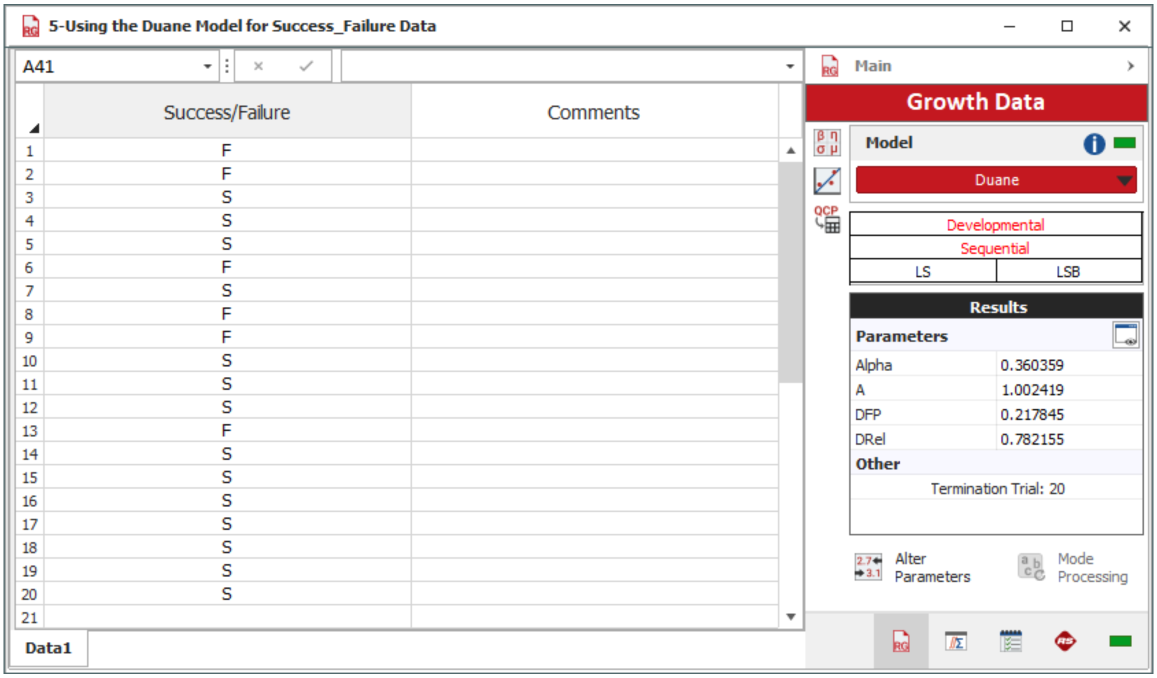
Sequential Data

Run Number	Result

1	F
2	F
3	S
4	S
5	S
6	F
7	S
8	F
9	F
10	S
11	S
12	S
13	F
14	S
15	S
16	S
17	S
18	S
19	S
20	S

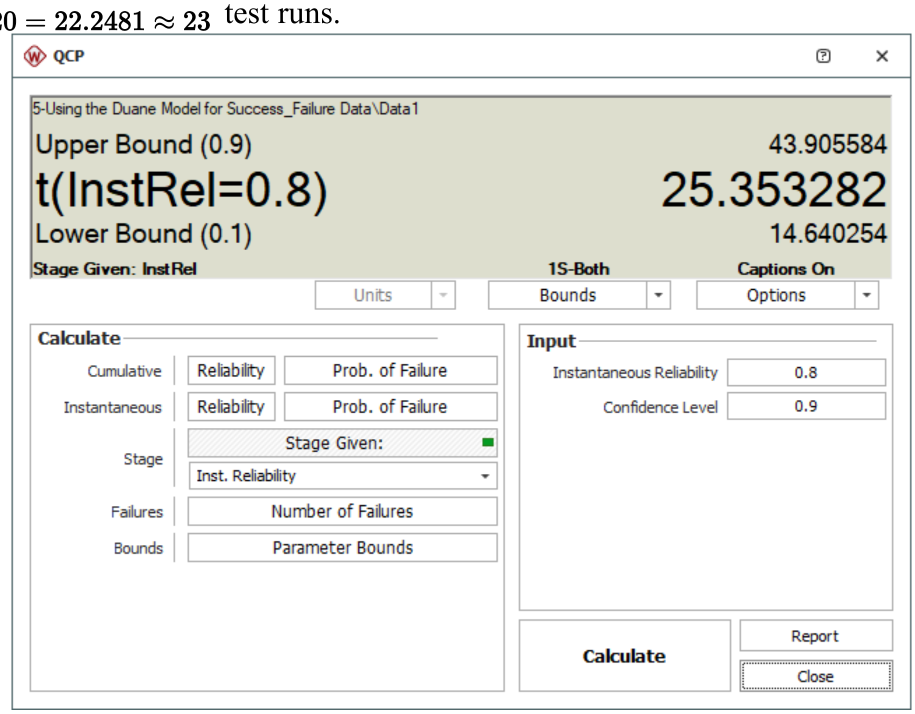
Solution

1. The following figure shows the data set entered into Weibull++ along with the estimated Duane parameters.



2. The Reliability at the end of the test is equal to 78.22%. Note that this is the DRel that is shown in the control panel in the above figure.

3. The figure below shows the number of test runs with both one-sided confidence bounds at 90% confidence level to achieve an instantaneous Reliability of 80%. Therefore, the number of additional test runs required with a 90% confidence level is equal to $42.2481 - 20 = 22.2481 \approx 23$ test runs.



Crow-AMSAA (NHPP)

Dr. Larry H. Crow [17] noted that the Duane Model could be stochastically represented as a Weibull process, allowing for statistical procedures to be used in the application of this model in reliability growth. This statistical extension became what is known as the Crow-AMSAA (NHPP) model. This method was first developed at the U.S. Army Materiel Systems Analysis Activity (AMSAA). It is frequently used on systems when usage is measured on a continuous scale. It can also be applied for the analysis of one shot items when there is high reliability and a large number of trials.

Test programs are generally conducted on a phase by phase basis. The Crow-AMSAA model is designed for tracking the reliability within a test phase and not across test phases. A development testing program may consist of several separate test phases. If corrective actions are introduced during a particular test phase, then this type of testing and the associated data are appropriate for analysis by the Crow-AMSAA model. The model analyzes the reliability growth progress within each test phase and can aid in determining the following:

- Reliability of the configuration currently on test
- Reliability of the configuration on test at the end of the test phase
- Expected reliability if the test time for the phase is extended
- Growth rate
- Confidence intervals
- Applicable goodness-of-fit tests

Background

The reliability growth pattern for the Crow-AMSAA model is exactly the same pattern as for the Duane postulate, that is, the cumulative number of failures is linear when plotted on ln-ln scale. Unlike the Duane postulate, the Crow-AMSAA model is statistically based. Under the Duane postulate, the failure rate is linear on ln-ln scale. However, for the Crow-AMSAA model statistical structure, the failure intensity of the underlying non-homogeneous Poisson process (NHPP) is linear when plotted on ln-ln scale.

Let $N(t)$ be the cumulative number of failures observed in cumulative test time t, and let $\rho(t)$ be the failure intensity for the Crow-AMSAA model. Under the NHPP model, $\rho(t)\Delta t$ is approximately the probably of a failure occurring over the interval $[t, t+\Delta t]$ for small Δt. In addition, the expected number of failures experienced over the test interval $[0, T]$ under the Crow-AMSAA model is given by:

$$E[N(T)] = \int_0^T \rho(t)dt$$

The Crow-AMSAA model assumes that $\rho(T)$ may be approximated by the Weibull failure rate function:

$$\rho(T) = \frac{\beta}{\eta^\beta}T^{\beta-1}$$

Therefore, if $\lambda = \frac{1}{\eta^\beta}$, the intensity function, $\rho(T)$, or the instantaneous failure intensity, $\lambda_i(T)$, is defined as:

$$\lambda_i(T) = \lambda\beta T^{\beta-1}, \text{with } T > 0,\ \lambda > 0 \text{ and } \beta > 0$$

In the special case of exponential failure times, there is no growth and the failure intensity, $\rho(t)$, is equal to λ. In this case, the expected number of failures is given by:

$$E[N(T)] = \int_0^T \rho(t)dt$$
$$= \lambda T$$

In order for the plot to be linear when plotted on ln-ln scale under the general reliability growth case, the following must hold true where the expected number of failures is equal to:

$$E[N(T)] = \int_0^T \rho(t)dt$$
$$= \lambda T^\beta$$

To put a statistical structure on the reliability growth process, consider again the special case of no growth. In this case the number of failures, $N(T)$, experienced during the testing over $[0, T]$ is random. The expected number of failures, $N(T)$, is said to follow the homogeneous (constant) Poisson process with mean λT and is given by:

$$\Pr\ [N(T) = n] = \frac{(\lambda T)^n e^{-\lambda T}}{n!};\ n = 0, 1, 2, \ldots$$

The Crow-AMSAA model generalizes this no growth case to allow for reliability growth due to corrective actions. This generalization keeps the Poisson distribution for the number of failures but allows for the expected number of failures, $E[N(T)]$, to be linear when plotted on ln-ln scale. The Crow-AMSAA model lets $E[N(T)] = \lambda T^\beta$. The probability that the number of failures, $N(T)$, will be equal to n under growth is then given by the Poisson distribution:

$$\Pr\ [N(T) = n] = \frac{(\lambda T^\beta)^n e^{-\lambda T^\beta}}{n!};\ n = 0, 1, 2, \ldots$$

This is the general growth situation, and the number of failures, $N(T)$, follows a non-homogeneous Poisson process. The exponential, "no growth" homogeneous Poisson process is a special case of the non-homogeneous Crow-AMSAA model. This is reflected in the Crow-AMSAA model parameter where $\beta = 1$. The cumulative failure rate, λ_c , is:

$$\lambda_c = \lambda T^{\beta-1}$$

The cumulative $MTBF_c$ is:

$$MTBF_c = \frac{1}{\lambda} T^{1-\beta}$$

As mentioned above, the local pattern for reliability growth within a test phase is the same as the growth pattern observed by Duane. The Duane $MTBF_c$ is equal to:

$$MTBF_{c_{DUANE}} = bT^{\alpha}$$

And the Duane cumulative failure rate, λ_c , is:

$$\lambda_{c_{DUANE}} = \frac{1}{b} T^{-\alpha}$$

Thus a relationship between Crow-AMSAA parameters and Duane parameters can be developed, such that:

$$b_{DUANE} = \frac{1}{\lambda_{AMSAA}}$$
$$\alpha_{DUANE} = 1 - \beta_{AMSAA}$$

Note that these relationships are not absolute. They change according to how the parameters (slopes, intercepts, etc.) are defined when the analysis of the data is performed. For the exponential case, $\beta = 1$, then $\lambda_i(T) = \lambda$, a constant. For $\beta > 1$, $\lambda_i(T)$ is increasing. This indicates a deterioration in system reliability. For $\beta < 1$, $\lambda_i(T)$ is decreasing. This is indicative of reliability growth. Note that the model assumes a Poisson process with the Weibull intensity function, not the Weibull distribution. Therefore, statistical procedures for the Weibull distribution do not apply for this model. The parameter λ is called a scale parameter because it depends upon the unit of measurement chosen for T , while β is the shape parameter that characterizes the shape of the graph of the intensity function.

The total number of failures, $N(T)$, is a random variable with Poisson distribution. Therefore, the probability that exactly n failures occur by time T is:

$$P[N(T) = n] = \frac{[\theta(T)]^n e^{-\theta(T)}}{n!}$$

The number of failures occurring in the interval from T_1 to T_2 is a random variable having a Poisson distribution with mean:

$$\theta(T_2) - \theta(T_1) = \lambda(T_2^\beta - T_1^\beta)$$

The number of failures in any interval is statistically independent of the number of failures in any interval that does not overlap the first interval. At time T_0, the failure intensity is $\lambda_i(T_0) = \lambda\beta T_0^{\beta-1}$. If improvements are not made to the system after time T_0, it is assumed that failures would continue to occur at the constant rate $\lambda_i(T_0) = \lambda\beta T_0^{\beta-1}$. Future failures would then follow an exponential distribution with mean $m(T_0) = \frac{1}{\lambda\beta T_0^{\beta-1}}$. The instantaneous MTBF of the system at time T is:

$$m(T) = \frac{1}{\lambda\beta T^{\beta-1}}$$

$m(T)$ is also called the demonstrated (or achieved) MTBF.

Note About Applicability

The Duane and Crow-AMSAA models are the most frequently used reliability growth models. Their relationship comes from the fact that both make use of the underlying observed linear relationship between the logarithm of cumulative MTBF and cumulative test time. However, the Duane model does not provide a capability to test whether the change in MTBF observed over time is significantly different from what might be seen due to random error between phases. The Crow-AMSAA model allows for such assessments. Also, the Crow-AMSAA allows for development of hypothesis testing procedures to determine growth presence in the data (where $\beta < 1$ indicates that there is growth in MTBF, $\beta = 1$ indicates a constant MTBF and $\beta > 1$ indicates a decreasing MTBF). Additionally, the Crow-AMSAA model views the process of reliability growth as probabilistic, while the Duane model views the process as deterministic.

Failure Times Data

A description of Failure Times Data is presented in the Reliability Growth Analysis Data Types page.

Parameter Estimation for Failure Times Data

The parameters for the Crow-AMSAA (NHPP) model are estimated using maximum likelihood estimation (MLE). The probability density function (*pdf*) of the i^{th} event given that the $(i-1)^{th}$ event occurred at T_{i-1} is:

$$f(T_i|T_{i-1}) = \frac{\beta}{\eta}\left(\frac{T_i}{\eta}\right)^{\beta-1} \cdot e^{-\frac{1}{\eta^\beta}\left(T_i^\beta - T_{i-1}^\beta\right)}$$

Let $\lambda = \frac{1}{\eta^\beta}$, the likelihood function is:

$$L = \lambda^n \beta^n e^{-\lambda T^{*\beta}} \prod_{i=1}^{n} T_i^{\beta-1}$$

where T^* is the termination time and is given by:

$$T^* = \begin{Bmatrix} T_n \text{ if the test is failure terminated} \\ T > T_n \text{ if the test is time terminated} \end{Bmatrix}$$

Taking the natural log on both sides:

$$\Lambda = n \ln \lambda + n \ln \beta - \lambda T^{*\beta} + (\beta - 1)\sum_{i=1}^{n} \ln T_i$$

And differentiating with respect to λ yields:

$$\frac{\partial \Lambda}{\partial \lambda} = \frac{n}{\lambda} - T^{*\beta}$$

Set equal to zero and solve for λ :

$$\hat{\lambda} = \frac{n}{T^{*\beta}}$$

Now differentiate with respect to β :

$$\frac{\partial \Lambda}{\partial \beta} = \frac{n}{\beta} - \lambda T^{*\beta} \ln T^* + \sum_{i=1}^{n} \ln T_i$$

Set equal to zero and solve for β :

$$\hat{\beta} = \frac{n}{n \ln T^* - \sum^{n}_{i=1} \ln T_i}$$

This equation is used for both failure terminated and time terminated test data.

BIASING AND UNBIASING OF BETA

The equation above returns the biased estimate, $\hat{\beta}$. The unbiased estimate, $\bar{\beta}$, can be calculated by using the following relationships. For time terminated data (the test ends after a specified test time):

$$\bar{\beta} = \frac{N-1}{N}\hat{\beta}$$

For failure terminated data (the test ends after a specified number of failures):

$$\bar{\beta} = \frac{N-2}{N-1}\hat{\beta}$$

By default $\hat{\beta}$ is returned. $\bar{\beta}$ can be returned by selecting the **Calculate unbiased beta** option on the Calculations tab of the Application Setup.

Cramer-von Mises Test

The Cramer-von Mises (CVM) goodness-of-fit test validates the hypothesis that the data follows a non-homogeneous Poisson process with a failure intensity equal to $u(t) = \lambda\beta t^{\beta-1}$. This test can be applied when the failure data is complete over the continuous interval $[0, T_q]$ with no gaps in the data. The CVM data type applies to all data types when the failure times are known, except for Fleet data.

If the individual failure times are known, a Cramer-von Mises statistic is used to test the null hypothesis that a non-homogeneous Poisson process with the failure intensity function $\rho(t) = \lambda\beta t^{\beta-1} (\lambda > 0, \beta > 0, t > 0)$ properly describes the reliability growth of a system. The Cramer-von Mises goodness-of-fit statistic is then given by the following expression:

$$C_M^2 = \frac{1}{12M} + \sum_{i=1}^{M}\left[\left(\frac{T_i}{T}\right)^{\bar{\beta}} - \frac{2i-1}{2M}\right]^2$$

where:

$$M = \begin{Bmatrix} N \text{ if the test is time terminated} \\ N-1 \text{ if the test is failure terminated} \end{Bmatrix}$$

$\bar{\beta}$ is the unbiased value of Beta.

The failure times, T_i , must be ordered so that $T_1 < T_2 < \ldots < T_M$. If the statistic C_M^2 is less than the critical value corresponding to M for a chosen significance level, then you can fail to reject the null hypothesis that the Crow-AMSAA model adequately fits the data.

CRITICAL VALUES

The following table displays the critical values for the Cramer-von Mises goodness-of-fit test given the sample size, M, and the significance level, α.

Critical values for Cramer-von Mises test

	α				
M	0.20	0.15	0.10	0.05	0.01
2	0.138	0.149	0.162	0.175	0.186
3	0.121	0.135	0.154	0.184	0.23
4	0.121	0.134	0.155	0.191	0.28
5	0.121	0.137	0.160	0.199	0.30
6	0.123	0.139	0.162	0.204	0.31
7	0.124	0.140	0.165	0.208	0.32
8	0.124	0.141	0.165	0.210	0.32
9	0.125	0.142	0.167	0.212	0.32
10	0.125	0.142	0.167	0.212	0.32
11	0.126	0.143	0.169	0.214	0.32
12	0.126	0.144	0.169	0.214	0.32
13	0.126	0.144	0.169	0.214	0.33
14	0.126	0.144	0.169	0.214	0.33
15	0.126	0.144	0.169	0.215	0.33
16	0.127	0.145	0.171	0.216	0.33
17	0.127	0.145	0.171	0.217	0.33
18	0.127	0.146	0.171	0.217	0.33
19	0.127	0.146	0.171	0.217	0.33
20	0.128	0.146	0.172	0.217	0.33
30	0.128	0.146	0.172	0.218	0.33
60	0.128	0.147	0.173	0.220	0.33
100	0.129	0.147	0.173	0.220	0.34

The significance level represents the probability of rejecting the hypothesis even if it's true. So, there is a risk associated with applying the goodness-of-fit test (i.e., there is a chance that the CVM will indicate that the model does not fit, when in fact it does). As the significance level is increased, the CVM test becomes more stringent. Keep in mind that the CVM test passes when the test statistic is less than the critical value. Therefore, the larger the critical value, the more room there is to work with (e.g., a CVM test with a significance level equal to 0.1 is more strict than a test with 0.01).

Confidence Bounds

The Weibull++ software provides two methods to estimate the confidence bounds for the Crow Extended model when applied to developmental testing data. The Fisher Matrix approach is based on the Fisher Information Matrix and is commonly employed in the reliability field. The Crow bounds were developed by Dr. Larry Crow. See the Crow-AMSAA Confidence Bounds chapter for details on how the confidence bounds are calculated.

Failure Times Data Examples

EXAMPLE - PARAMETER ESTIMATION

A prototype of a system was tested with design changes incorporated during the test. The following table presents the data collected over the entire test. Find the Crow-AMSAA parameters and the intensity function using maximum likelihood estimators.

Developmental Test Data

Row	Time to Event (hr)	$ln(T)$
1	2.7	0.99325
2	10.3	2.33214
3	12.5	2.52573
4	30.6	3.42100
5	57.0	4.04305
6	61.3	4.11578
7	80.0	4.38203
8	109.5	4.69592
9	125.0	4.82831

10	128.6	4.85671
11	143.8	4.96842
12	167.9	5.12337
13	229.2	5.43459
14	296.7	5.69272
15	320.6	5.77019
16	328.2	5.79362
17	366.2	5.90318
18	396.7	5.98318
19	421.1	6.04287
20	438.2	6.08268
21	501.2	6.21701
22	620.0	6.42972

Solution

For the failure terminated test, β is:

$$\begin{aligned}\widehat{\beta} &= \frac{n}{n\ln T^{*} - \sum^{n}{}_{i=1} \ln T_i} \\ &= \frac{22}{22\ln 620 - \sum^{22}{}_{i=1} \ln T_i}\end{aligned}$$

where:

$$\sum_{i=1}^{22} \ln T_i = 105.6355$$

Then:

$$\widehat{\beta} = \frac{22}{22\ln 620 - 105.6355} = 0.6142$$

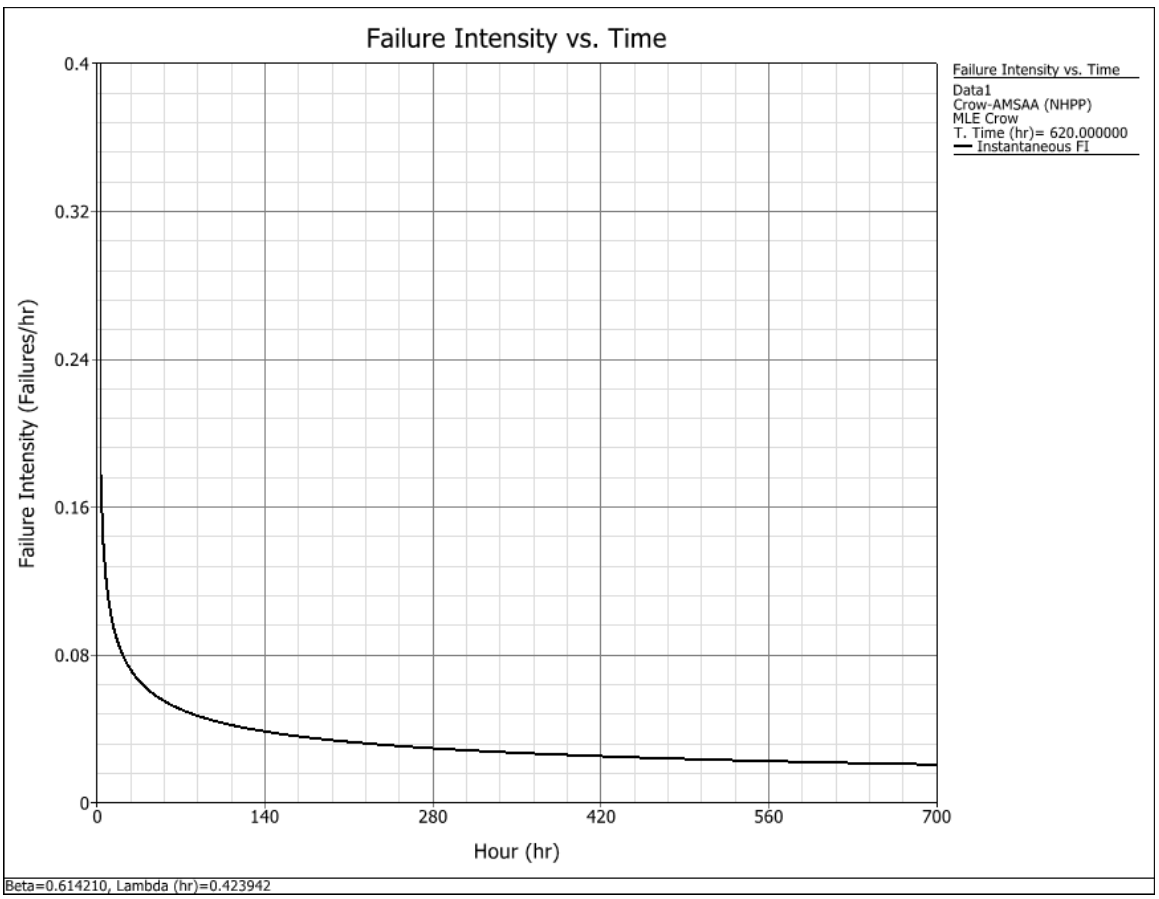
And for λ :

$$\widehat{\lambda} = \frac{n}{T^{*\beta}}$$
$$= \frac{22}{620^{0.6142}} = 0.4239$$

Therefore, $\lambda_i(T)$ becomes:

$$\widehat{\lambda}_i(T) = 0.4239 \cdot 0.6142 \cdot 620^{-0.3858}$$
$$= 0.0217906 \frac{\text{failures}}{\text{hr}}$$

The next figure shows the plot of the failure rate. If no further changes are made, the estimated MTBF is $\frac{1}{0.0217906}$ or 46 hours.



EXAMPLE - CONFIDENCE BOUNDS ON FAILURE INTENSITY

Using the values of $\hat{\beta}$ and $\hat{\lambda}$ estimated in the example given above, calculate the 90% 2-sided confidence bounds on the cumulative and instantaneous failure intensity.

Solution

Fisher Matrix Bounds

The partial derivatives for the Fisher Matrix confidence bounds are:

$$\frac{\partial^2 \Lambda}{\partial \lambda^2} = -\frac{22}{0.4239^2} = -122.43$$
$$\frac{\partial^2 \Lambda}{\partial \beta^2} = -\frac{22}{0.6142^2} - 0.4239 \cdot 620^{0.6142} (\ln 620)^2 = -967.68$$
$$\frac{\partial^2 \Lambda}{\partial \lambda \partial \beta} = -620^{0.6142} \ln 620 = -333.64$$

The Fisher Matrix then becomes:

$$\begin{bmatrix} 122.43 & 333.64 \\ 333.64 & 967.68 \end{bmatrix}^{-1} = \begin{bmatrix} Var(\hat{\lambda}) & Cov(\hat{\beta}, \hat{\lambda}) \\ Cov(\hat{\beta}, \hat{\lambda}) & Var(\hat{\beta}) \end{bmatrix}$$
$$= \begin{bmatrix} 0.13519969 & -0.046614609 \\ -0.046614609 & 0.017105343 \end{bmatrix}$$

For $T = 620$ hours, the partial derivatives of the cumulative and instantaneous failure intensities are:

$$\frac{\partial \lambda_c(T)}{\partial \beta} = \hat{\lambda} T^{\hat{\beta}-1} \ln(T)$$
$$= 0.4239 \cdot 620^{-0.3858} \ln 620$$
$$= 0.22811336$$
$$\frac{\partial \lambda_c(T)}{\partial \lambda} = T^{\hat{\beta}-1}$$
$$= 620^{-0.3858}$$
$$= 0.083694185$$

$$\frac{\partial \lambda_i(T)}{\partial \beta} = \hat{\lambda} T^{\hat{\beta}-1} + \hat{\lambda}\hat{\beta} T^{\hat{\beta}-1} \ln T$$
$$= 0.4239 \cdot 620^{-0.3858} + 0.4239 \cdot 0.6142 \cdot 620^{-0.3858} \ln 620$$
$$= 0.17558519$$

$$\frac{\partial \lambda_i(T)}{\partial \lambda} = \hat{\beta} T^{\hat{\beta}-1}$$
$$= 0.6142 \cdot 620^{-0.3858}$$
$$= 0.051404969$$

Therefore, the variances become:

$$Var(\hat{\lambda_c}(T)) = 0.22811336^2 \cdot 0.017105343 + 0.083694185^2 \cdot 0.13519969 - 2 \cdot 0.22811336 \cdot 0.083694185 \cdot 0.046614609$$
$$= 0.00005721408$$
$$Var(\hat{\lambda_i}(T)) = 0.17558519^2 \cdot 0.01715343 + 0.051404969^2 \cdot 0.13519969 - 2 \cdot 0.17558519 \cdot 0.051404969 \cdot 0.046614609$$
$$= 0.0000431393$$

The cumulative and instantaneous failure intensities at $T = 620$ hours are:

$\lambda_c(T)$ =0.03548
$\lambda_i(T)$ =0.02179

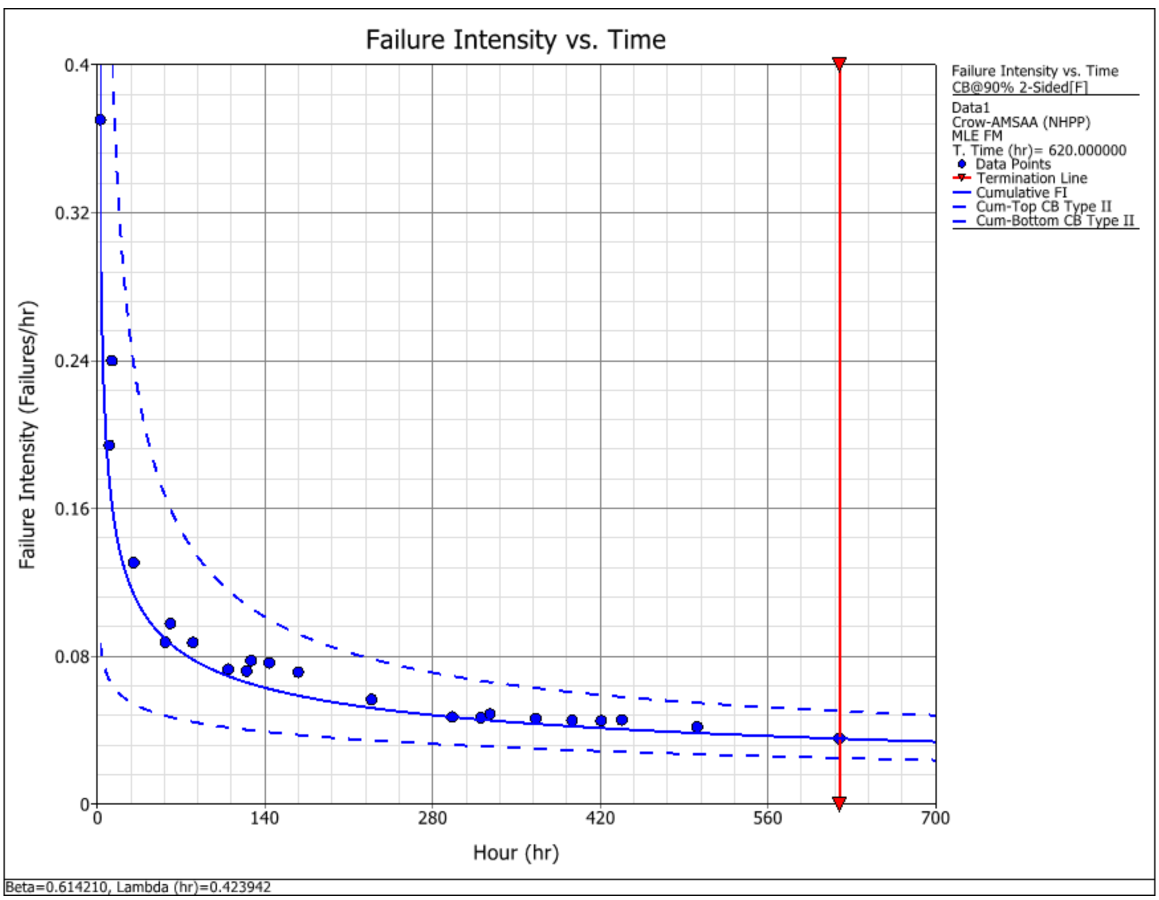
So, at the 90% confidence level and for $T = 620$ hours, the Fisher Matrix confidence bounds for the cumulative failure intensity are:

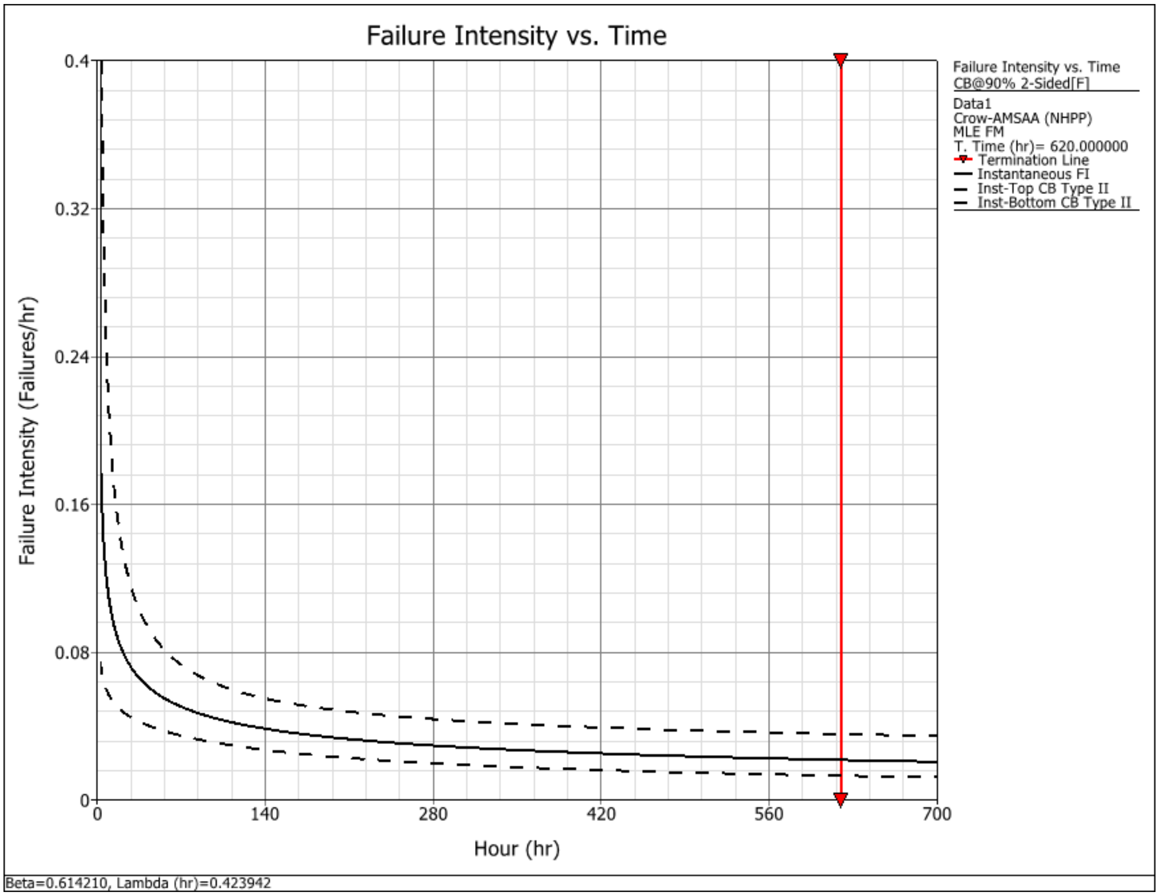
$[\lambda_c(T)]_L$ =0.02499
$[\lambda_c(T)]_U$ =0.05039

The confidence bounds for the instantaneous failure intensity are:

$[\lambda_i(T)]_L$ =0.01327
$[\lambda_i(T)]_U$ =0.03579

The following figures display plots of the Fisher Matrix confidence bounds for the cumulative and instantaneous failure intensity, respectively.





Crow Bounds

Given that the data is failure terminated, the Crow confidence bounds for the cumulative failure intensity at the 90% confidence level and for $T = 620$ hours are:

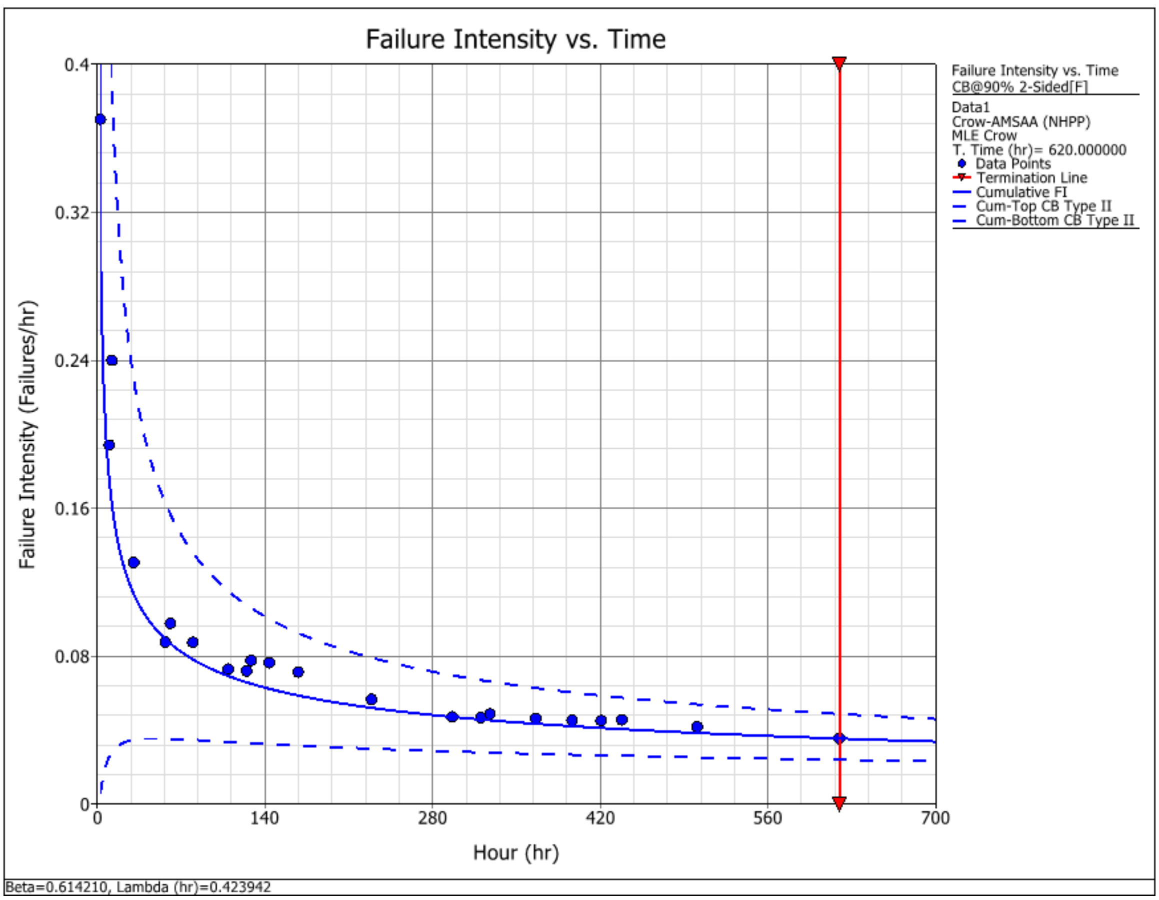
$$\begin{aligned}[\lambda_c(T)]_L &= \frac{\chi^2_{\frac{\alpha}{2},2N}}{2 \cdot t} \\ &= \frac{29.787476}{2 * 620} \\ &= 0.02402 \\ [\lambda_c(T)]_U &= \frac{\chi^2_{1-\frac{\alpha}{2},2N}}{2 \cdot t} \\ &= \frac{60.48089}{2 * 620} \\ &= 0.048775\end{aligned}$$

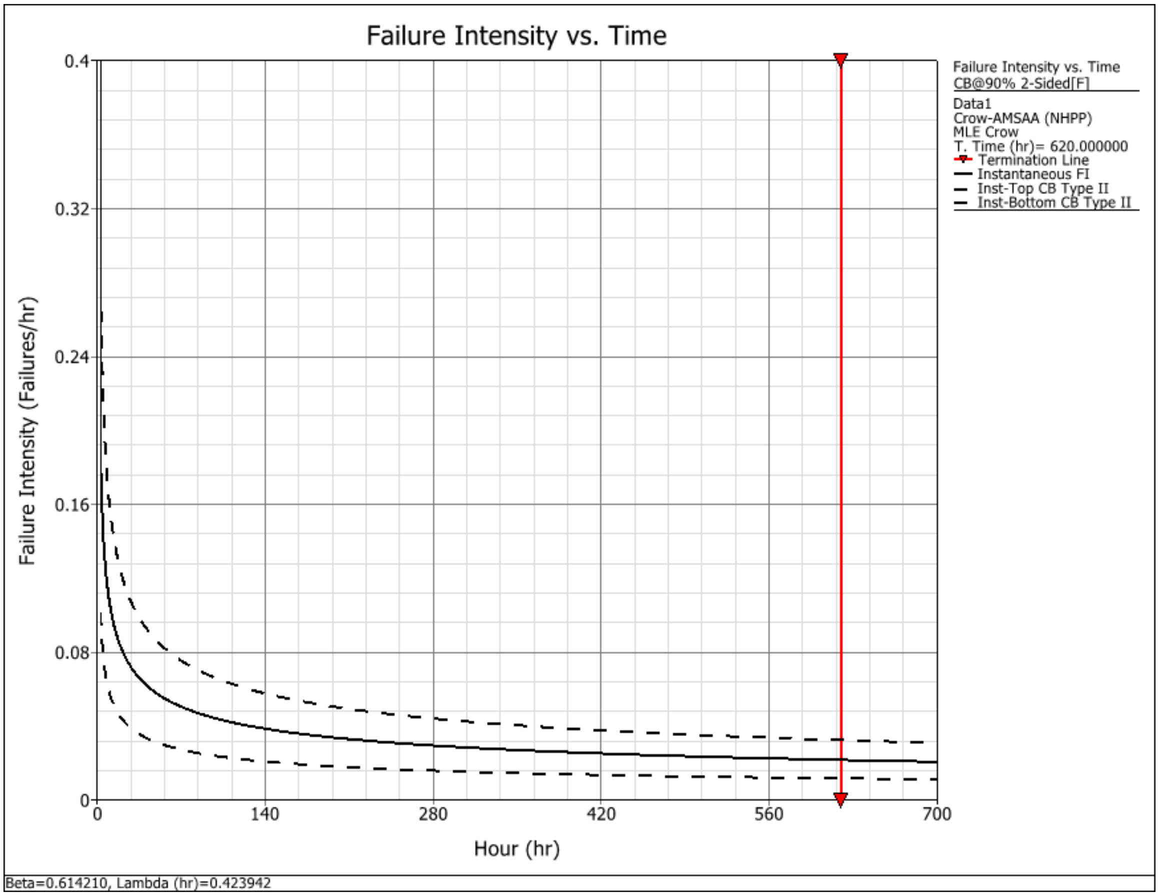
The Crow confidence bounds for the instantaneous failure intensity at the 90% confidence level and for $T = 620$ hours are calculated by first estimating the bounds on the instantaneous MTBF. Once these are calculated, take the inverse as shown below. Details on the confidence bounds for instantaneous MTBF are presented here.

$$
\begin{aligned}
[\lambda_i(t)]_L &= \frac{1}{[MTBF_i]_U} \\
&= \frac{1}{MTBF_i \cdot U} \\
&= 0.01179
\end{aligned}
$$

$$
\begin{aligned}
[\lambda_i(t)]_U &= \frac{1}{[MTBF_i]_L} \\
&= \frac{1}{MTBF_i \cdot L} \\
&= 0.03253
\end{aligned}
$$

The following figures display plots of the Crow confidence bounds for the cumulative and instantaneous failure intensity, respectively.





EXAMPLE - CONFIDENCE BOUNDS ON MTBF

Calculate the confidence bounds on the cumulative and instantaneous MTBF for the data from the example given above.

Solution

Fisher Matrix Bounds

From the previous example:

$$Var(\hat{\lambda}) = 0.13519969$$
$$Var(\hat{\beta}) = 0.017105343$$
$$Cov(\hat{\beta}, \hat{\lambda}) = -0.046614609$$

And for $T = 620$ hours, the partial derivatives of the cumulative and instantaneous MTBF are:

$$\begin{aligned}
\frac{\partial m_c(T)}{\partial \beta} &= -\frac{1}{\hat{\lambda}} T^{1-\hat{\beta}} \ln T \\
&= -\frac{1}{0.4239} 620^{0.3858} \ln 620 \\
&= -181.23135 \\
\frac{\partial m_c(T)}{\partial \lambda} &= -\frac{1}{\hat{\lambda}^2} T^{1-\hat{\beta}} \\
&= -\frac{1}{0.4239^2} 620^{0.3858} \\
&= -66.493299 \\
\frac{\partial m_i(T)}{\partial \beta} &= -\frac{1}{\hat{\lambda}\hat{\beta}^2} T^{1-\beta} - \frac{1}{\hat{\lambda}\hat{\beta}} T^{1-\hat{\beta}} \ln T \\
&= -\frac{1}{0.4239 \cdot 0.6142^2} 620^{0.3858} - \frac{1}{0.4239 \cdot 0.6142} 620^{0.3858} \ln 620 \\
&= -369.78634 \\
\frac{\partial m_i(T)}{\partial \lambda} &= -\frac{1}{\hat{\lambda}^2 \hat{\beta}} T^{1-\hat{\beta}} \\
&= -\frac{1}{0.4239^2 \cdot 0.6142} \cdot 620^{0.3858} \\
&= -108.26001
\end{aligned}$$

Therefore, the variances become:

$$\begin{aligned}
Var(\hat{m}_c(T)) =& (-181.23135)^2 \cdot 0.017105343 + (-66.493299)^2 \cdot 0.13519969 \\
& - 2 \cdot (-181.23135) \cdot (-66.493299) \cdot 0.046614609 \\
=& 36.113376
\end{aligned}$$

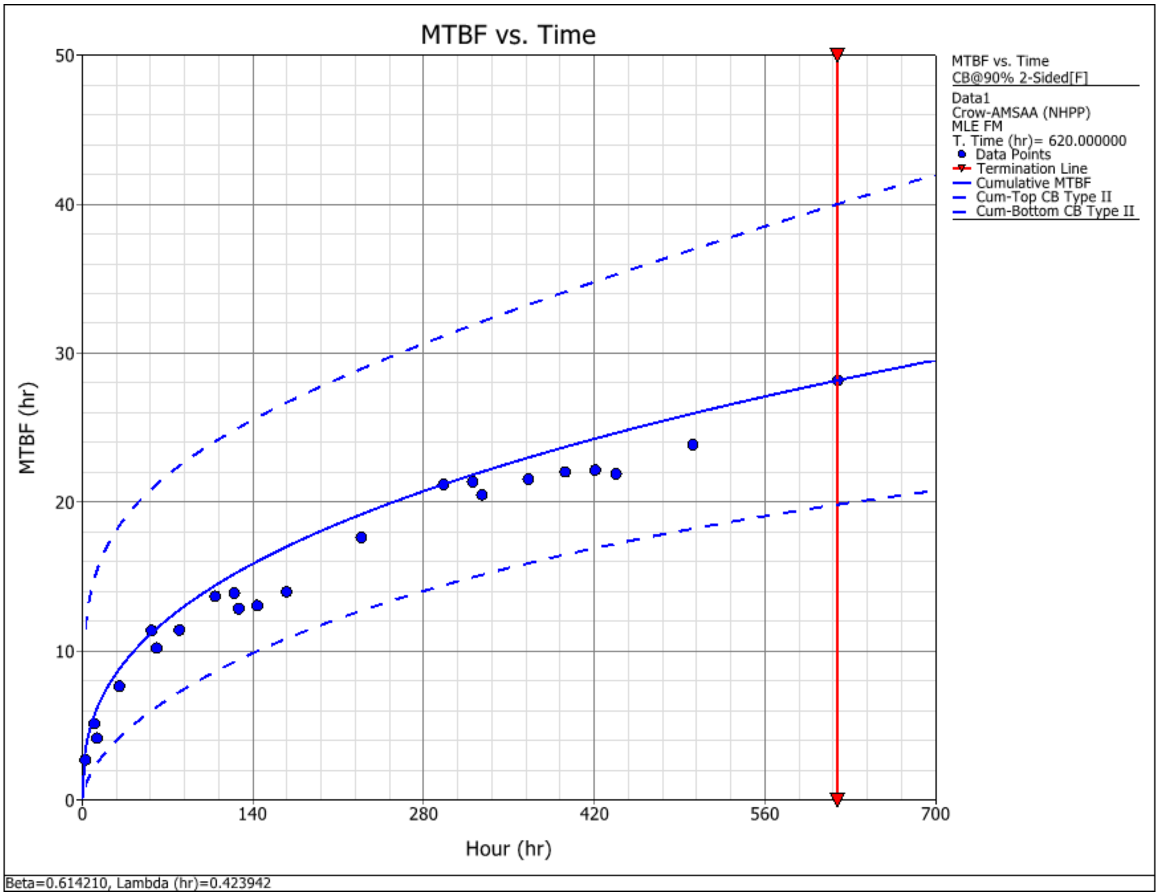
$$\begin{aligned}
Var(\hat{m}_i(T)) =& (-369.78634)^2 \cdot 0.017105343 + (-108.26001)^2 \cdot 0.13519969 \\
& - 2 \cdot (-369.78634) \cdot (-108.26001) \cdot 0.046614609 \\
=& 191.33709
\end{aligned}$$

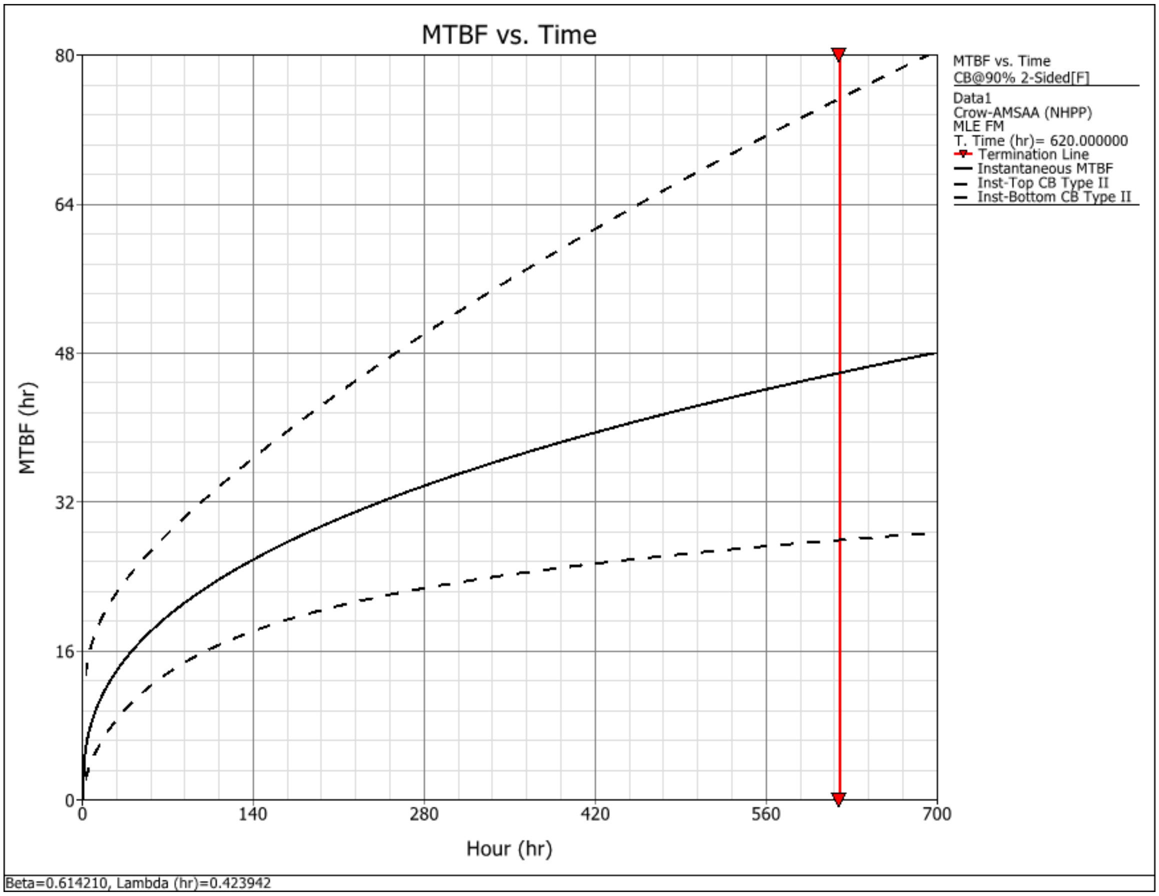
So, at 90% confidence level and $T = 620$ hours, the Fisher Matrix confidence bounds are:

$$\begin{aligned}
[m_c(T)]_L =& \hat{m}_c(t) e^{-z_\alpha \sqrt{Var(\hat{m}_c(t))}/\hat{m}_c(t)} \\
=& 19.84581 \\
[m_c(T)]_U =& \hat{m}_c(t) e^{z_\alpha \sqrt{Var(\hat{m}_c(t))}/\hat{m}_c(t)} \\
=& 40.01927
\end{aligned}$$

$$\begin{aligned}
[m_i(T)]_L =& \hat{m}_i(t) e^{-z_\alpha \sqrt{Var(\hat{m}_i(t))}/\hat{m}_i(t)} \\
=& 27.94261 \\
[m_i(T)]_U =& \hat{m}_i(t) e^{z_\alpha \sqrt{Var(\hat{m}_i(t))}/\hat{m}_i(t)} \\
=& 75.34193
\end{aligned}$$

The following two figures show plots of the Fisher Matrix confidence bounds for the cumulative and instantaneous MTBFs.


MTBF vs. Time
MTBF (hr)
Hour (hr)
0
140
280
420
560
700
0
10
20
30
40
50
MTBF vs. Time
CB@90% 2-Sided[F]
Data1
Crow-AMSAA (NHPP)
MLE FM
T. Time (hr)= 620.000000
Data Points
Termination Line
Cumulative MTBF
Cum-Top CB Type II
Cum-Bottom CB Type II
Beta=0.614210, Lambda (hr)=0.423942



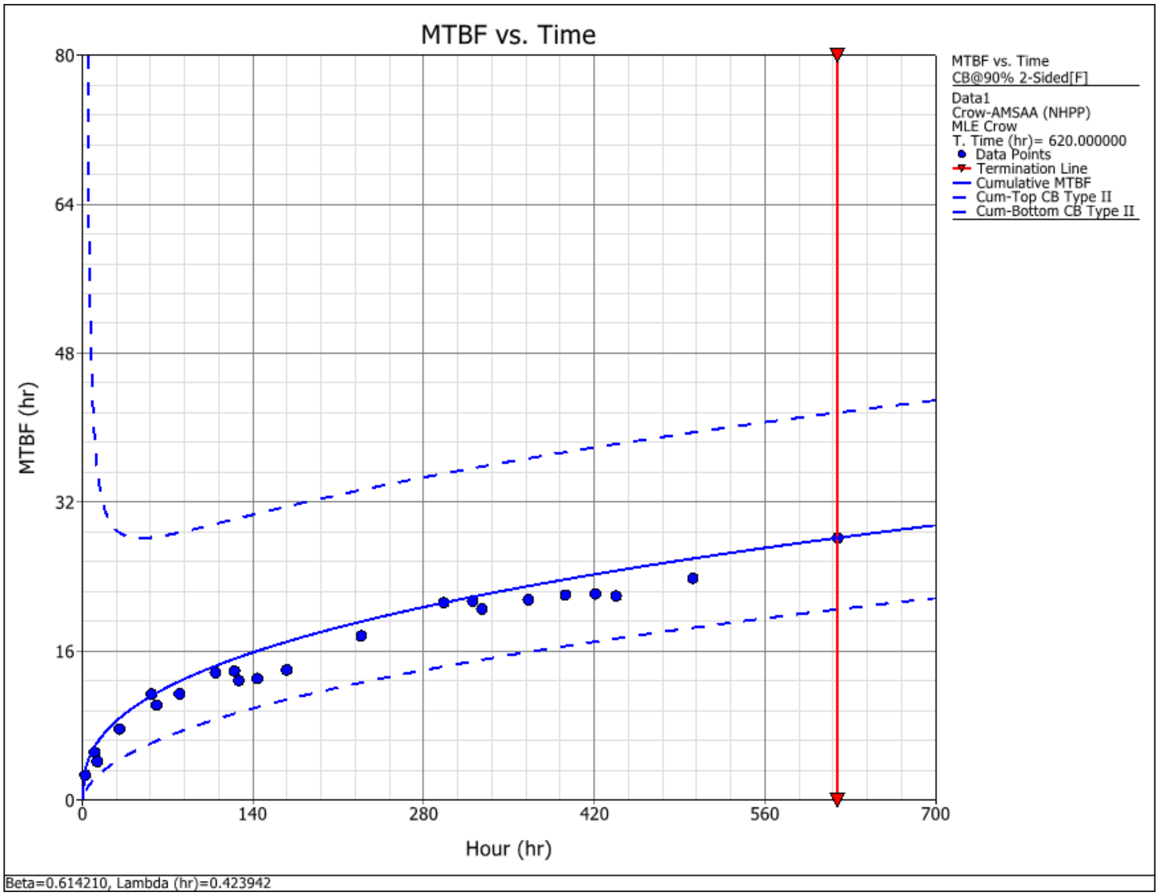
Crow Bounds

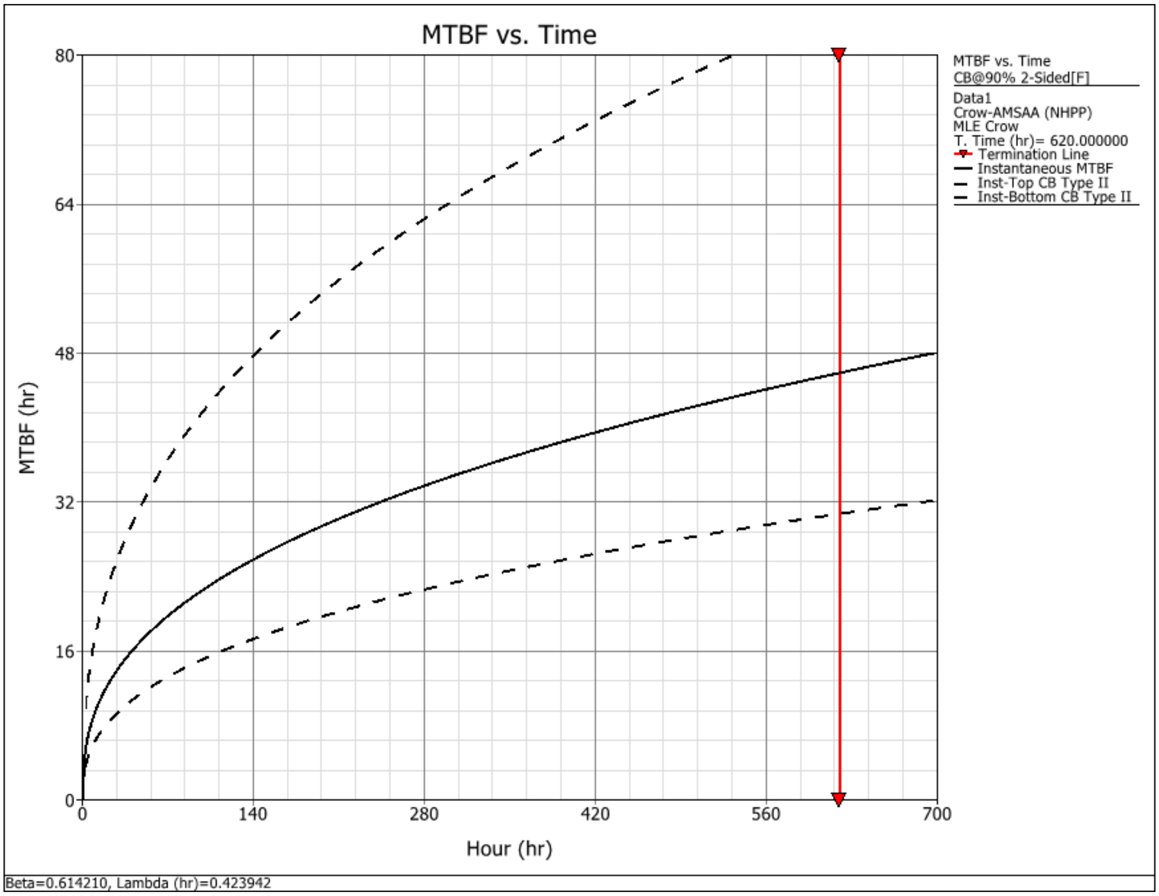
The Crow confidence bounds for the cumulative MTBF and the instantaneous MTBF at the 90% confidence level and for $T = 620$ hours are:

$$\begin{aligned}[m_c(T)]_L &= \frac{1}{[\lambda_c(T)]_U} \\ &= 20.5023 \\ [m_c(T)]_U &= \frac{1}{[\lambda_c(T)]_L} \\ &= 41.6282\end{aligned}$$

$$\begin{aligned}[MTBF_i]_L &= MTBF_i \cdot \Pi_1 \\ &= 30.7445 \\ [MTBF_i]_U &= MTBF_i \cdot \Pi_2 \\ &= 84.7972\end{aligned}$$

The figures below show plots of the Crow confidence bounds for the cumulative and instantaneous MTBF.


MTBF vs. Time
MTBF (hr)
Hour (hr)
80
64
48
32
16
0
140
280
420
560
700
MTBF vs. Time
CB@90% 2-Sided[F]
Data1
Crow-AMSAA (NHPP)
MLE Crow
T. Time (hr)= 620.000000
Data Points
Termination Line
Cumulative MTBF
Cum-Top CB Type II
Cum-Bottom CB Type II
Beta=0.614210, Lambda (hr)=0.423942



Confidence bounds can also be obtained on the parameters $\hat{\beta}$ and $\hat{\lambda}$. For Fisher Matrix confidence bounds:

$$\beta_L = \hat{\beta} e^{z_\alpha \sqrt{Var(\hat{\beta})}/\hat{\beta}}$$
$$= 0.4325$$
$$\beta_U = \hat{\beta} e^{-z_\alpha \sqrt{Var(\hat{\beta})}/\hat{\beta}}$$
$$= 0.8722$$

and:

$$\lambda_L = \hat{\lambda} e^{z_\alpha \sqrt{Var(\hat{\lambda})}/\hat{\lambda}}$$
$$= 0.1016$$
$$\lambda_U = \hat{\lambda} e^{-z_\alpha \sqrt{Var(\hat{\lambda})}/\hat{\lambda}}$$
$$= 1.7691$$

For Crow confidence bounds:

$$\beta_L = 0.4527$$
$$\beta_U = 0.9350$$

and:

$$\lambda_L = 0.2870$$
$$\lambda_U = 0.5827$$

Multiple Systems

When more than one system is placed on test during developmental testing, there are multiple data types which are available depending on the testing strategy and the format of the data. The data types that allow for the analysis of multiple systems using the Crow-AMSAA (NHPP) model are given below:

- Multiple Systems (Known Operating Times)
- Multiple Systems (Concurrent Operating Times)
- Multiple Systems with Dates

Goodness-of-fit Tests

For all multiple systems data types, the Cramer-von Mises (CVM) Test is available. For Multiple Systems (Concurrent Operating Times) and Multiple Systems with Dates, two additional tests are also available: Laplace Trend Test and Common Beta Hypothesis.

Multiple Systems (Known Operating Times)

A description of Multiple Systems (Known Operating Times) is presented on the Reliability Growth Analysis Data Types page.

Consider the data in the table below for two prototypes that were placed in a reliability growth test.

Developmental Test Data for Two Identical Systems

Failure Number	Failed Unit	Test Time Unit 1 (hr)	Test Time Unit 2 (hr)	Total Test Time (hr)	$ln(T)$
1	1	1.0	1.7	2.7	0.99325
2	1	7.3	3.0	10.3	2.33214
3	2	8.7	3.8	12.5	2.52573
4	2	23.3	7.3	30.6	3.42100

5	2	46.4	10.6	57.0	4.04305
6	1	50.1	11.2	61.3	4.11578
7	1	57.8	22.2	80.0	4.38203
8	2	82.1	27.4	109.5	4.69592
9	2	86.6	38.4	125.0	4.82831
10	1	87.0	41.6	128.6	4.85671
11	2	98.7	45.1	143.8	4.96842
12	1	102.2	65.7	167.9	5.12337
13	1	139.2	90.0	229.2	5.43459
14	1	166.6	130.1	296.7	5.69272
15	2	180.8	139.8	320.6	5.77019
16	1	181.3	146.9	328.2	5.79362
17	2	207.9	158.3	366.2	5.90318
18	2	209.8	186.9	396.7	5.98318
19	2	226.9	194.2	421.1	6.04287
20	1	232.2	206.0	438.2	6.08268
21	2	267.5	233.7	501.2	6.21701
22	2	330.1	289.9	620.0	6.42972

The Failed Unit column indicates the system that failed and is meant to be informative, but it does not affect the calculations. To combine the data from both systems, the system ages are added together at the times when a failure occurred. This is seen in the Total Test Time column above. Once the single timeline is generated, then the calculations for the parameters Beta and Lambda are the same as the process presented for Failure Times Data. The results of this analysis would match the results of Failure Times - Example 1.

Multiple Systems (Concurrent Operating Times)

A description of Multiple Systems (Concurrent Operating Times) is presented on the Reliability Growth Analysis Data Types page.

PARAMETER ESTIMATION FOR MULTIPLE SYSTEMS (CONCURRENT OPERATING TIMES)

To estimate the parameters, the equivalent system must first be determined. The equivalent single system (ESS) is calculated by summing the usage across all systems when a failure occurs. Keep in mind that Multiple Systems (Concurrent Operating Times) assumes that the systems are running simultaneously and accumulate the same usage. If the systems have different end times then the equivalent system must only account for the systems that are operating when a failure occurred. Systems with a start time greater than zero are shifted back to $t = 0$. This is the same as having a start time equal to zero and the converted end time is equal to the end time minus the start time. In addition, all failures times are adjusted by subtracting the start time from each value to ensure that all values occur within $t = 0$ and the adjusted end time. A start time greater than zero indicates that it is not known as to what events occurred at a time less than the start time. This may have been caused by the events during this period not being tracked and/or recorded properly.

As an example, consider two systems have entered a reliability growth test. Both systems have a start time equal to zero and both begin the test with the same configuration. System 1 operated for 100 hours and System 2 operated for 125 hours. The failure times for each system are given below:

- System 1: 25, 47, 80
- System 2: 15, 62, 89, 110

To build the ESS, the total accumulated hours across both systems is taken into account when a failure occurs. Therefore, given the data for Systems 1 and 2, the ESS is comprised of the following events: 30, 50, 94, 124, 160, 178, 210.

The ESS combines the data from both systems into a single timeline. The termination time for the ESS is (100 + 125) = 225 hours. The parameter estimates for $\hat{\beta}$ and $\hat{\lambda}$ are then calculated using the ESS. This process is the same as the method for Failure Times data.

EXAMPLE - CONCURRENT OPERATING TIMES

Six systems were subjected to a reliability growth test, and a total of 82 failures were observed. Given the data in the table below, which presents the start/end times and times-to-failure for each system, do the following:

1. Estimate the parameters of the Crow-AMSAA model using maximum likelihood estimation.
2. Determine how many additional failures would be generated if testing continues until 3,000 hours.

Multiple Systems (Concurrent Operating Times) Data

System #	1	2	3	4	5	6
Start Time (Hr)	0	0	0	0	0	0
End Time (Hr)	504	541	454	474	436	500

Failure Times (Hr)	21	83	26	36	23	7
	29	83	26	306	46	13
	43	83	57	306	127	13
	43	169	64	334	166	31
	43	213	169	354	169	31
	66	299	213	395	213	82
	115	375	231	403	213	109
	159	431	231	448	255	137
	199		231	456	369	166
	202		231	461	374	200
	222		304		380	210
	248		383		415	220
	248					301
	255					422
	286					437
	286					469
	304					469
	320					
	348					
	364					
	404					
	410					
	429					

Solution

1. To estimate the parameters $\hat{\beta}$ and $\hat{\lambda}$, the equivalent single system (ESS) must first be determined. The ESS is given below:

Equivalent Single System

Row	Time to Event (hr)		Row	Time to Event (hr)		Row	Time to Event (hr)		Row	Time to Event (hr)
1	42		22	498		43	1386		64	2214
2	78		23	654		44	1386		65	2244
3	78		24	690		45	1386		66	2250
4	126		25	762		46	1386		67	2280
5	138		26	822		47	1488		68	2298
6	156		27	954		48	1488		69	2370
7	156		28	996		49	1530		70	2418
8	174		29	996		50	1530		71	2424
9	186		30	1014		51	1716		72	2460
10	186		31	1014		52	1716		73	2490
11	216		32	1014		53	1794		74	2532
12	258		33	1194		54	1806		75	2574
13	258		34	1200		55	1824		76	2586
14	258		35	1212		56	1824		77	2621
15	276		36	1260		57	1836		78	2676
16	342		37	1278		58	1836		79	2714
17	384		38	1278		59	1920		80	2734
18	396		39	1278		60	2004		81	2766

19	492		40	1278		61	2088		82	2766
20	498		41	1320		62	2124			
21	498		42	1332		63	2184			

Given the ESS data, the value of $\hat{\beta}$ is calculated using:

$$\hat{\beta} = \frac{n}{n \ln T^* - \sum^{n}{}_{i=1} \ln T_i}$$

$$\hat{\beta} = 0.8939$$

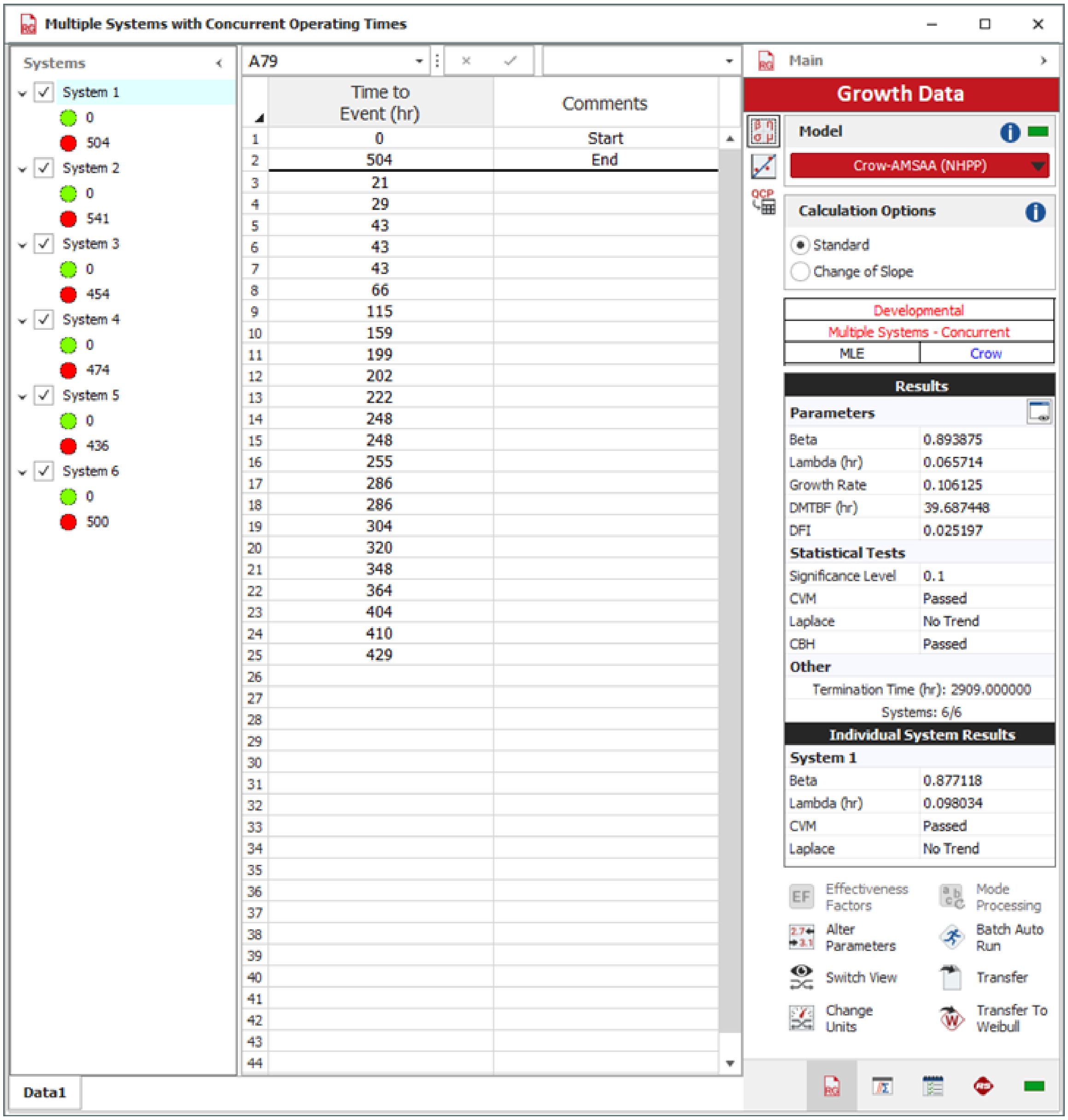
where n is the number of failures and T^* is the termination time. The termination time is the sum of end times for each of the systems, which equals 2,909.

$\hat{\lambda}$ is estimated with:

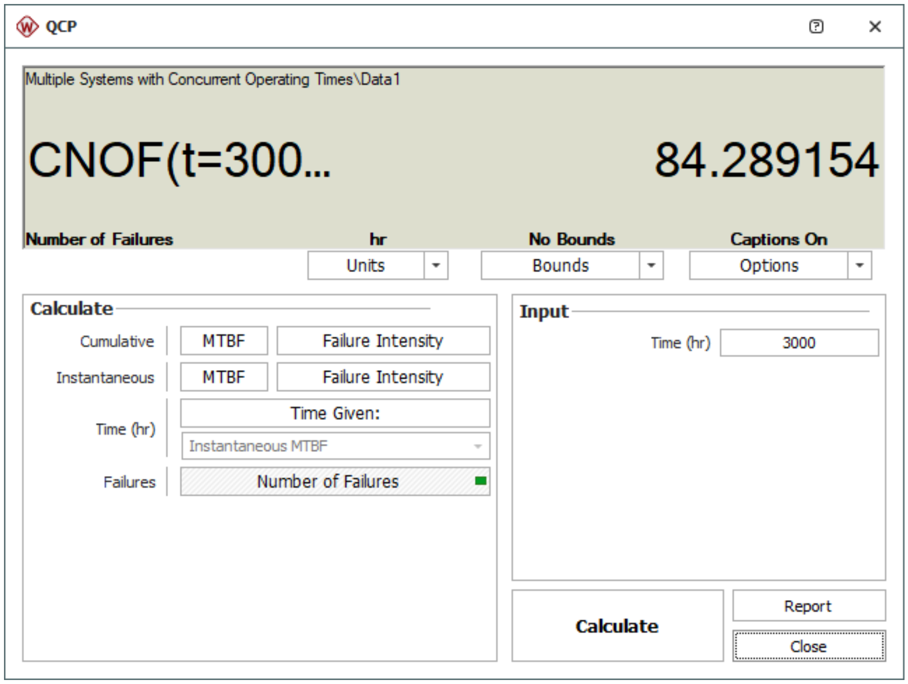
$$\hat{\lambda} = \frac{n}{T^{*\beta}}$$

$$\hat{\lambda} = 0.0657$$

The next figure shows the parameters estimated using Weibull++.



2. The number of failures can be estimated using the Quick Calculation Pad, as shown next. The estimated number of failures at 3,000 hours is equal to 84.2892 and 82 failures were observed during testing. Therefore, the number of additional failures generated if testing continues until 3,000 hours is equal to $84.2892 - 82 = 2.2892 \approx 3$



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Multiple Systems with Dates

An overview of the Multiple Systems with Dates data type is presented on the Reliability Growth Analysis Data Types page. While Multiple Systems with Dates requires a date for each event, including the start and end times for each system, once the equivalent single system is determined, the parameter estimation is the same as it is for Multiple Systems (Concurrent Operating Times). See Parameter Estimation for Multiple Systems (Concurrent Operating Times) for details.

Grouped Data

A description of Grouped Data is presented in the Reliability Growth Analysis Data Types page.

Parameter Estimation for Grouped Data

For analyzing grouped data, we follow the same logic described previously for the Duane model. If the $E[N(T)]$ equation from the Background section above is linearized:

$$\ln[E(N(T))] = \ln\lambda + \beta\ln T$$

According to Crow [9], the likelihood function for the grouped data case, (where $n_1, n_2, n_3, \ldots,$ n_k failures are observed and k is the number of groups), is:

$$\prod_{i=1}^{k} \Pr\ (N_i = n_i) = \prod_{i=1}^{k} \frac{\left(\lambda T_i^{\beta} - \lambda T_{i-1}^{\beta}\right)^{n_i} \cdot e^{-(\lambda T_i^{\beta} - \lambda T_{i-1}^{\beta})}}{n_i!}$$

And the MLE of λ based on this relationship is:

$$\hat{\lambda} = \frac{n}{T_k^{\hat{\beta}}}$$

where n is the total number of failures from all the groups.

The estimate of β is the value $\hat{\beta}$ that satisfies:

$$\sum_{i=1}^{k} n_i \left[\frac{T_i^{\hat{\beta}} \ln T_i - T_{i-1}^{\hat{\beta}} \ln T_{i-1}}{T_i^{\hat{\beta}} - T_{i-1}^{\hat{\beta}}} - \ln T_k \right] = 0$$

See Crow-AMSAA Confidence Bounds for details on how confidence bounds for grouped data are calculated.

Chi-Squared Test

A chi-squared goodness-of-fit test is used to test the null hypothesis that the Crow-AMSAA reliability model adequately represents a set of grouped data. This test is applied only when the data is grouped. The expected number of failures in the interval from T_{i-1} to T_i is approximated by:

$$\hat{\theta}_i = \hat{\lambda} \left(T_i^{\hat{\beta}} - T_{i-1}^{\hat{\beta}} \right)$$

For each interval, $\hat{\theta}_i$ shall not be less than 5 and, if necessary, adjacent intervals may have to be combined so that the expected number of failures in any combined interval is at least 5. Let the number of intervals after this recombination be d, and let the observed number of failures in the i^{th} new interval be N_i. Finally, let the expected number of failures in the i^{th} new interval be $\hat{\theta}_i$. Then the following statistic is approximately distributed as a chi-squared random variable with degrees of freedom $d-2$.

$$\chi^2 = \sum_{i=1}^{d} \frac{\left(N_i - \hat{\theta}_i\right)^2}{\hat{\theta}_i}$$

The null hypothesis is rejected if the χ^2 statistic exceeds the critical value for a chosen significance level. In this case, the hypothesis that the Crow-AMSAA model adequately fits the grouped data shall be rejected. Critical values for this statistic can be found in chi-squared distribution tables.

Grouped Data Examples

EXAMPLE - SIMPLE GROUPED

Consider the grouped failure times data given in the following table. Solve for the Crow-AMSAA parameters using MLE.

Grouped Failure Times Data

Run Number	Cumulative Failures	End Time (hours)	$\ln(T_i)$	$\ln(T_i)^2$	$\ln(\theta_i)$	$\ln(T_i)\cdot\ln(\theta_i)$
1	2	200	5.298	28.072	0.693	3.673
2	3	400	5.991	35.898	1.099	6.582
3	4	600	6.397	40.921	1.386	8.868
4	11	3000	8.006	64.102	2.398	19.198
		Sum =	25.693	168.992	5.576	38.321

Solution

Using Weibull++, the value of $\hat{\beta}$, which must be solved numerically, is 0.6315. Using this value, the estimator of λ is:

$$\begin{aligned}\hat{\lambda} =& \frac{11}{3,000^{0.6315}} \\ =& 0.0701\end{aligned}$$

Therefore, the intensity function becomes:

$$\hat{\rho}(T) = 0.0701 \cdot 0.6315 \cdot T^{-0.3685}$$

EXAMPLE - HELICOPTER SYSTEM

A new helicopter system is under development. System failure data has been collected on five helicopters during the final test phase. The actual failure times cannot be determined since the failures are not discovered until after the helicopters are brought into the maintenance area. However, total flying hours are known when the helicopters are brought in for service, and every 2 weeks each helicopter undergoes a thorough inspection to uncover any failures that may have occurred since the last inspection. Therefore, the cumulative total number of flight hours and the cumulative total number of failures for the 5 helicopters are known for each 2-week period. The total number of flight hours from the test phase is 500, which was accrued over a period of 12 weeks (six 2-week intervals). For each 2-week interval, the total number of flight hours and total number of failures for the 5 helicopters were recorded. The grouped data set is displayed in the following table.

Grouped Data for a New Helicopter System

Interval	Interval Length	Failures in Interval
1	0 - 62	12
2	62 -100	6
3	100 - 187	15
4	187 - 210	3
5	210 - 350	18
6	350 - 500	16

Do the following:

1. Estimate the parameters of the Crow-AMSAA model using maximum likelihood estimation.
2. Calculate the confidence bounds on the cumulative and instantaneous MTBF using the Fisher Matrix and Crow methods.

Solution

1. Using Weibull++, the value of $\hat{\beta}$, must be solved numerically. Once $\hat{\beta}$ has been estimated then the value of $\hat{\lambda}$ can be determined. The parameter values are displayed below:

$$\hat{\beta} = 0.81361$$

$$\hat{\lambda} = 0.44585$$

The grouped Fisher Matrix confidence bounds can be obtained on the parameters $\hat{\beta}$ and $\hat{\lambda}$ at the 90% confidence level by:

$$\begin{aligned}\beta_L &= \hat{\beta} e^{z_\alpha \sqrt{Var(\hat{\beta})}/\hat{\beta}} \\ &= 0.6546 \\ \beta_U &= \hat{\beta} e^{-z_\alpha \sqrt{Var(\hat{\beta})}/\hat{\beta}} \\ &= 1.0112\end{aligned}$$

and:

$$\begin{aligned}\lambda_L =& \hat{\lambda} e^{z_\alpha \sqrt{Var(\hat{\lambda})}/\hat{\lambda}} \\ =& 0.14594 \\ \lambda_U =& \hat{\lambda} e^{-z_\alpha \sqrt{Var(\hat{\lambda})}/\hat{\lambda}} \\ =& 1.36207\end{aligned}$$

Crow confidence bounds can also be obtained on the parameters $\hat{\beta}$ and $\hat{\lambda}$ at the 90% confidence level, as:

$$\begin{aligned}\beta_L =& \hat{\beta}(1-S) \\ =& 0.63552 \\ \beta_U =& \hat{\beta}(1+S) \\ =& 0.99170\end{aligned}$$

and:

$$\begin{aligned}\lambda_L =& \frac{\chi^2_{\frac{\alpha}{2},2N}}{2 \cdot T_k^{\beta}} \\ =& 0.36197 \\ \lambda_U =& \frac{\chi^2_{1-\frac{\alpha}{2},2N+2}}{2 \cdot T_k^{\beta}} \\ =& 0.53697\end{aligned}$$

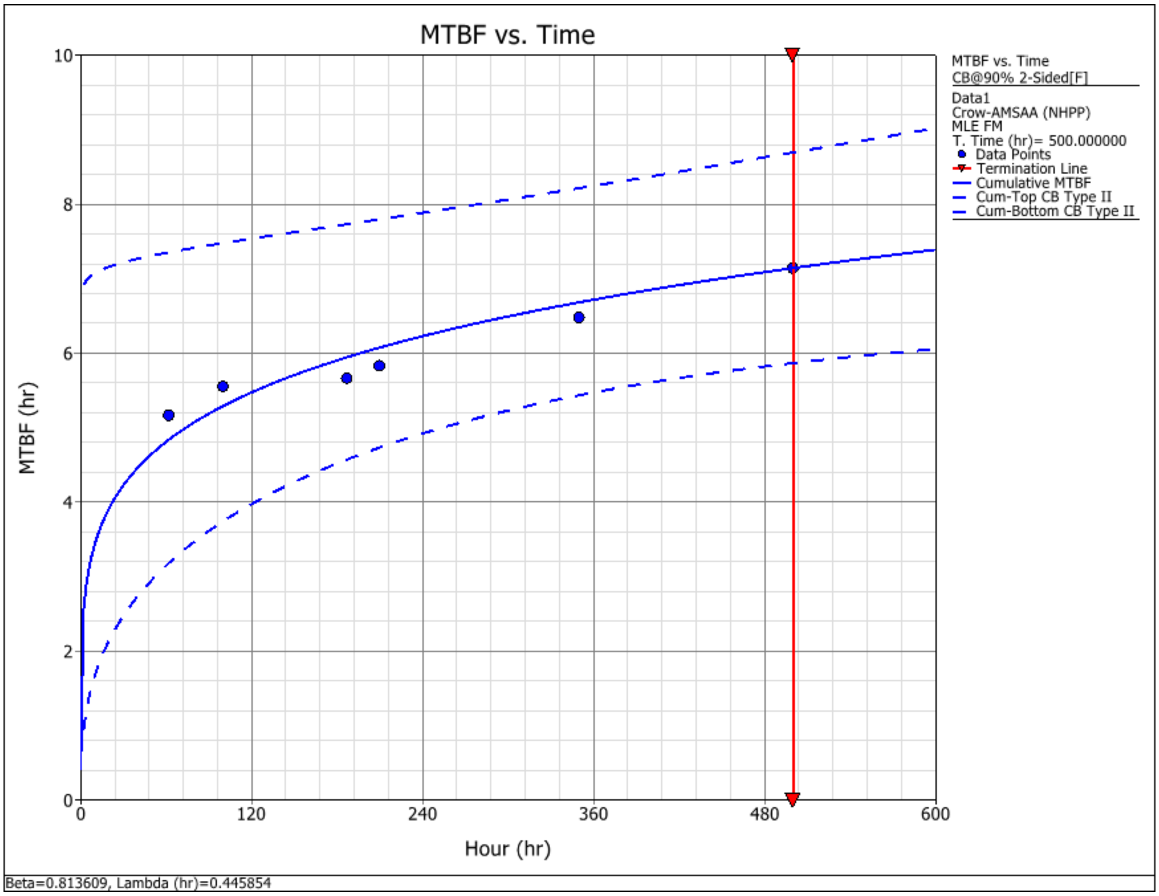
2. The Fisher Matrix confidence bounds for the cumulative MTBF and the instantaneous MTBF at the 90% 2-sided confidence level and for $T = 500$ hour are:

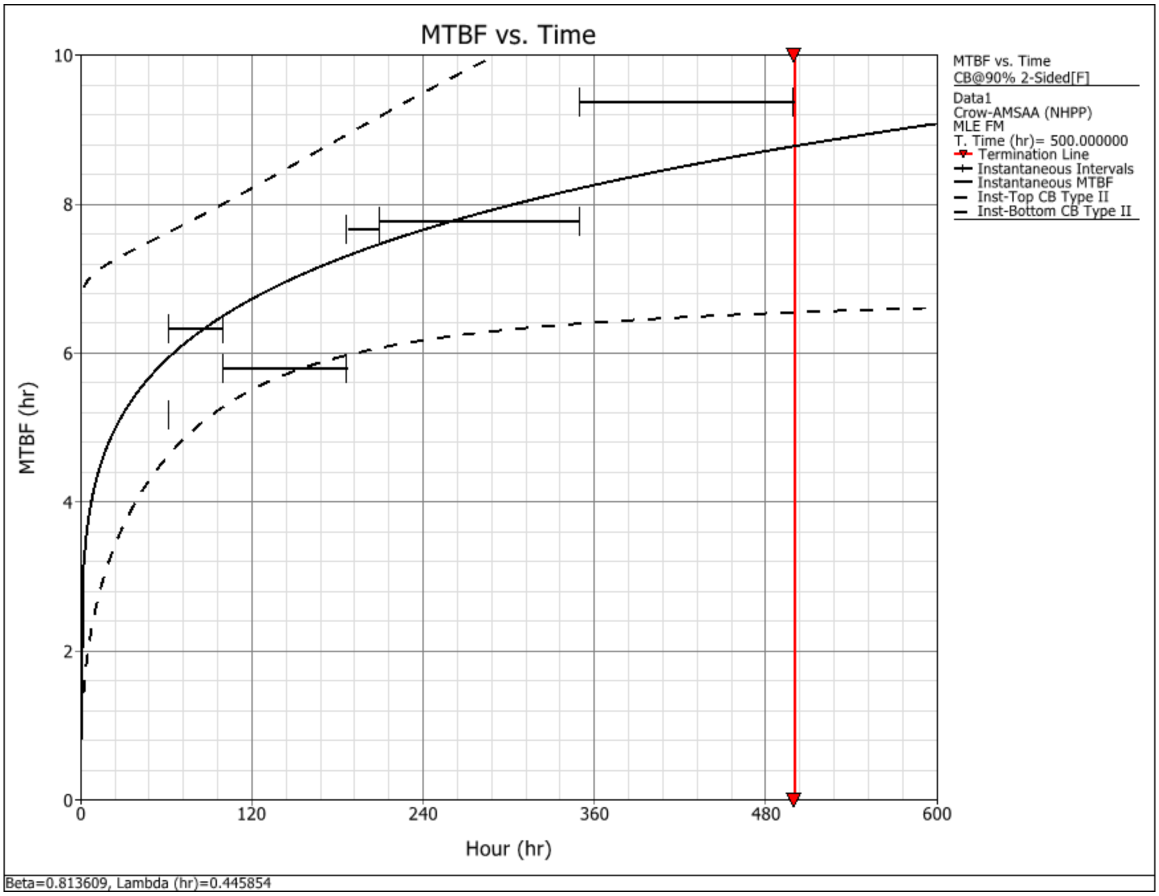
$$\begin{aligned}[m_c(T)]_L =& \hat{m}_c(t) e^{z_{\alpha/2} \sqrt{Var(\hat{m}_c(t))}/\hat{m}_c(t)} \\ =& 5.8680 \\ [m_c(T)]_U =& \hat{m}_c(t) e^{-z_{\alpha/2} \sqrt{Var(\hat{m}_c(t))}/\hat{m}_c(t)} \\ =& 8.6947\end{aligned}$$

and:

$$\begin{aligned}[MTBF_i]_L =& \hat{m}_i(t) e^{z_{\alpha/2} \sqrt{Var(\hat{m}_i(t))}/\hat{m}_i(t)} \\ =& 6.6483 \\ [MTBF_i]_U =& \hat{m}_i(t) e^{-z_{\alpha/2} \sqrt{Var(\hat{m}_i(t))}/\hat{m}_i(t)} \\ =& 11.5932\end{aligned}$$

The next two figures show plots of the Fisher Matrix confidence bounds for the cumulative and instantaneous MTBF.


MTBF vs. Time
MTBF (hr)
Hour (hr)
MTBF vs. Time
CB@90% 2-Sided[F]
Data1
Crow-AMSAA (NHPP)
MLE FM
T. Time (hr)= 500.000000
Data Points
Termination Line
Cumulative MTBF
Cum-Top CB Type II
Cum-Bottom CB Type II
Beta=0.813609, Lambda (hr)=0.445854



The Crow confidence bounds for the cumulative and instantaneous MTBF at the 90% 2-sided confidence level and for $T = 500$ hours are:

$$[m_c(T)]_L = \frac{1}{C(t)_U} = 5.85449$$

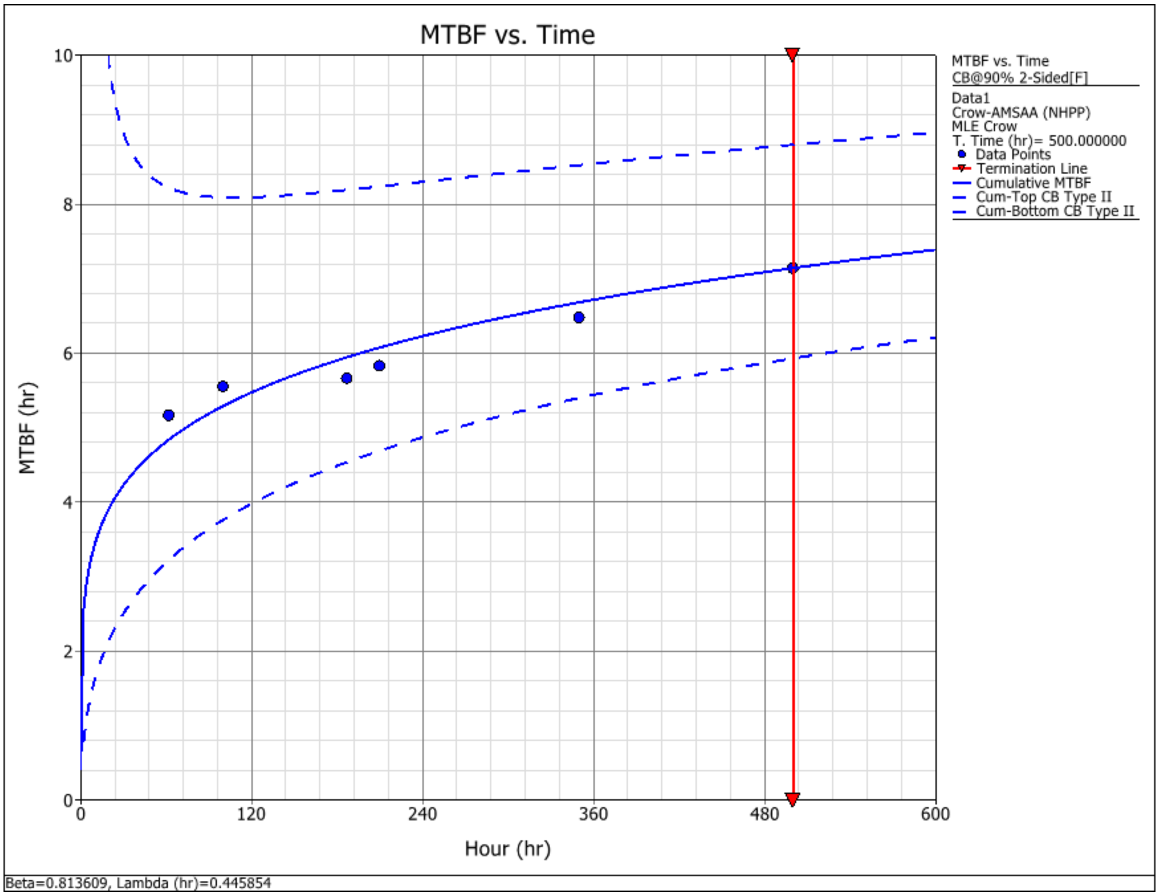
$$[m_c(T)]_U = \frac{1}{C(t)_L} = 8.79822$$

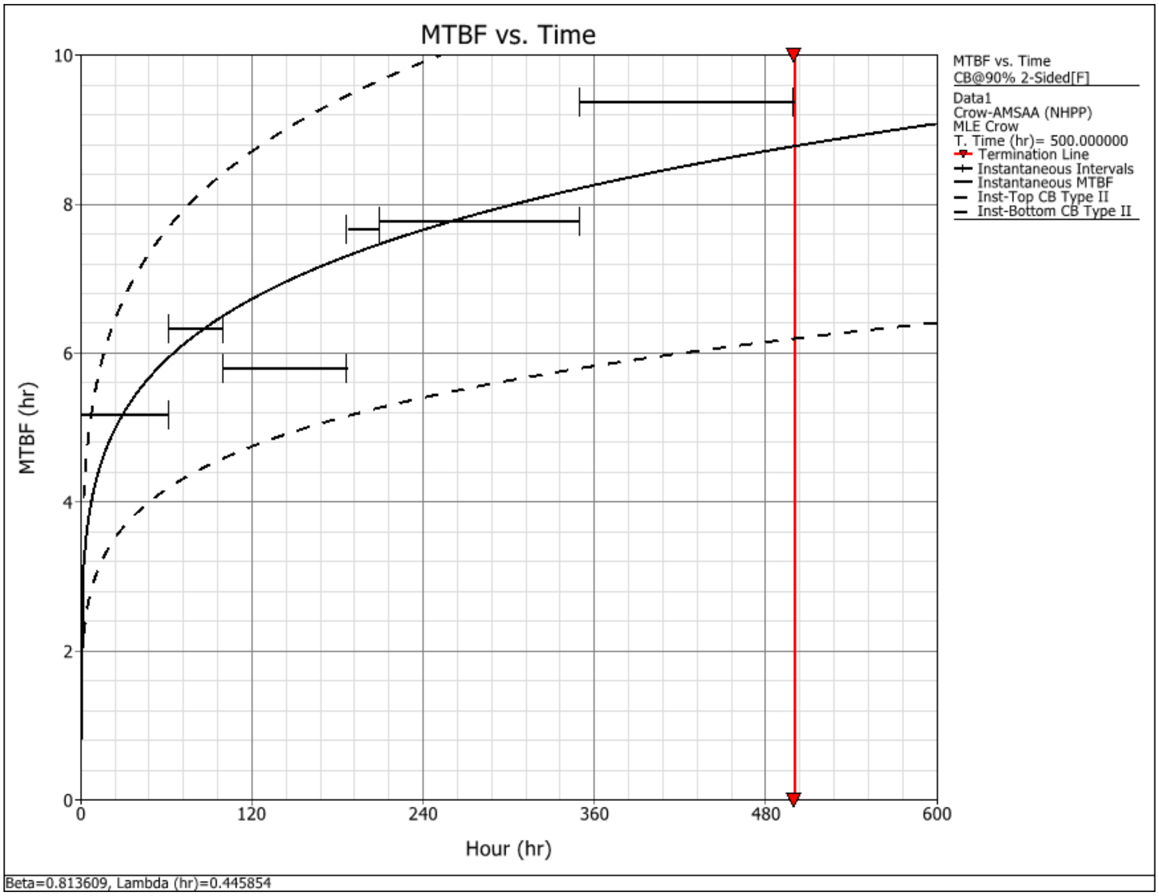
and:

$$[MTBF_i]_L = \hat{m}_i(1 - W) = 6.19623$$

$$[MTBF_i]_U = \hat{m}_i(1 + W) = 11.36223$$

The next two figures show plots of the Crow confidence bounds for the cumulative and instantaneous MTBF.


MTBF vs. Time
MTBF vs. Time
CB@90% 2-Sided[F]
Data1
Crow-AMSAA (NHPP)
MLE Crow
T. Time (hr)= 500.000000
Data Points
Termination Line
Cumulative MTBF
Cum-Top CB Type II
Cum-Bottom CB Type II
MTBF (hr)
Hour (hr)
0
2
4
6
8
10
0
120
240
360
480
600
Beta=0.813609, Lambda (hr)=0.445854



Missing Data

Most of the reliability growth models used for estimating and tracking reliability growth based on test data assume that the data set represents all actual system failure times consistent with a uniform definition of failure (complete data). In practice, this may not always be the case and may result in too few or too many failures being reported over some interval of test time. This may result in distorted estimates of the growth rate and current system reliability. This section discusses a practical reliability growth estimation and analysis procedure based on the assumption that anomalies may exist within the data over some interval of the test period but the remaining failure data follows the Crow-AMSAA reliability growth model. In particular, it is assumed that the beginning and ending points in which the anomalies lie are generated independently of the underlying reliability growth process. The approach for estimating the parameters of the growth model with problem data over some interval of time is basically to not use this failure information. The analysis retains the contribution of the interval to the total test time, but no assumptions are made regarding the actual number of failures over the interval. This is often referred to as *gap analysis*.

Consider the case where a system is tested for time T and the actual failure times are recorded. The time T may possibly be an observed failure time. Also, the end points of the gap interval may or may not correspond to a recorded failure time. The underlying assumption is that the data used in the maximum likelihood estimation follows the Crow-AMSAA model with a Weibull intensity function $\lambda\beta t^{\beta-1}$. It is not assumed that zero failures occurred during the gap interval, rather, it is assumed that the actual number of failures is unknown, and hence no information at all regarding these failure is used to estimate λ and β.

Let S_1 , S_2 denote the end points of the gap interval, $S_1 < S_2$. Let $0 < X_1 < X_2 < \ldots < X_{N_1} \leq S_1$ be the failure times over $(0,\, S_1)$ and let $S_2 < Y_1 < Y_2 < \ldots < Y_{N_1} \leq T$ be the failure times over (S_2, T). The maximum likelihood estimates of λ and β are values $\widehat{\lambda}$ and $\widehat{\beta}$ satisfying the following equations.

$$\widehat{\lambda} = \frac{N_1 + N_2}{S_1^{\widehat{\beta}} + T^{\widehat{\beta}} - S_2^{\widehat{\beta}}}$$

$$\widehat{\beta} = \frac{N_1 + N_2}{\widehat{\lambda}\left[S_1^{\widehat{\beta}} \ln S_1 + T^{\widehat{\beta}} \ln T - S_2^{\widehat{\beta}} \ln S_2\right] - \left[\sum^{N_1}{}_{i=1} \ln X_i + \sum^{N_2}{}_{i=1} \ln Y_i\right]}$$

In general, these equations cannot be solved explicitly for $\widehat{\lambda}$ and $\widehat{\beta}$, but must be solved by an iterative procedure.

Example - Gap Analysis

Consider a system under development that was subjected to a reliability growth test for $T = 1{,}000$ hours. Each month, the successive failure times, on a cumulative test time basis, were reported. According to the test plan, 125 hours of test time were accumulated on each prototype system each month. The total reliability growth test program lasted for 7 months. One prototype was tested for each of the months 1, 3, 4, 5, 6 and 7 with 125 hours of test time. During the second month, two prototypes were tested for a total of 250 hours of test time. The next table shows the successive $N = 86$ failure times that were reported for $T = 1{,}000$ hours of testing.

$X_i, i = 1, 2, \ldots, 86$, $N = 86, T = 1000$

.5	.6	10.7	16.6	18.3	19.2	19.5	25.3
39.2	39.4	43.2	44.8	47.4	65.7	88.1	97.2
104.9	105.1	120.8	195.7	217.1	219	257.5	260.4
281.3	283.7	289.8	306.6	328.6	357.0	371.7	374.7

393.2	403.2	466.5	500.9	501.5	518.4	520.7	522.7
524.6	526.9	527.8	533.6	536.5	542.6	543.2	545.0
547.4	554.0	554.1	554.2	554.8	556.5	570.6	571.4
574.9	576.8	578.8	583.4	584.9	590.6	596.1	599.1
600.1	602.5	613.9	616.0	616.2	617.1	621.4	622.6
624.7	628.8	642.4	684.8	731.9	735.1	753.6	792.5
803.7	805.4	832.5	836.2	873.2	975.1		

The observed and cumulative number of failures for each month are:

Month	Time Period	Observed Failure Times	Cumulative Failure Times
1	0-125	19	19
2	125-375	13	32
3	375-500	3	35
4	500-625	38	73
5	625-750	5	78
6	750-875	7	85
7	875-1000	1	86

1. Determine the maximum likelihood estimators for the Crow-AMSAA model.
2. Evaluate the goodness-of-fit for the model.
3. Consider $(500, 625)$ as the gap interval and determine the maximum likelihood estimates of λ and β.

Solution

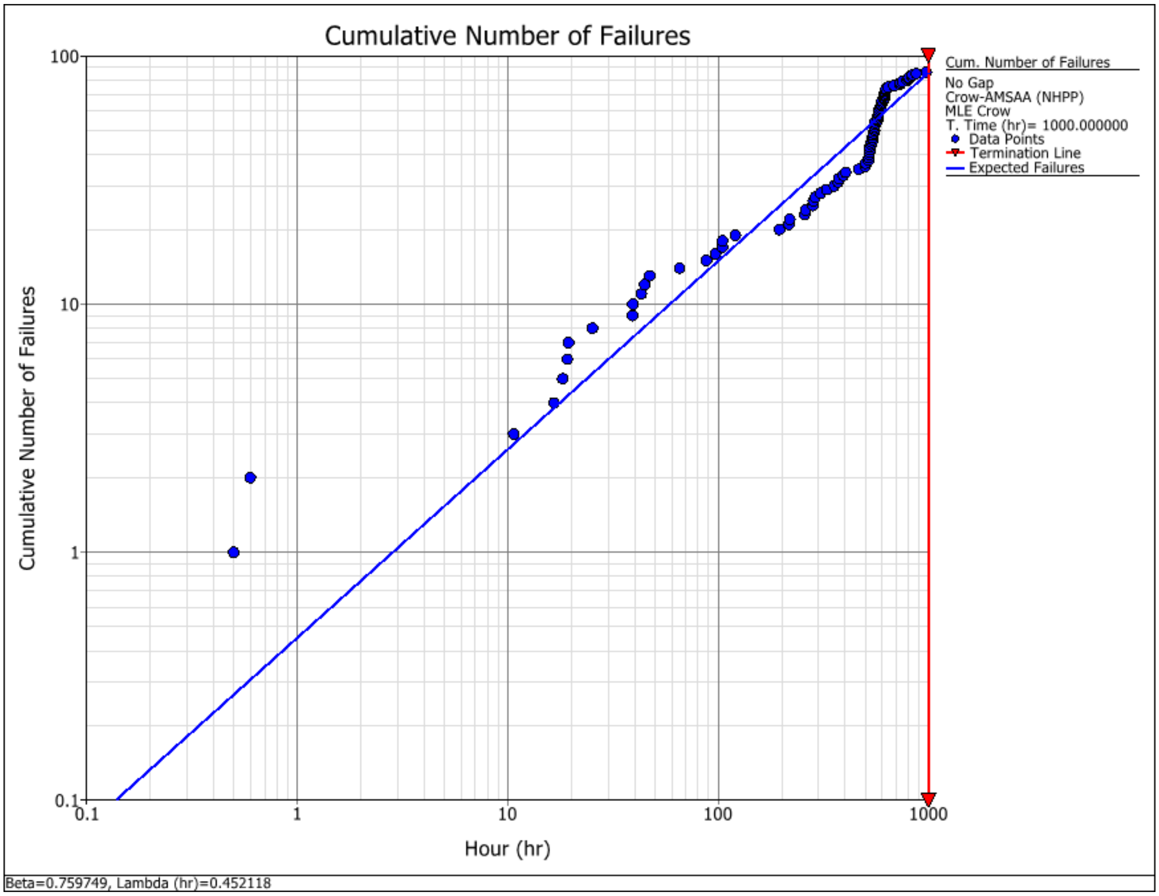
1. For the time terminated test:

$$\widehat{\beta} = 0.7597$$
$$\widehat{\lambda} = 0.4521$$

2. The Cramer-von Mises goodness-of-fit test for this data set yields:

$$C_M^2 = \frac{1}{12M} + \sum_{i=1}^{M} \left[\left(\frac{T_i}{T}\right)^{\hat{\beta}} - \frac{2i-1}{2M} \right]^2 = 0.6989$$

The critical value at the 10% significance level is 0.173. Therefore, the test indicated that the analyst should reject the hypothesis that the data set follows the Crow-AMSAA reliability growth model. The following plot shows $\ln N(t)$ versus $\ln t$ with the fitted line $\ln \hat{\lambda} + \hat{\beta} \ln t$, where $\hat{\lambda} = 0.4521$ and $\hat{\beta} = 0.7597$ are the maximum likelihood estimates.

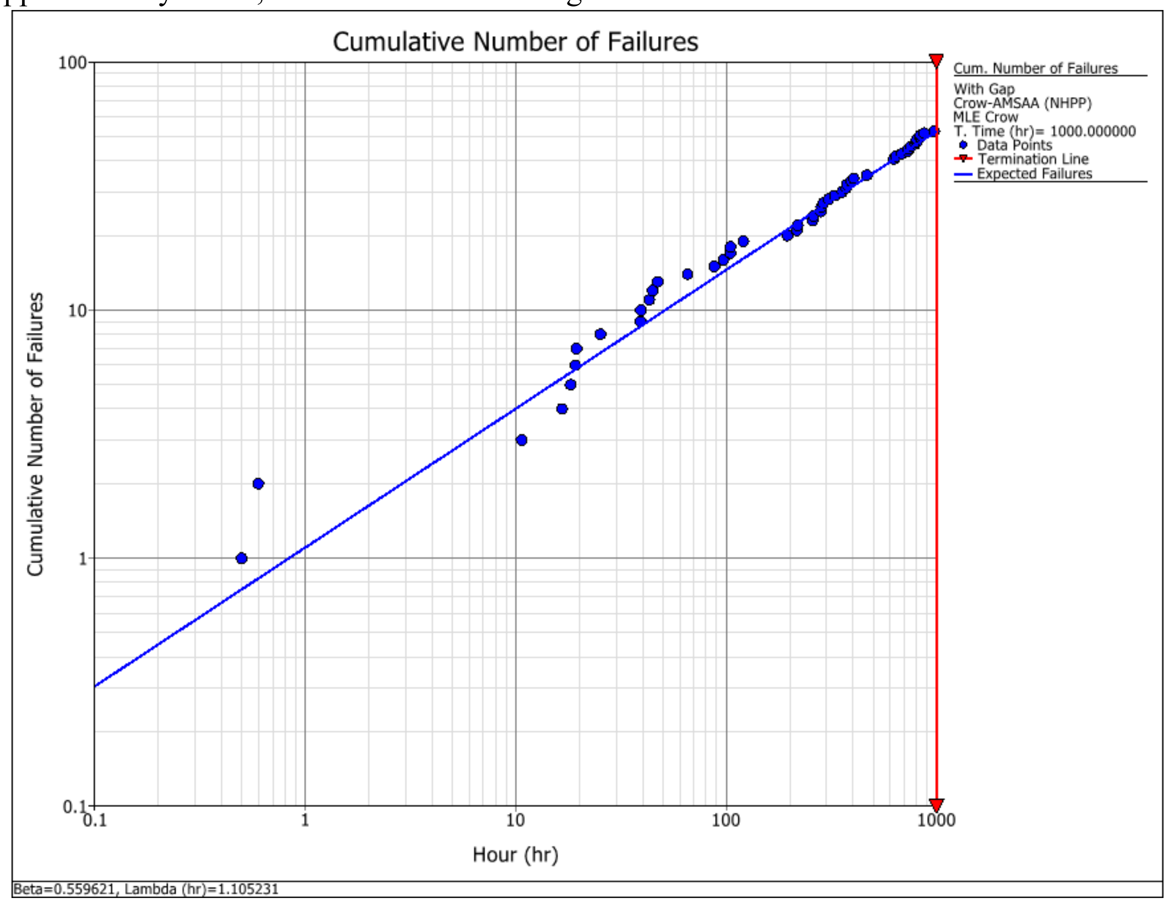


Observing the data during the fourth month (between 500 and 625 hours), 38 failures were reported. This number is very high in comparison to the failures reported in the other months. A quick investigation found that a number of new data collectors were assigned to the project during this month. It was also discovered that extensive design changes were made during this period, which involved the removal of a large number of parts. It is possible that these removals, which were not failures, were incorrectly reported as failed parts. Based on knowledge of the system and the test program, it was clear that such a large number of actual system failures was extremely unlikely. The consensus was that this anomaly was due to the failure reporting. For this analysis, it was decided that the actual number of failures over this month is assumed to be unknown, but consistent with the remaining data and the Crow-AMSAA reliability growth model.

3. Considering the problem interval $(500, 625)$ as the gap interval, we will use the data over the interval $(0, 500)$ and over the interval $(625, 1000)$. The equations for analyzing missing data are the appropriate equations to estimate λ and β because the failure times are known. In this case $S_1 = 500,\ S_2 = 625$ and $T = 1000,\ N_1 = 35,\ N_2 = 13$. The maximum likelihood estimates of λ and β are:

$$\widehat{\beta} = 0.5596$$
$$\widehat{\lambda} = 1.1052$$

The next figure is a plot of the cumulative number of failures versus time. This plot is approximately linear, which also indicates a good fit of the model.



Discrete Data

The Crow-AMSAA model can be adapted for the analysis of *success/failure* data (also called *discrete* or *attribute* data). The following discrete data types are available:

- Sequential
- Grouped per Configuration
- Mixed

Sequential data and Grouped per Configuration are very similar as the parameter estimation methodology is the same for both data types. Mixed data is a combination of Sequential Data and Grouped per Configuration and is presented in Mixed Data.

Grouped per Configuration

Suppose system development is represented by i configurations. This corresponds to $i-1$ configuration changes, unless fixes are applied at the end of the test phase, in which case there would be i configuration changes. Let N_i be the number of trials during configuration i and let M_i be the number of failures during configuration i. Then the cumulative number of trials through configuration i, namely T_i, is the sum of the N_i for all i, or:

$$T_i = \sum N_i$$

And the cumulative number of failures through configuration i, namely K_i, is the sum of the M_i for all i, or:

$$K_i = \sum M_i$$

The expected value of K_i can be expressed as $E[K_i]$ and defined as the expected number of failures by the end of configuration i. Applying the learning curve property to $E[K_i]$ implies:

$$E\left[K_i\right] = \lambda T_i^{\beta}$$

Denote f_1 as the probability of failure for configuration 1 and use it to develop a generalized equation for f_i in terms of the T_i and N_i. From the equation above, the expected number of failures by the end of configuration 1 is:

$$E\left[K_1\right] = \lambda T_1^{\beta} = f_1 N_1$$

$$\therefore f_1 = \frac{\lambda T_1^{\beta}}{N_1}$$

Applying the $E\left[K_i\right]$ equation again and noting that the expected number of failures by the end of configuration 2 is the sum of the expected number of failures in configuration 1 and the expected number of failures in configuration 2:

$$\begin{aligned}E[K_2] &= \lambda T_2^{\beta} \\ &= f_1 N_1 + f_2 N_2 \\ &= \lambda T_1^{\beta} + f_2 N_2\end{aligned}$$

$$\therefore f_2 = \frac{\lambda T_2^{\beta} - \lambda T_1^{\beta}}{N_2}$$

By this method of inductive reasoning, a generalized equation for the failure probability on a configuration basis, f_i , is obtained, such that:

$$f_i = \frac{\lambda T_i^{\beta} - \lambda T_{i-1}^{\beta}}{N_i}$$

In this equation, i represents the trial number. Thus, an equation for the reliability (probability of success) for the i^{th} configuration is obtained:

$$R_i = 1 - f_i$$

Sequential Data

From the Grouped per Configuration section, the following equation is given:

$$f_i = \frac{\lambda T_i^{\beta} - \lambda T_{i-1}^{\beta}}{N_i}$$

For the special case where $N_i = 1$ for all i, the equation above becomes a smooth curve, g_i , that represents the probability of failure for trial by trial data, or:

$$g_i = \lambda \cdot i^{\beta} - \lambda \cdot (i-1)^{\beta}$$

When $N_i = 1$, this is the same as Sequential Data where systems are tested on a trial-by-trial basis. The equation for the reliability for the i^{th} trial is:

$$R_i = 1 - g_i$$

Parameter Estimation for Discrete Data

This section describes procedures for estimating the parameters of the Crow-AMSAA model for success/failure data which includes Sequential data and Grouped per Configuration. An example is presented illustrating these concepts. The estimation procedures provide maximum likelihood estimates (MLEs) for the model's two parameters, λ and β. The MLEs for λ and β allow for point estimates for the probability of failure, given by:

$$\hat{f}_i = \frac{\hat{\lambda} T_i^{\hat{\beta}} - \hat{\lambda} T_{i-1}^{\hat{\beta}}}{N_i} = \frac{\hat{\lambda}\left(T_i^{\hat{\beta}} - T_{i-1}^{\hat{\beta}}\right)}{N_i}$$

And the probability of success (reliability) for each configuration i is equal to:

$$\hat{R}_i = 1 - \hat{f}_i$$

The likelihood function is:

$$\prod_{i=1}^{k} \binom{N_i}{M_i} \left(\frac{\lambda T_i^{\beta} - \lambda T_{i-1}^{\beta}}{N_i}\right)^{M_i} \left(\frac{N_i - \lambda T_i^{\beta} + \lambda T_{i-1}^{\beta}}{N_i}\right)^{N_i - M_i}$$

Taking the natural log on both sides yields:

$$\begin{aligned}\Lambda = \quad & \sum_{i=1}^{K} \left[\ln\binom{N_i}{M_i} + M_i \left[\ln(\lambda T_i^{\beta} - \lambda T_{i-1}^{\beta}) - \ln N_i\right]\right] \\ & + \sum_{i=1}^{K} \left[(N_i - M_i)\left[\ln(N_i - \lambda T_i^{\beta} + \lambda T_{i-1}^{\beta}) - \ln N_i\right]\right]\end{aligned}$$

Taking the derivative with respect to λ and β respectively, exact MLEs for λ and β are values satisfying the following two equations:

$$\sum_{i=1}^{K} H_i \times S_i = 0$$
$$\sum_{i=1}^{K} U_i \times S_i = 0$$

where:

$$H_i = \left[T_i^{\beta} \ln T_i - T_{i-1}^{\beta} \ln T_{i-1}\right]$$
$$S_i = \frac{M_i}{\left[\lambda T_i^{\beta} - \lambda T_{i-1}^{\beta}\right]} - \frac{N_i - M_i}{\left[N_i - \lambda T_i^{\beta} + \lambda T_{i-1}^{\beta}\right]}$$
$$U_i = T_i^{\beta} - T_{i-1}^{\beta}$$

Example - Grouped per Configuration

A one-shot system underwent reliability growth development testing for a total of 68 trials. Delayed corrective actions were incorporated after the 14th, 33rd and 48th trials. From trial 49 to trial 68, the configuration was not changed.

- Configuration 1 experienced 5 failures,
- Configuration 2 experienced 3 failures,
- Configuration 3 experienced 4 failures and
- Configuration 4 experienced 4 failures.

Do the following:

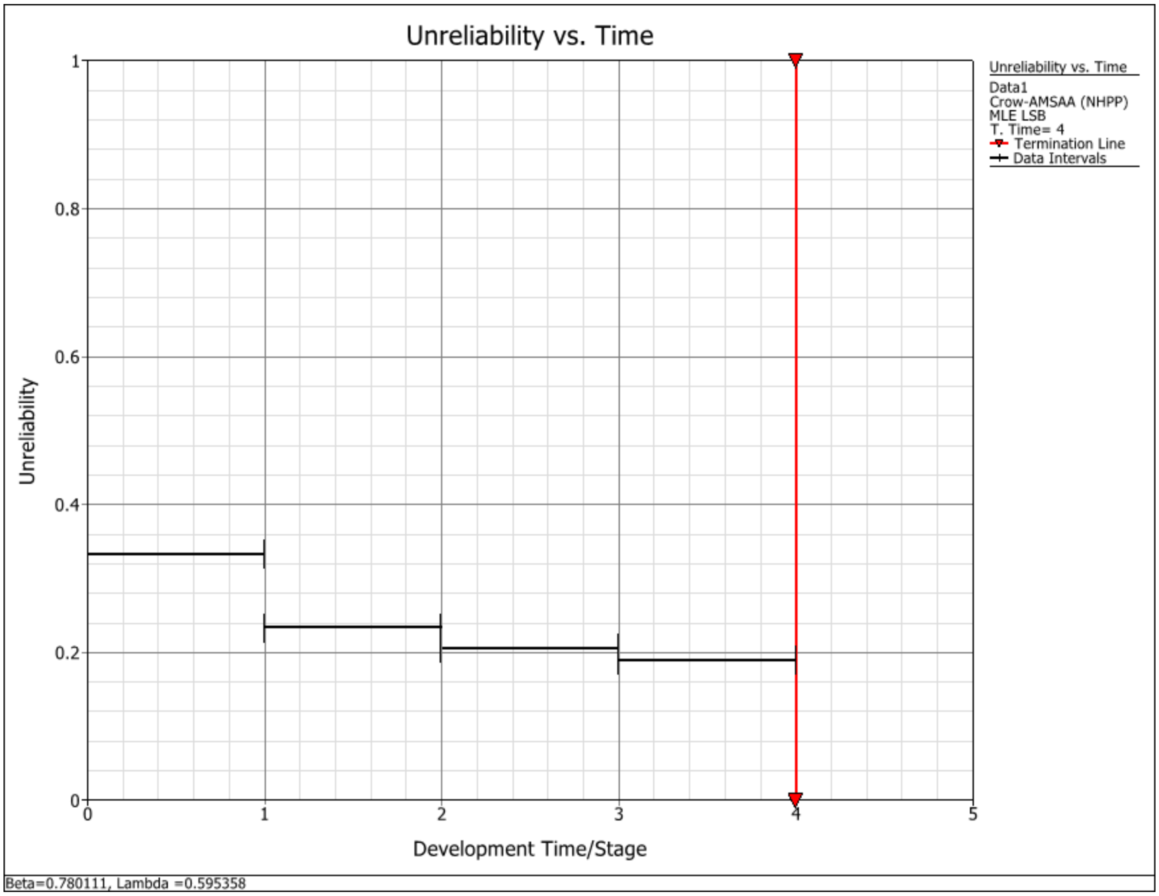
1. Estimate the parameters of the Crow-AMSAA model using maximum likelihood estimation.
2. Estimate the unreliability and reliability by configuration.

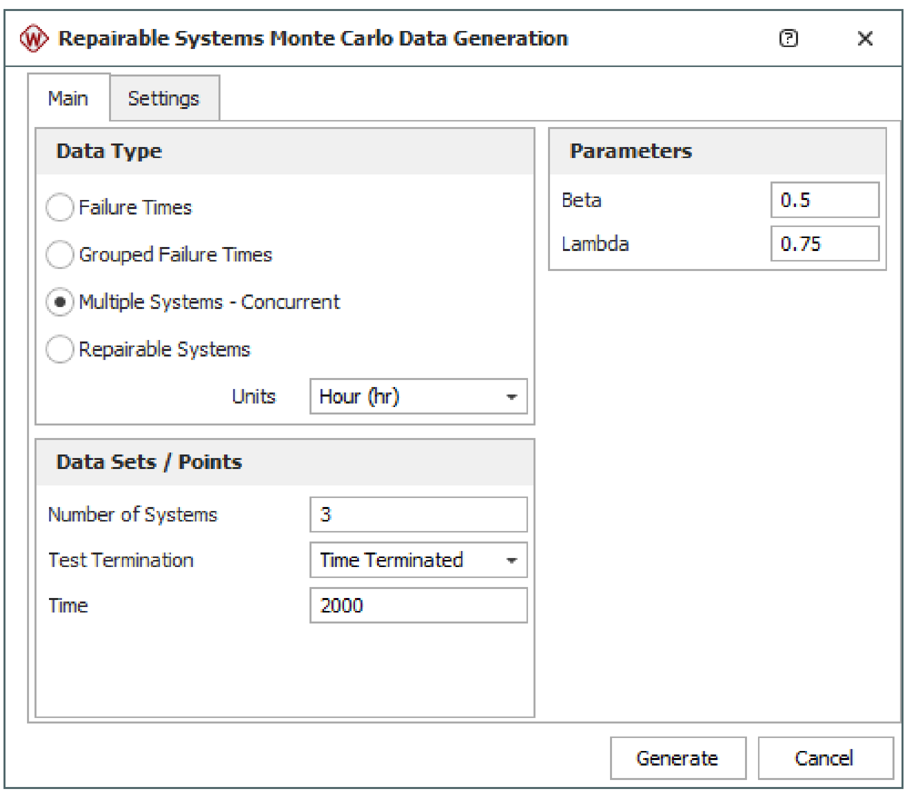
Solution

1. The parameter estimates for the Crow-AMSAA model using the parameter estimation for discrete data methodology yields $\lambda = 0.5954$ and $\beta = 0.7801$.
2. The following table displays the results for probability of failure and reliability, and these results are displayed in the next two plots.

Estimated Failure Probability and Reliability by Configuration

Configuration($_i$)	Estimated Failure Probability	Estimated Reliability
1	0.333	0.667
2	0.234	0.766
3	0.206	0.794
4	0.190	0.810


Unreliability vs. Time
Unreliability
Development Time/Stage
Unreliability vs. Time
Data1
Crow-AMSAA (NHPP)
MLE LSB
T. Time= 4
Termination Line
Data Intervals
Beta=0.780111, Lambda =0.595358


Repairable Systems Monte Carlo Data Generation
Main
Settings
Data Type
Failure Times
Grouped Failure Times
Multiple Systems - Concurrent
Repairable Systems
Units
Hour (hr)
Data Sets / Points
Number of Systems
3
Test Termination
Time Terminated
Time
2000
Parameters
Beta
0.5
Lambda
0.75
Generate
Cancel

Mixed Data

The Mixed data type provides additional flexibility in terms of how it can handle different testing strategies. Systems can be tested using different configurations in groups or individual trial by trial, or a mixed combination of individual trials and configurations of more than one trial. The Mixed data type allows you to enter the data so that it represents how the systems were tested within the total number of trials. For example, if you launched five (5) missiles for a given configuration and none of them failed during testing, then there would be a row within the data sheet indicating that this configuration operated successfully for these five trials. If the very next trial, the sixth, failed then this would be a separate row within the data. The flexibility with the data entry allows for a greater understanding in terms of how the systems were tested by simply examining the data. The methodology for estimating the parameters $\hat{\beta}$ and $\hat{\lambda}$ are the same as those presented in the Grouped Data section. With Mixed data, the average reliability and average unreliability within a given interval can also be calculated.

The average unreliability is calculated as:

$$\text{Average Unreliability } (t_1, t_2) = \frac{\lambda t_2^{\beta} - \lambda t_1^{\beta}}{t_2 - t_1}$$

and the average reliability is calculated as:

$$\text{Average Reliability } (t_1, t_2) = 1 - \frac{\lambda t_2^{\beta} - \lambda t_1^{\beta}}{t_2 - t_1}$$

MIXED DATA CONFIDENCE BOUNDS

Bounds on Average Failure Probability

The process to calculate the average unreliability confidence bounds for Mixed data is as follows:

1. Calculate the average failure probability (t_1, t_2) .
2. There will exist a t^* between t_1 and t_2 such that the instantaneous unreliability at t^* equals the average unreliability (t_1, t_2) . The confidence intervals for the instantaneous unreliability at t^* are the confidence intervals for the average unreliability (t_1, t_2) .

Bounds on Average Reliability

The process to calculate the average reliability confidence bounds for Mixed data is as follows:

1. Calculate confidence bounds for average unreliability (t_1, t_2) as described above.
2. The confidence bounds for reliability are 1 minus these confidence bounds for average unreliability.

EXAMPLE - MIXED DATA

The table below shows the number of failures of each interval of trials and the cumulative number of trials in each interval for a reliability growth test. For example, the first row indicates that for an interval of 14 trials, 5 failures occurred.

Mixed Data

Failures in Interval	Cumulative Trials
5	14
3	33
4	48
0	52
1	53
0	57
1	58
0	62
1	63
0	67
1	68

Using the Weibull++ software, the parameters of the Crow-AMSAA model are estimated as follows:

$$\hat{\beta} = 0.7950$$

and:

$$\hat{\lambda} = 0.5588$$

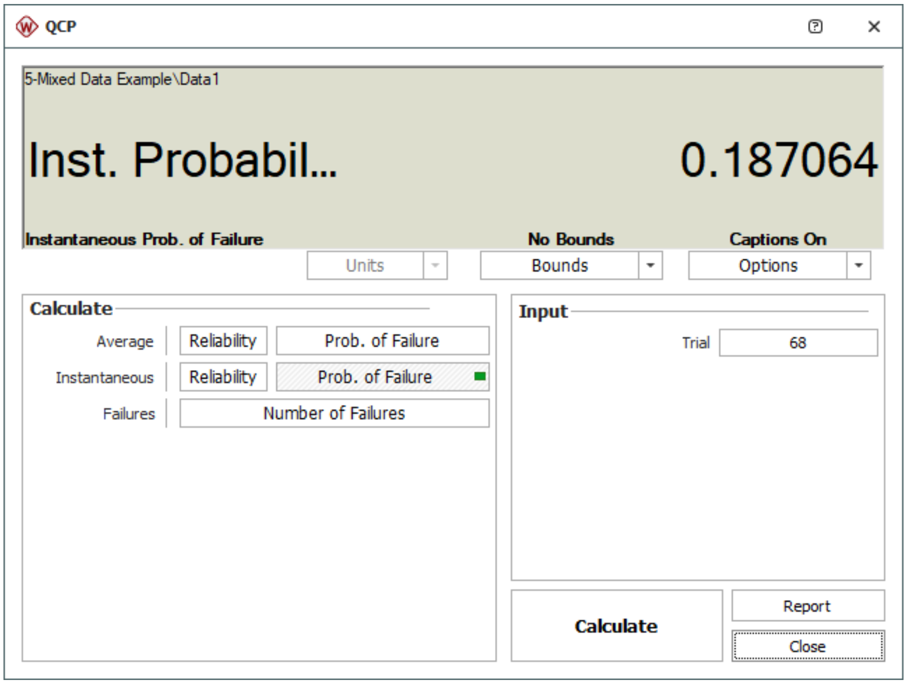
As we have seen, the Crow-AMSAA instantaneous failure intensity, $\lambda_i(T)$, is defined as:

$$\lambda_i(T) = \lambda\beta T^{\beta-1}, \text{with } T > 0,\ \lambda > 0 \text{ and } \beta > 0$$

Using the parameter estimates, we can calculate the instantaneous unreliability at the end of the test, or $T = 68$.

$$R_i(68) = 0.5588 \cdot 0.7950 \cdot 68^{0.7950-1} = 0.1871$$

This result that can be obtained from the Quick Calculation Pad (QCP), for $T = 68$, as seen in the following picture.



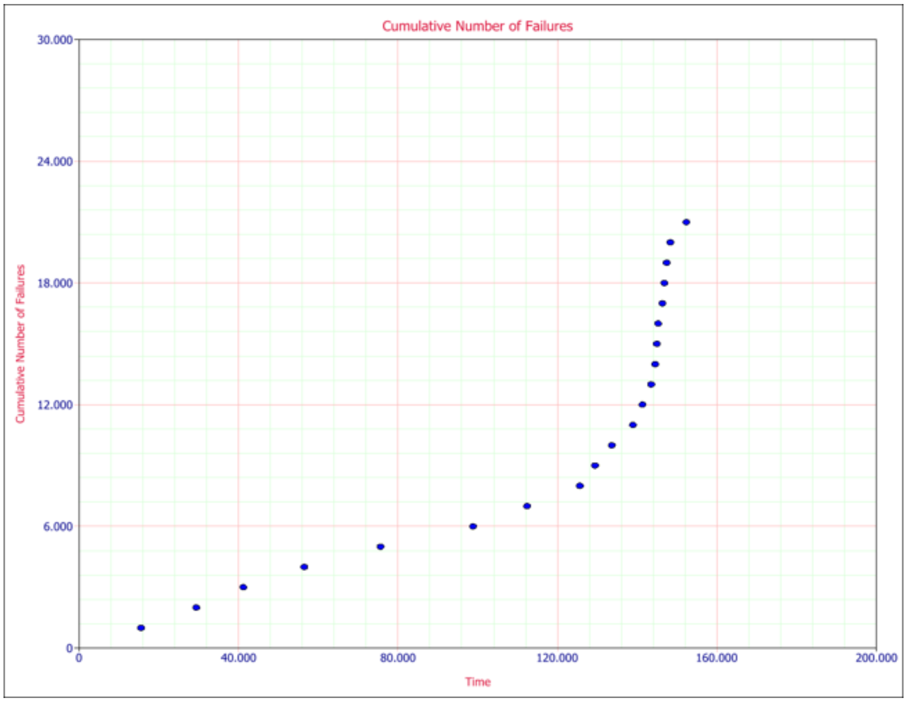
The instantaneous reliability can then be calculated as:

$$R_{inst} = 1 - 0.1871 = 0.8129$$

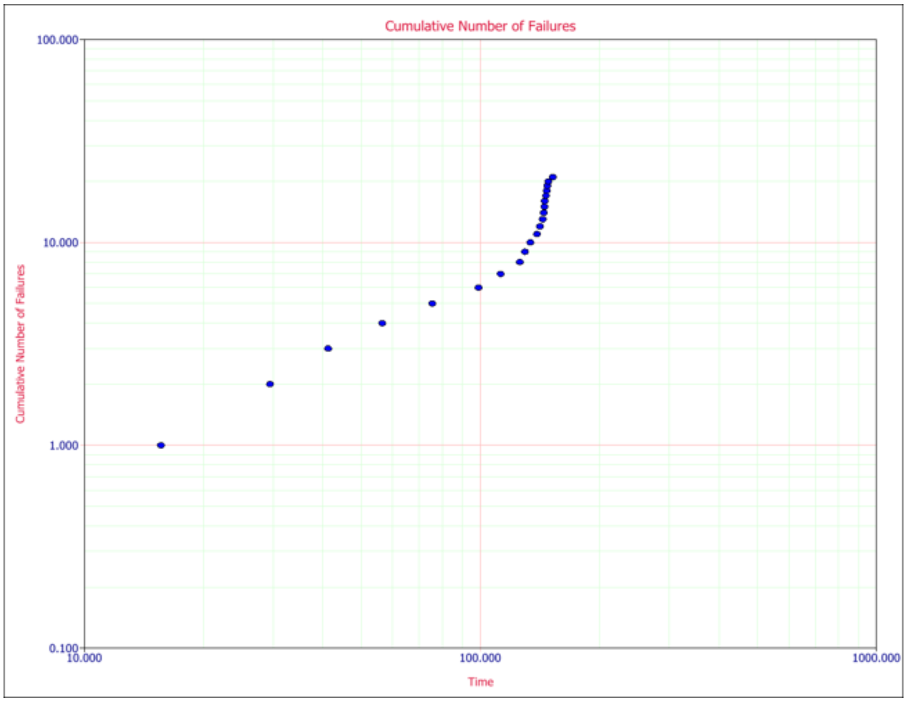
Change of Slope

The assumption of the Crow-AMSAA (NHPP) model is that the failure intensity is monotonically increasing, decreasing or remaining constant over time. However, there might be cases in which the system design or the operational environment experiences major changes during the observation period and, therefore, a single model will not be appropriate to describe the failure behavior for the entire timeline. Weibull++ incorporates a methodology that can be applied to scenarios where a major change occurs during a reliability growth test. The test data can be broken into two segments with a separate Crow-AMSAA (NHPP) model applied to each segment.

Consider the data in the following plot from a reliability growth test.



As discussed above, the cumulative number of failures vs. the cumulative time should be linear on logarithmic scales. The next figure shows the data plotted on logarithmic scales.



One can easily recognize that the failure behavior is not constant throughout the duration of the test. Just by observing the data, it can be asserted that a major change occurred at around 140 hours that resulted in a change in the rate of failures. Therefore, using a single model to analyze this data set likely will not be appropriate.

The Change of Slope methodology proposes to split the data into two segments and apply a Crow-AMSAA (NHPP) model to each segment. The time of change that will be used to split the data into the two segments (it will be referred to as T_1) could be estimated just by observing the data, but will most likely be dictated by engineering knowledge of the specific change to the system design or operating conditions. It is important to note that although two separate models will be applied to each segment, the information collected in the first segment (i.e., data up to T_1) will be considered when creating the model for the second segment (i.e., data after T_1). The models presented next can be applied to the reliability growth analysis of a single system or multiple systems.

Model for First Segment (Data up to *T1*)

The data up to the point of the change that occurs at T_1 will be analyzed using the Crow-AMSAA (NHPP) model. Based on the ML equations for λ and β (in the section Maximum Likelihood Estimators), the ML estimators of the model are:

$$\widehat{\lambda_1} = \frac{n_1}{T_1^{\beta_1}}$$

and

$$\widehat{\beta}_1 = \frac{n_1}{n_1 \ln T_1 - \sum^{n_1}{}_{i=1} \ln t_i}$$

where:

- T_1 is the time when the change occurs
- n_1 is the number of failures observed up to time T_1
- t_i is the time at which each corresponding failure was observed

The equation for $\widehat{\beta_1}$ can be rewritten as follows:

$$\begin{aligned}\widehat{\beta}_1 &= \frac{n_1}{n_1 \ln T_1 - (\ln t_1 + \ln t_2 + \ldots + \ln t_{n_1})} \\ &= \frac{n_1}{(\ln T_1 - \ln t_1) + (\ln T_1 - \ln t_2) + (\ldots) + (\ln T_1 - \ln t_{n_1})} \\ &= \frac{n_1}{\ln \frac{T_1}{t_1} + \ln \frac{T_1}{t_2} + \ldots + \ln \frac{T_1}{t_{n_1}}}\end{aligned}$$

or

$$\widehat{\beta}_1 = \frac{n_1}{\sum^{n_1}{}_{i=1} \ln \frac{T_1}{t_i}}$$

Model for Second Segment (Data after *T1*)

The Crow-AMSAA (NHPP) model will be used again to analyze the data after T_1 . However, the information collected during the first segment will be used when creating the model for the second segment. Given that, the ML estimators of the model parameters in the second segment are:

$$\widehat{\lambda_2} = \frac{n}{T_2^{\beta_2}}$$

and:

$$\widehat{\beta}_2 = \frac{n_2}{n_1 \ln \frac{T_2}{T_1} + \sum^{n}_{i=n_1+1} \ln \frac{T_2}{t_i}}$$

where:

- n_2 is the number of failures that were observed after T_1
- $n = n_1 + n_2$ is the total number of failures observed throughout the test
- T_2 is the end time of the test. The test can either be failure terminated or time terminated

Example - Multiple MLE

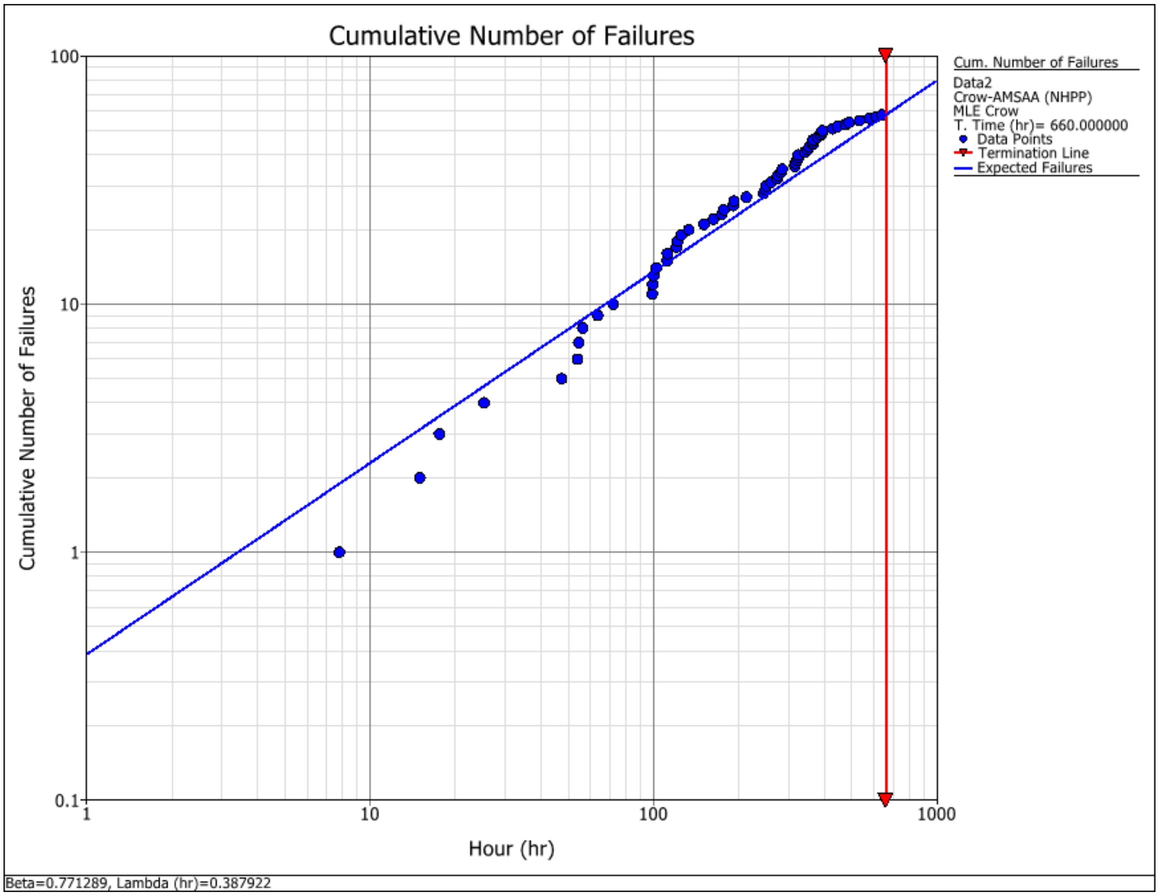
The following table gives the failure times obtained from a reliability growth test of a newly designed system. The test has a duration of 660 hours.

Failure Times From a Reliability Growth Test

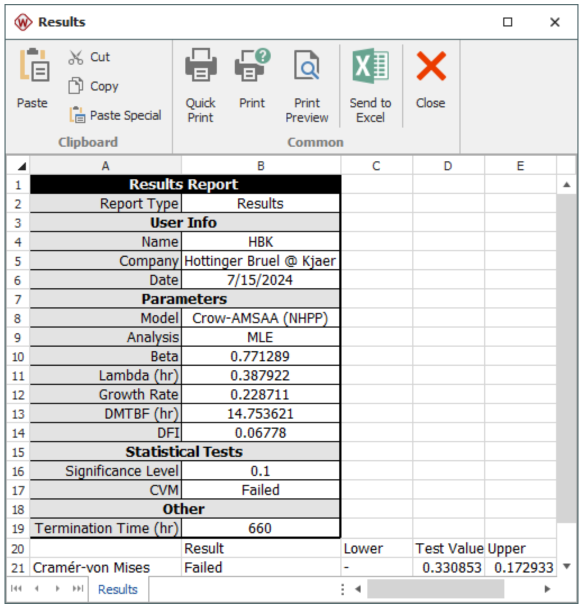
7.8	99.2	151	260.1	342	430.2
17.6	99.6	163	273.1	350.2	445.7
25.3	100.3	174.5	274.7	355.2	475.9
15	102.5	177.4	282.8	364.6	490.1
47.5	112	191.6	285	364.9	535
54	112.2	192.7	315.4	366.3	580.3
54.5	120.9	213	317.1	379.4	610.6
56.4	121.9	244.8	320.6	389	640.5

63.6	125.5	249	324.5	394.9	
72.2	133.4	250.8	324.9	395.2	

First, apply a single Crow-AMSAA (NHPP) model to all of the data. The following plot shows the expected failures obtained from the model (the line) along with the observed failures (the points).



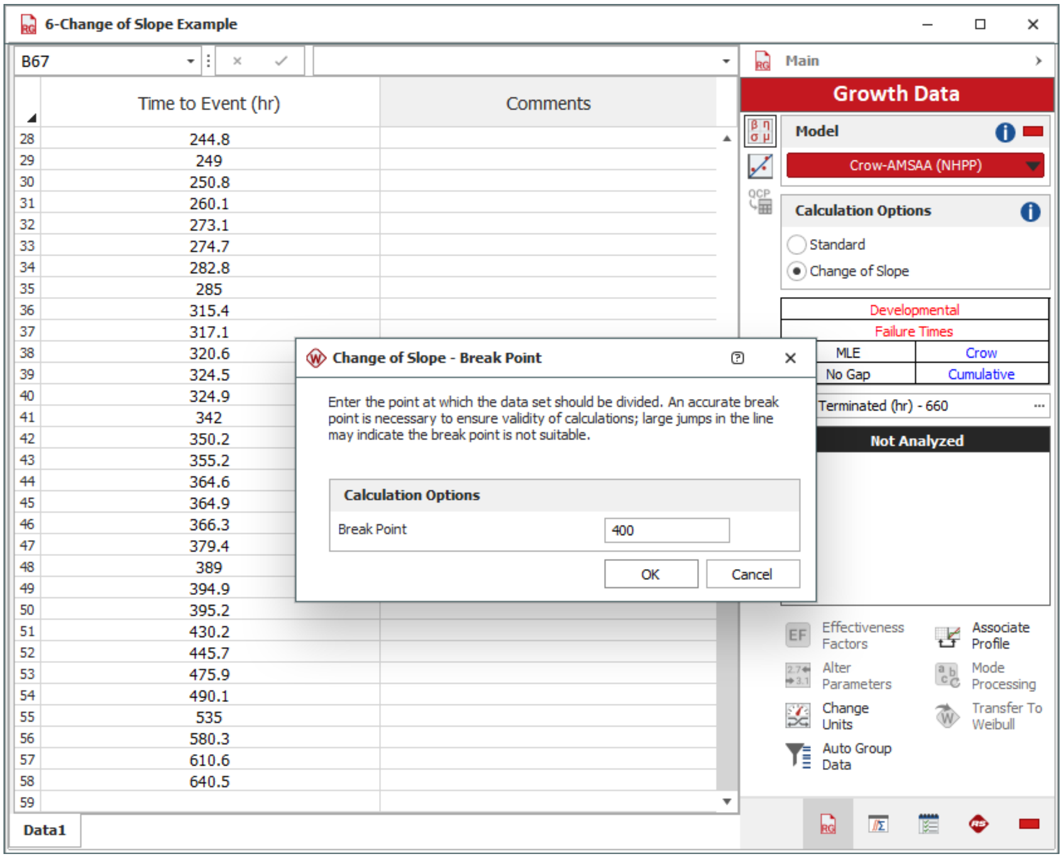
The plot shows that the model does not seem to accurately track the data. This is confirmed by performing the Cramer-von Mises goodness-of-fit test, which checks the hypothesis that the data follows a non-homogeneous Poisson process with a power law failure intensity. The model fails the goodness-of-fit test because the test statistic (0.3309) is higher than the critical value (0.1729) at the 0.1 significance level. The next figure shows a customized report that displays both the calculated parameters and the statistical test results.



Results Report				
Report Type	Results			
User Info				
Name	HBK			
Company	Hottinger Bruel @ Kjaer			
Date	7/15/2024			
Parameters				
Model	Crow-AMSAA (NHPP)			
Analysis	MLE			
Beta	0.771289			
Lambda (hr)	0.387922			
Growth Rate	0.228711			
DMTBF (hr)	14.753621			
DFI	0.06778			
Statistical Tests				
Significance Level	0.1			
CVM	Failed			
Other				
Termination Time (hr)	660			
	Result	Lower	Test Value	Upper
Cramér-von Mises	Failed	-	0.330853	0.172933

Through further investigation, it is discovered that a significant design change occurred at 400 hours of test time. It is suspected that this modification is responsible for the change in the failure behavior.

In Weibull++ , you have the option to perform a standard Crow-AMSAA (NHPP) analysis, or perform a Change of Slope analysis where you specify a specific breakpoint, as shown in the following figure. Weibull++ actually creates a grouped data set where the data in Segment 1 is included and defined by a single interval to calculate the Segment 2 parameters. However, these results are equivalent to the parameters estimated using the equations presented here.



Therefore, the Change of Slope methodology is applied to break the data into two segments for analysis. The first segment is set from 0 to 400 hours and the second segment is from 401 to 660 hours (which is the end time of the test). The Crow-AMSAA (NHPP) parameters for the first segment (0-400 hours) are:

$$\widehat{\lambda_1} = \frac{n_1}{T_1^{\beta_1}} = \frac{50}{400^{1.0359}} = 0.1008$$

and

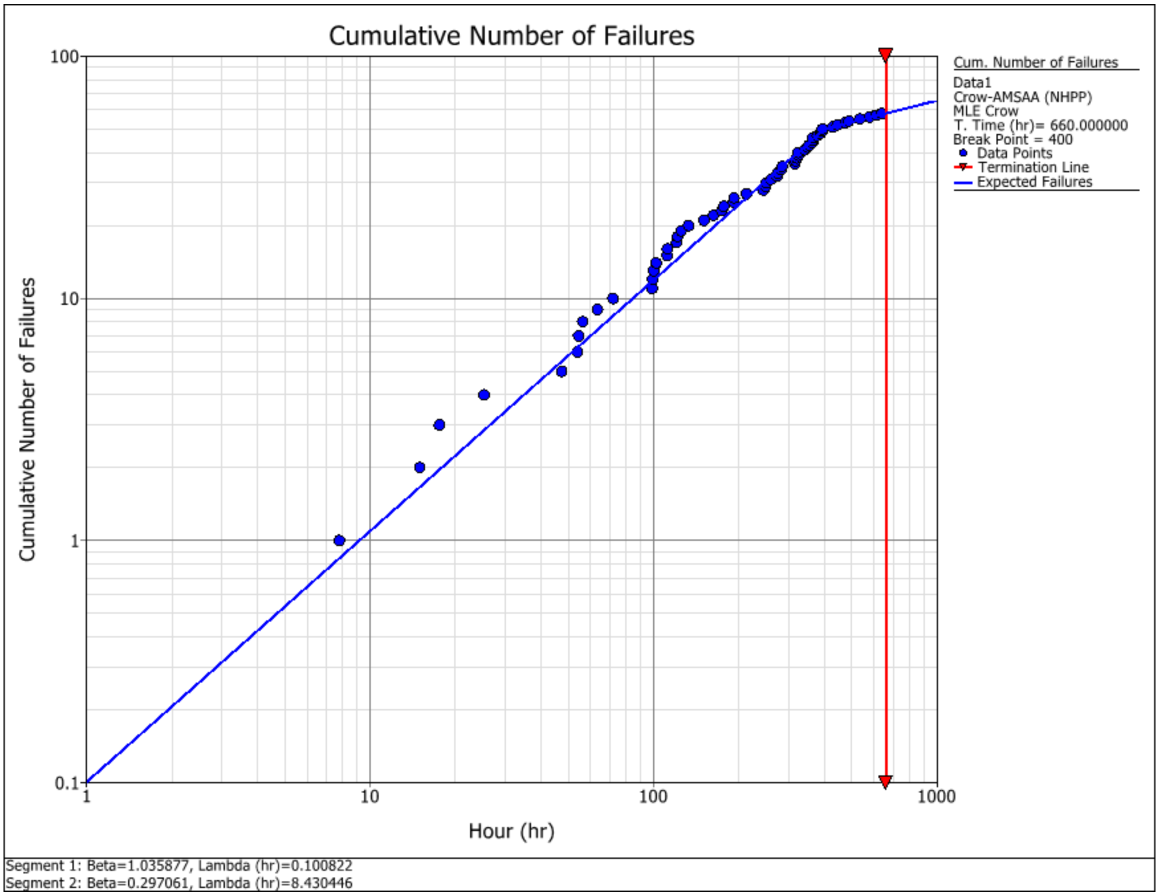
$$\widehat{\beta}_1 = \frac{n_1}{\sum^{n_1}{}_{i=1} \ln \frac{T_1}{t_i}} = \frac{50}{\sum^{50}{}_{i=1} \ln \frac{400}{t_i}} = 1.0359$$

The Crow-AMSAA (NHPP) parameters for the second segment (401-660 hours) are:

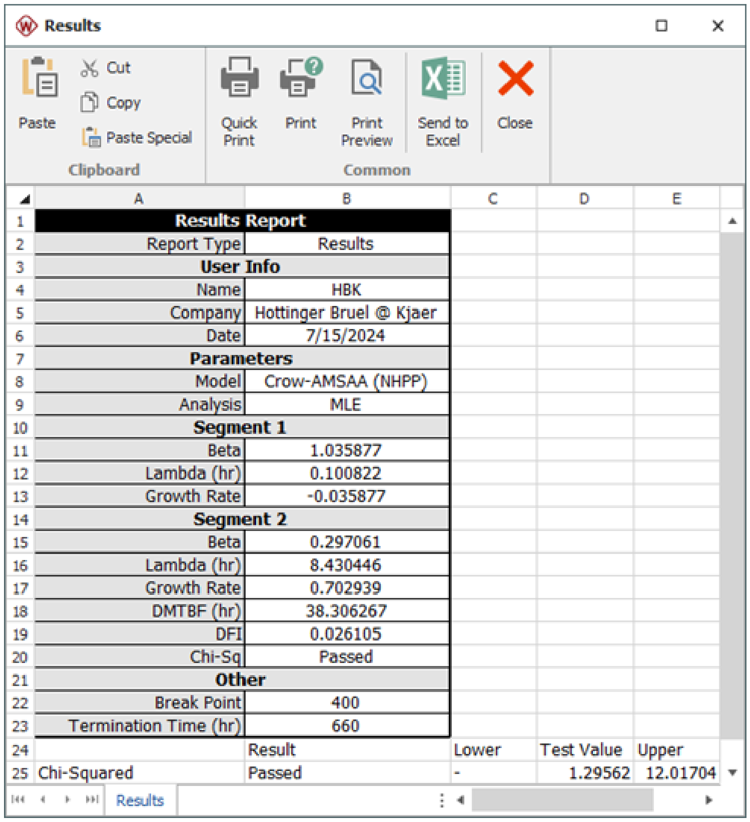
$$\widehat{\lambda_2} = \frac{n}{T_2^{\beta_2}} = \frac{58}{660^{0.2971}} = 8.4304$$

$$\widehat{\beta}_2 = \frac{n_2}{n_1 \ln \frac{T_2}{T_1} + \sum^{n}{}_{i=n_1+1} \ln \frac{T_2}{t_i}} = \frac{8}{50 \ln \frac{660}{400} + \sum^{58}{}_{i=51} \ln \frac{660}{T_i}} = 0.2971$$

The following figure shows a plot of the two-segment analysis along with the observed data. It is obvious that the Change of Slope method tracks the data more accurately.

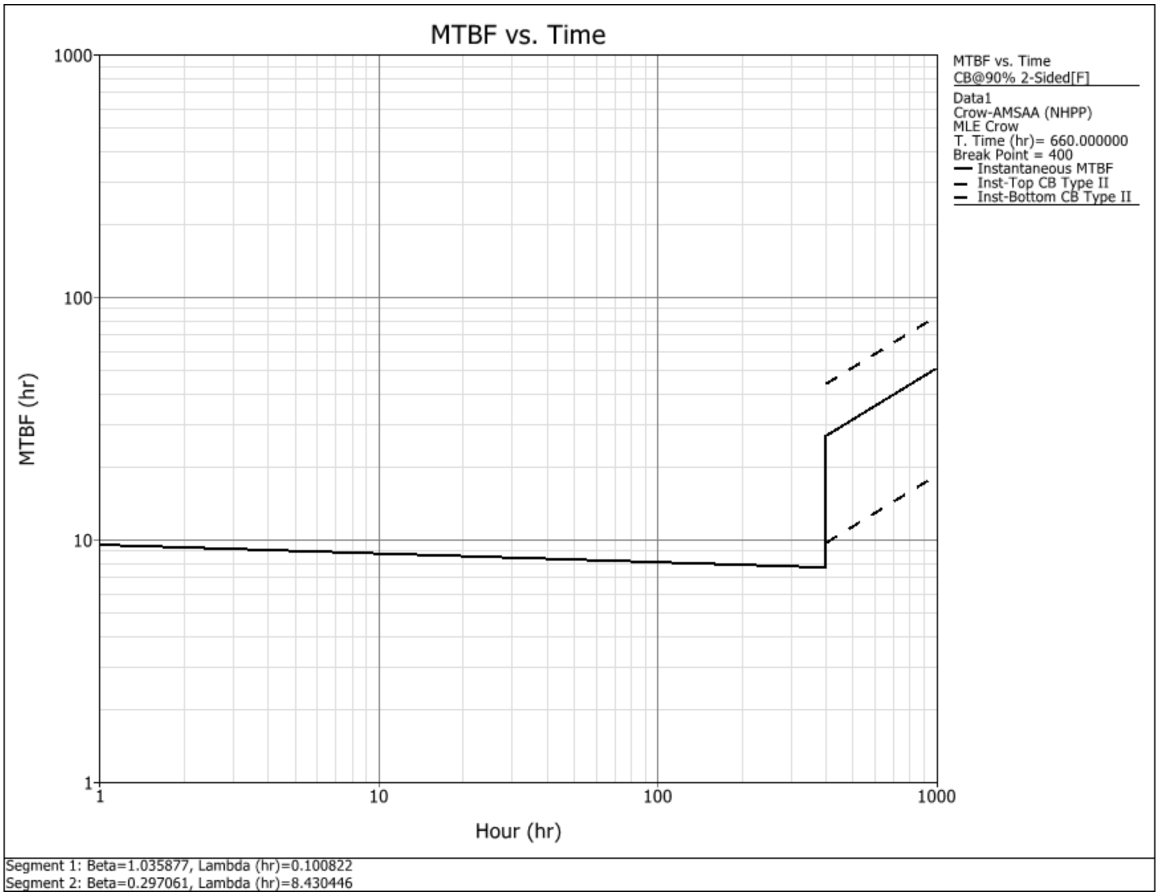


This can also be verified by performing a chi-squared goodness-of-fit test. The chi-squared statistic is 1.2956, which is lower than the critical value of 12.017 at the 0.1 significance level; therefore, the analysis passes the test. The next figure shows a customized report that displays both the calculated parameters and the statistical test results.

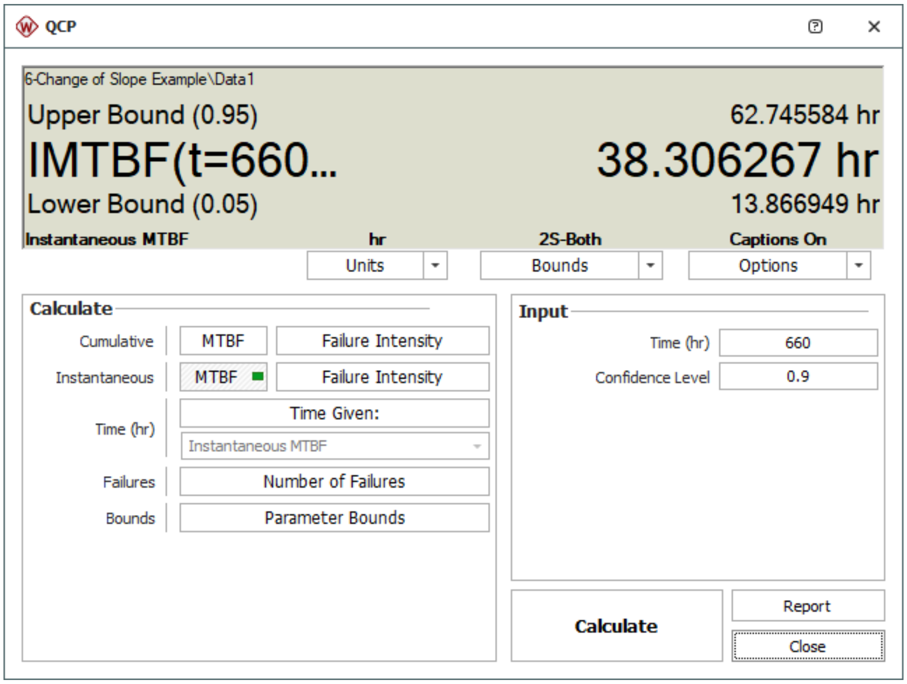


	A	B	C	D	E
1	Results Report				
2	Report Type	Results			
3	User Info				
4	Name	HBK			
5	Company	Hottinger Bruel @ Kjaer			
6	Date	7/15/2024			
7	Parameters				
8	Model	Crow-AMSAA (NHPP)			
9	Analysis	MLE			
10	Segment 1				
11	Beta	1.035877			
12	Lambda (hr)	0.100822			
13	Growth Rate	-0.035877			
14	Segment 2				
15	Beta	0.297061			
16	Lambda (hr)	8.430446			
17	Growth Rate	0.702939			
18	DMTBF (hr)	38.306267			
19	DFI	0.026105			
20	Chi-Sq	Passed			
21	Other				
22	Break Point	400			
23	Termination Time (hr)	660			
24		Result	Lower	Test Value	Upper
25	Chi-Squared	Passed	-	1.29562	12.01704

When you have a model that fits the data, it can be used to make accurate predictions and calculations. Metrics such as the demonstrated MTBF at the end of the test or the expected number of failures at later times can be calculated. For example, the following plot shows the instantaneous MTBF vs. time, together with the two-sided 90% confidence bounds. Note that confidence bounds are available for the second segment only. For times up to 400 hours, the parameters of the first segment were used to calculate the MTBF, while the parameters of the second segment were used for times after 400 hours. Also note that the number of failures at the end of segment 1 is not assumed to be equal to the number of failures at the start of segment 2. This can result in a visible jump in the plot, as in this example.



The next figure shows the use of the Quick Calculation Pad (QCP) in the Weibull++ software to calculate the Demonstrated MTBF at the end of the test (instantaneous MTBF at time = 660), together with the two-sided 90% confidence bounds. All the calculations were based on the parameters of the second segment.



More Examples

Determining Whether a Design Meets the MTBF Goal

A prototype of a system was tested at the end of one of its design stages. The test was run for a total of 300 hours and 27 failures were observed. The table below shows the collected data set. The prototype has a design specification of an MTBF equal to 10 hours with a 90% confidence level at 300 hours. Do the following:

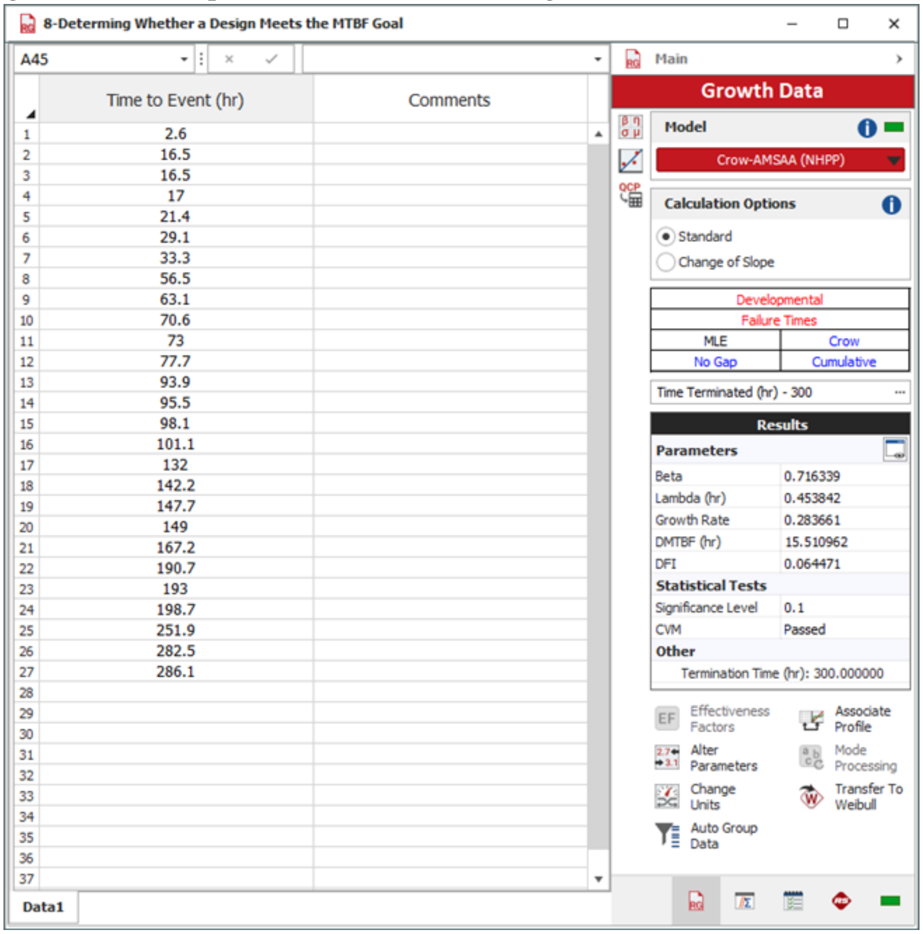
1. Estimate the parameters of the Crow-AMSAA model using maximum likelihood estimation.
2. Does the prototype meet the specified goal?

Failure Times Data

2.6	56.5	98.1	190.7
16.5	63.1	101.1	193
16.5	70.6	132	198.7
17	73	142.2	251.9
21.4	77.7	147.7	282.5
29.1	93.9	149	286.1
33.3	95.5	167.2	

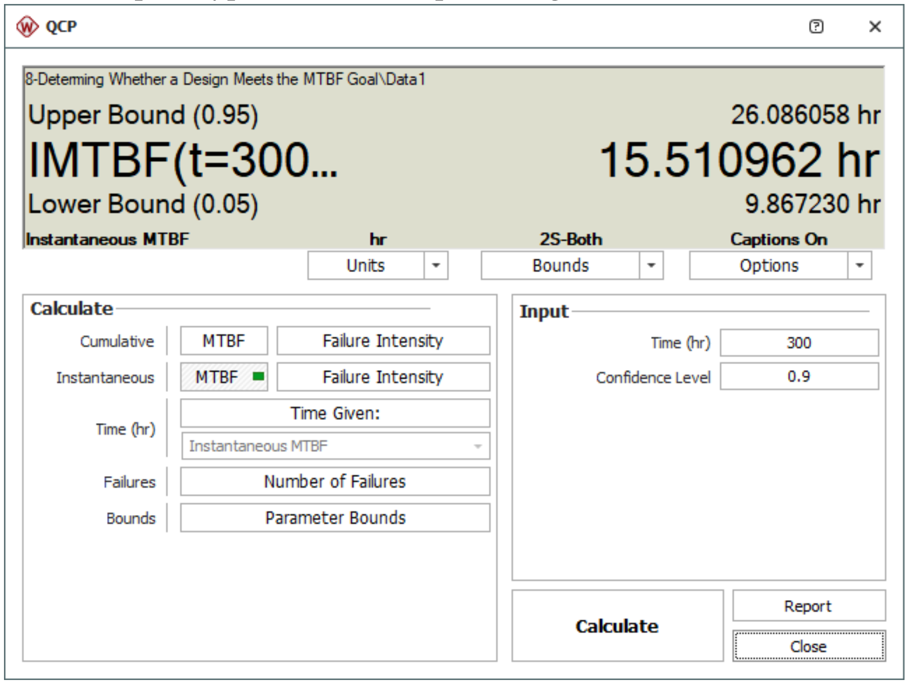
Solution

1. The next figure shows the parameters estimated using Weibull++.



2. The instantaneous MTBF with one-sided 90% confidence bounds can be calculated using the Quick Calculation Pad (QCP), as shown next. From the QCP, it is estimated that the lower limit on the MTBF at 300 hours with a 90% confidence level is equal to 10.8170

hours. Therefore, the prototype has met the specified goal.



Analyzing Mixed Data for a One-Shot System

A one-shot system underwent reliability growth development for a total of 50 trials. The test was performed as a combination of configuration in groups and individual trial by trial. The table below shows the data set obtained from the test. The first column specifies the number of failures that occurred in each interval, and the second column shows the cumulative number of trials in that interval. Do the following:

1. Estimate the parameters of the Crow-AMSAA model using maximum likelihood estimators.
2. What are the instantaneous reliability and the 2-sided 90% confidence bounds at the end of the test?
3. Plot the cumulative reliability with 2-sided 90% confidence bounds.
4. If the test was continued for another 25 trials what would the expected number of additional failures be?

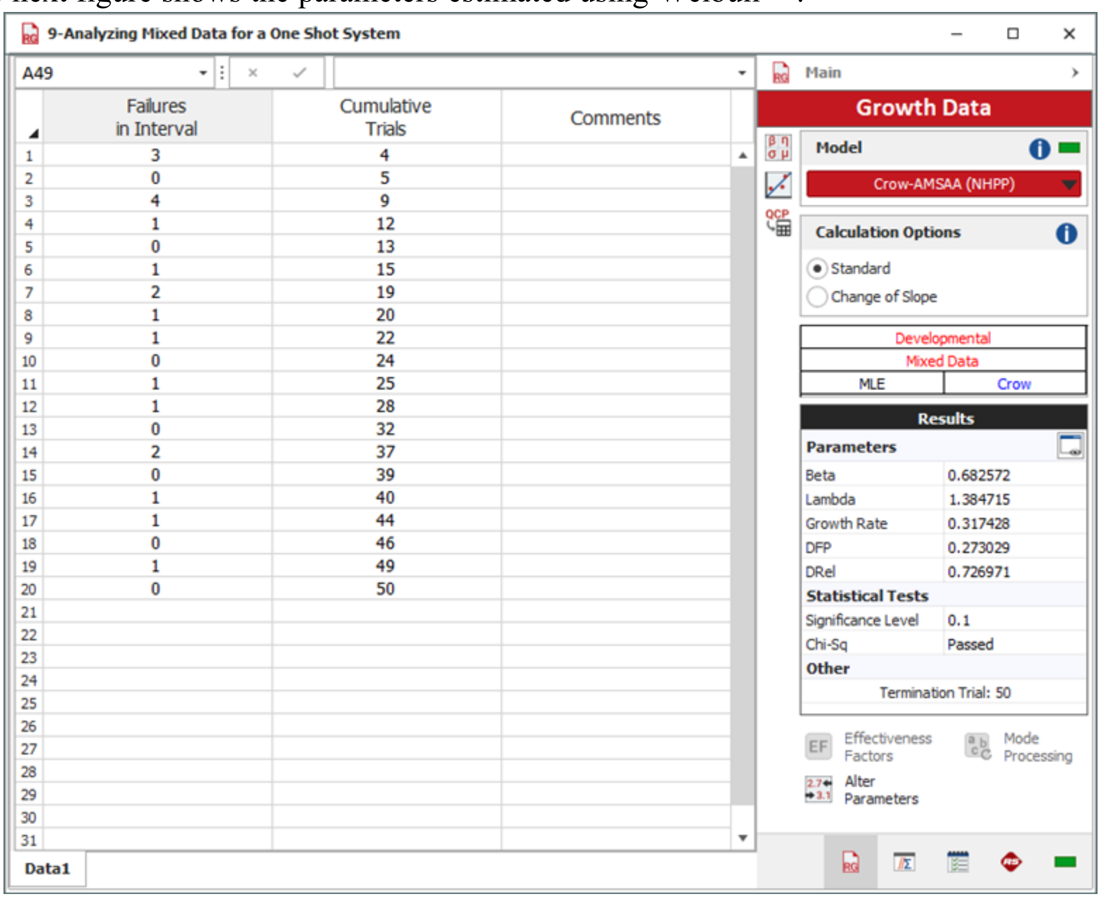
Mixed Data

Failures in Interval	Cumulative Trials	Failures in Interval	Cumulative Trials
3	4	1	25
0	5	1	28

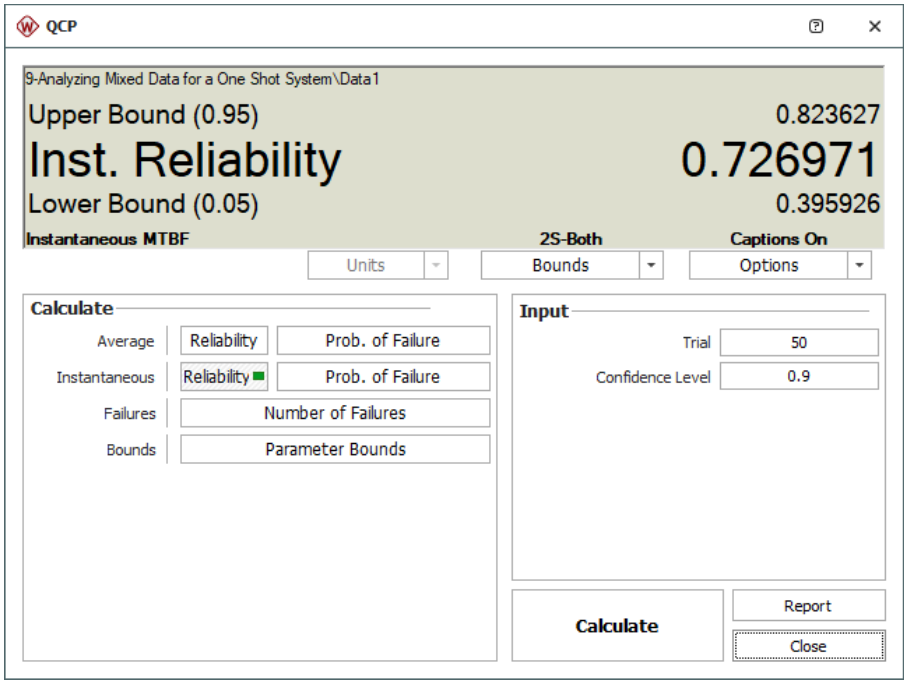
4	9	0	32
1	12	2	37
0	13	0	39
1	15	1	40
2	19	1	44
1	20	0	46
1	22	1	49
0	24	0	50

Solution

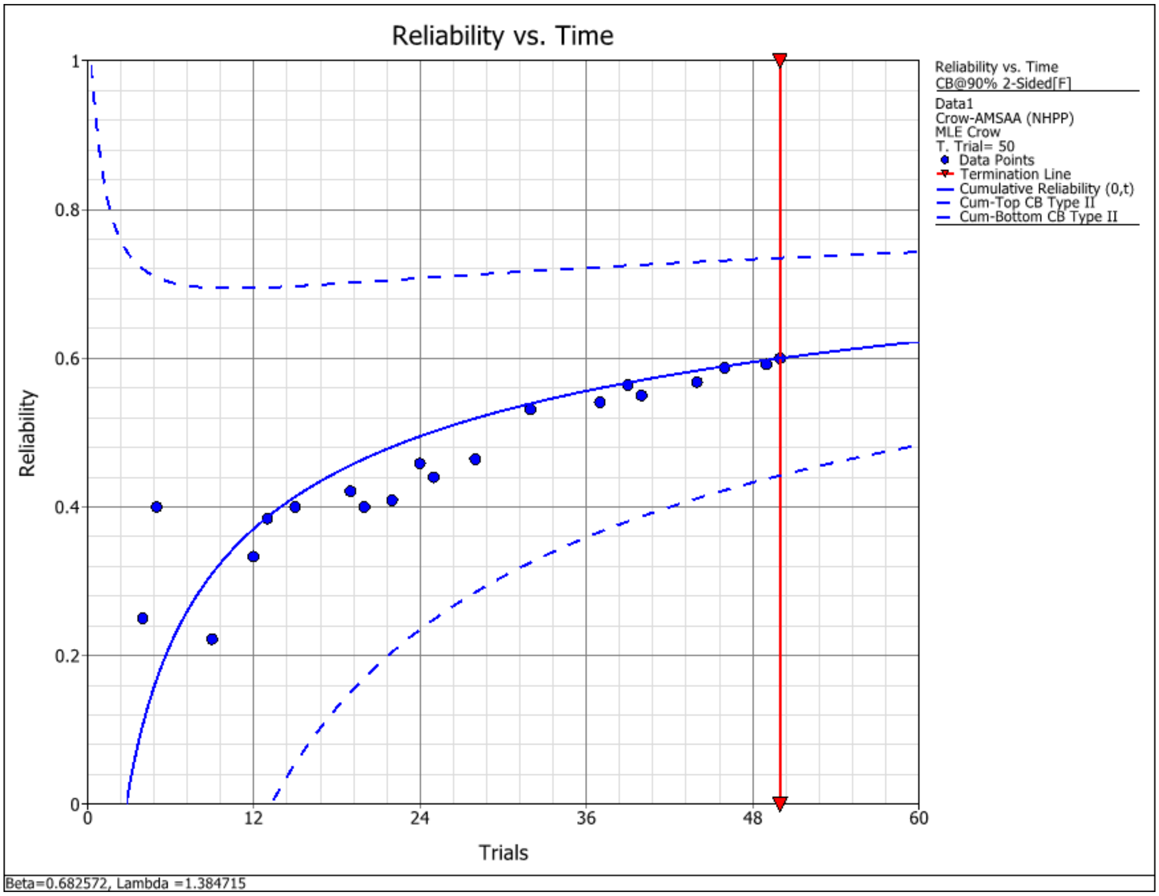
1. The next figure shows the parameters estimated using Weibull++.



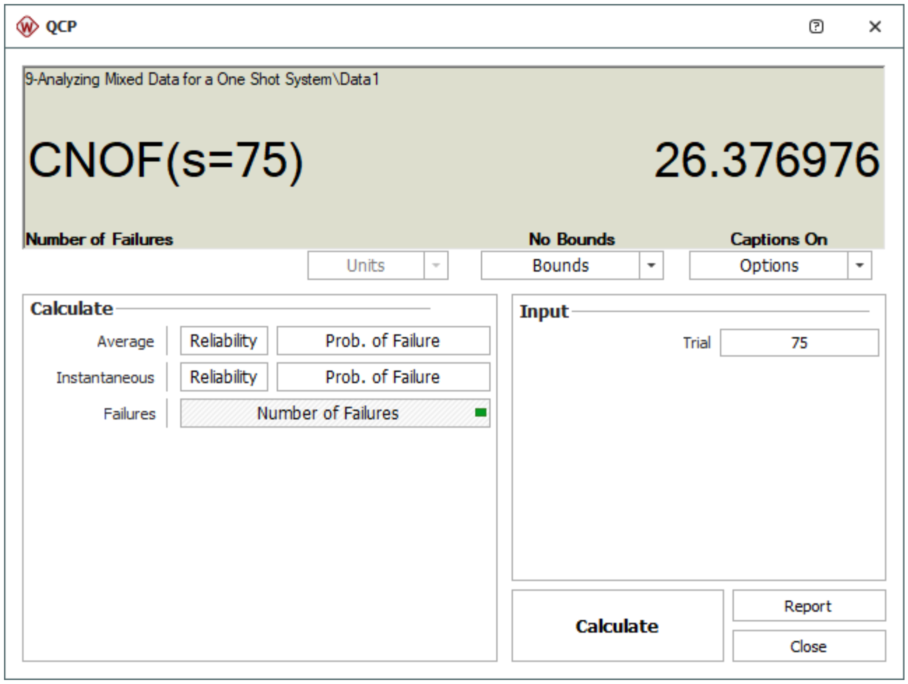
2. The figure below shows the calculation of the instantaneous reliability with the 2-sided 90% confidence bounds. From the QCP, it is estimated that the instantaneous reliability at stage 50 (or at the end of the test) is 72.70% with an upper and lower 2-sided 90% confidence bound of 82.36% and 39.59%, respectively.



3. The following plot shows the cumulative reliability with the 2-sided 90% confidence bounds.



4. The last figure shows the calculation of the expected number of failures after 75 trials. From the QCP, it is estimated that the cumulative number of failures after 75 trials is $26.3770 \approx 27$. Since 20 failures occurred in the first 50 trials, the estimated number of additional failures is 7.



Crow Extended

In reliability growth analysis, the Crow-AMSAA (NHPP) model assumes that the corrective actions for the observed failure modes are incorporated during the test (test-fix-test). However, in actual practice, fixes may be delayed until after the completion of the test (test-find-test) or some fixes may be implemented during the test while others are delayed (test-fix-find-test). At the end of a test phase, two reliability estimates are of concern: *demonstrated reliability* and *projected reliability*. The demonstrated reliability, which is based on data generated during the test phase, is an estimate of the system reliability for its configuration at the end of the test phase. The projected reliability measures the impact of the delayed fixes at the end of the current test phase.

Most of the reliability growth literature are concerned with procedures and models for calculating the demonstrated reliability, and very little attention has been paid to techniques for reliability projections. The procedure for making reliability projections utilizes engineering assessments of the effectiveness of the delayed fixes for each observed failure mode. These effectiveness factors are then used with the data generated during the test phase to obtain a projected estimate for the updated configuration by adjusting the number of failures observed during the test phase. The process of estimating the projected reliability is accomplished using the Crow Extended model. The Crow Extended model allows for a flexible growth strategy that can include corrective actions performed during the test, as well as delayed corrective actions. The test-find-test and test-fix-find-test scenarios are simply subsets of the Crow Extended model.

For developmental testing, the Crow Extended model can be applied when using any of the following data types:

- Failure Times Data
- Multiple Systems (Known Operating Times)
- Multiple Systems (Concurrent Operating Times)
- Multiple Systems with Dates
- Multiple Systems with Event Codes
- Grouped Failure Times
- Mixed Data

As the name implies, Crow Extended is simply an "extension" of the Crow-AMSAA (NHPP) model. The calculations for Crow Extended still incorporate the methods for Crow-AMSAA based on the data type, as described on the Crow-AMSAA (NHPP) page. The Crow-AMSAA model estimates the growth during the test, while the Crow Extended model accounts for the growth after the test based on the delayed corrective actions. Additional details regarding the calculations for the Crow Extended model are presented in the sections below.

Background

When a system is tested and failure modes are observed, management can make one of two possible decisions: to fix or to not fix the failure modes. Failure modes that are not fixed are called *A modes* and failure modes that receive a corrective action are called *B modes*. The A modes account for all failure modes that management considers to be not economical or not justified to receive corrective action. The B modes provide the assessment and management metric structure for corrective actions during and after a test. There are two types of B modes: BC modes, which are corrected during the test, and BD modes, which are corrected only at the end of the test. The management strategy is defined by how the corrective actions, if any, will be implemented. In summary, the classifications are defined as follows:

- **A** indicates that no corrective action was performed or will be performed (management chooses not to address for technical, financial or other reasons).
- **BC** indicates that the corrective action was implemented during the test. The analysis assumes that the effect of the corrective action was experienced during the test (as with other test-fix-test reliability growth analyses).

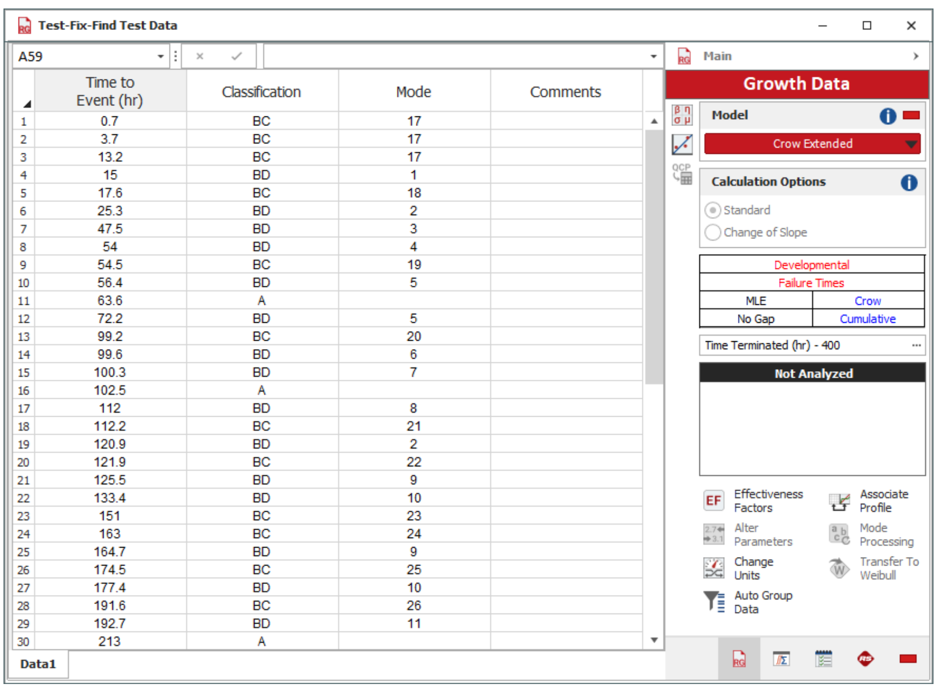
- **BD** indicates that the corrective action will be delayed until after the completion of the current test. BD modes will provide a jump in the system's MTBF at the termination time.

In terms of assessing a system's reliability, there are three specific metrics of interest:

- **Demonstrated (or achieved) MTBF (DMTBF)** is the system's instantaneous MTBF at the termination time.
- **Projected MTBF (PMTBF)** is the system's expected instantaneous MTBF after the implementation of the delayed corrective actions.
- **Growth Potential MTBF (GPMTBF)** is the maximum MTBF that can be attained for the system design and management strategy.

These metrics can also be displayed in Weibull++ in terms of failure intensity (FI). The demonstrated MTBF is calculated using basically the same methods, depending on data type, as presented on the Crow-AMSAA (NHPP) page. Projected MTBF/FI and growth potential MTBF/FI are presented in detail in the sections below.

The following picture shows an example of data entered for the Crow Extended model.

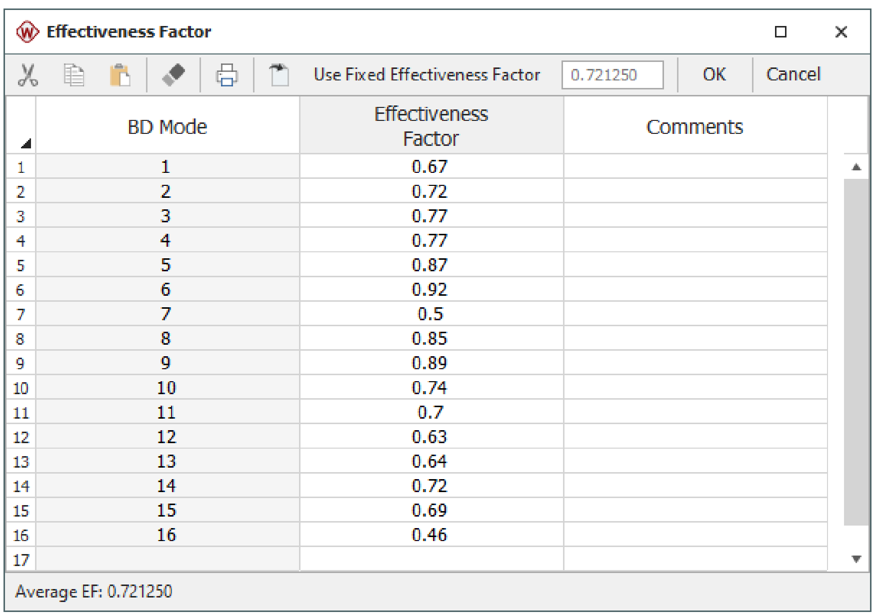


	Time to Event (hr)	Classification	Mode	Comments
1	0.7	BC	17	
2	3.7	BC	17	
3	13.2	BC	17	
4	15	BD	1	
5	17.6	BC	18	
6	25.3	BD	2	
7	47.5	BD	3	
8	54	BD	4	
9	54.5	BC	19	
10	56.4	BD	5	
11	63.6	A		
12	72.2	BD	5	
13	99.2	BC	20	
14	99.6	BD	6	
15	100.3	BD	7	
16	102.5	A		
17	112	BD	8	
18	112.2	BC	21	
19	120.9	BD	2	
20	121.9	BC	22	
21	125.5	BD	9	
22	133.4	BD	10	
23	151	BC	23	
24	163	BC	24	
25	164.7	BD	9	
26	174.5	BC	25	
27	177.4	BD	10	
28	191.6	BC	26	
29	192.7	BD	11	
30	213	A		

As you can see, each failure is indicated with A, BC or BD in the **Classification** column. In addition, any number or text can be used to specify the failure mode. In this example, numbers were used in the **Mode** column for simplicity, but you could just as easily use "Seal Leak," or whatever designation you deem appropriate for identifying the mode. In Weibull++, a failure mode is defined as a problem and a cause.

Reliability growth is achieved by decreasing the failure intensity. The failure intensity for the A failure modes will not change; therefore, reliability growth can only be achieved by decreasing the BC and BD mode failure intensity. In general, the only part of the BD mode failure intensity that can be decreased is that which has been seen during testing, since the failure intensity due to BD modes that were unseen during testing still remains. The BC failure modes are corrected during test, and the BC failure intensity will not change any more at the end of test.

It is very important to note that once a BD failure mode is in the system, it is rarely totally eliminated by a corrective action. After a BD mode has been found and fixed, a certain percentage of the failure intensity will be removed, but a certain percentage of the failure intensity will generally remain. For each BD mode, an *effectiveness factor* (EF) is required to estimate how effective the corrective action will be in eliminating the failure intensity due to the failure mode. The EF is the fractional decrease in a mode's failure intensity after a corrective action has been made, and it must be a value between 0 and 1. It has been shown empirically that an average EF, d, is about 70%. Therefore, about 30 percent, (i.e., 100 $(1-d)$ percent), of the BD mode failure intensity will typically remain in the system after all of the corrective actions have been implemented. However, individual EFs for the failure modes may be larger or smaller than the average. The next figure displays the Weibull++ software's Effectiveness Factor window where the effectiveness factors for each unique BD failure mode can be specified.



	BD Mode	Effectiveness Factor	Comments
1	1	0.67	
2	2	0.72	
3	3	0.77	
4	4	0.77	
5	5	0.87	
6	6	0.92	
7	7	0.5	
8	8	0.85	
9	9	0.89	
10	10	0.74	
11	11	0.7	
12	12	0.63	
13	13	0.64	
14	14	0.72	
15	15	0.69	
16	16	0.46	
17			

Average EF: 0.721250

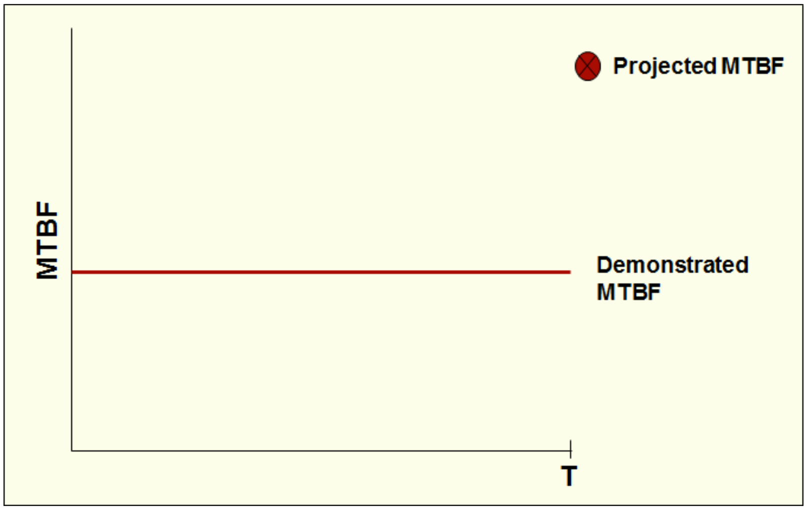
The EF does not account for the scenario where additional failure modes are created from the corrective actions. If this happens, then the actual EF will be lower than the assumed value. An EF greater than or equal to 0.9 indicates a significant improvement in a failure mode's MTBF (or reduction in failure intensity). This is indicative of a design change. The increase in a failure mode's MTBF, given an EF, can be calculated using:

$$Multiplier = \frac{1}{1 - EF}$$

Using this equation, an $EF = 0.7$ corresponds to an increase of 3.3X. Therefore, if a failure mode had an MTBF of 100 and a delayed fix was applied, an estimate of the failure mode's MTBF after the fix is equal to ≈ 333. An $EF = 0.9$ corresponds to a 10X increase in the failure mode's MTBF. So there is a large increase between EF = 0.7 and EF = 0.9. Before assigning an EF to a BD mode, it is recommended to run a test to verify the fix and to validate the assumed EF value. Ideally, you want to have some sort of justification for using the entered EF value. If you do not know the EF value for each BD mode, then you can also specify a fixed EF which is then applied to all modes. The average value of EF = 0.7 is a good place to start, but if you want to be a bit more conservative, then an EF = 0.4, or a smaller value, could be used.

Test-Find-Test

Test-find-test is a testing strategy where all corrective actions are delayed until after the test. Therefore, there are not any BC modes when analyzing test-find-test data (only A and BD modes). This scenario is also called the *Crow-AMSAA Projection* model, but for the purposes of the Weibull++ software, it is simply a special case of the Crow Extended model. The picture below presents the test-find-test scenario.



Since there are no fixes applied during the test, the assumption is that the system's MTBF does not change during the test. In other words, the system's MTBF is constant. The system should not be exhibiting an increasing or decreasing trend in its reliability if changes are not being made. Therefore, the assumption is that $\beta = 1$. Weibull++ will return two β values; Beta (hyp) which always shows a value equal to 1 since this is the underlying assumption, and Beta which is the Crow-AMSAA (NHPP) estimate that considers all of the failure times. The assumption of $\beta = 1$ can be verified by looking at the confidence bounds on β. If the confidence bounds on β include one, then you can fail to reject the hypothesis that $\beta = 1$. Weibull++ does this check automatically and uses a confidence level equal to 1 minus the specified significance level to check if the confidence bounds on β include one. If the $\beta = 1$ assumption is violated, then the value for Beta (hyp) will be displayed in red text. If the confidence bounds on β do not include 1, then the following questions should be considered as they relate to the data:

- If multiple systems were tested, did they have the same configuration throughout the test? Were the corrective actions applied to all systems?
- Were the test conditions consistent for each system?
- Was there a change in the failure definition?
- Any issues with data entry?

$\beta = 1$ is then used to estimate the demonstrated MTBF of the system and it is also used in the goodness-of-fit tests.

Projected Failure Intensity

Suppose a system is subjected to development testing for a period of time, T. The system can be considered as consisting of two types of failure modes: A modes and BD modes. It is assumed that all BD modes are in series and fail independently according to the exponential distribution. Also assume that the rate of occurrence of A modes follows an exponential distribution with failure intensity λ_A. The system MTBF is constant throughout the test phase since all of the corrective actions are delayed until after the completion of the test. After the delayed fixes have been implemented, the system MTBF will then jump to a higher value.

Let K denote the total number of BD modes in the system, and let λ_i denote the failure intensity for the i^{th} BD mode, such that $i = 1, 2, \ldots, K$. Then, at time equal to zero, the system failure intensity $r(0)$ is:

$$r(0) = \lambda_A + \lambda_{BD}$$

where:

$$\lambda_{BD} = \sum_{i=1}^{K} \lambda_i \, .$$

During the test $(0, T)$, a random number of M distinct BD modes will be observed, such that $M \leq K$. Denote the effectiveness factor (EF) for the i^{th} BD mode as d_i, $i = 1, 2, \ldots, K$. The effectiveness factor d_i is the percent decrease in λ_i after a corrective action has been made for the i^{th} BD mode. That is, the corrective action for the i^{th} BD mode removes $100 \times d_i$ percent of the failure rate, and $100 \times (1 - d_i)$ percent remains. The failure intensity for the i^{th} BD failure mode after a corrective action is $(1-d_i)\lambda_i$. If corrective actions are taken on the M BD modes observed by time T , then the system failure intensity is reduced from $r(0)$ to:

$$r(T) = \lambda_A + \sum_{i=1}^{M} (1-d_i)\lambda_i + (\lambda_{BD} - \sum_{i=1}^{M} \lambda_i)$$
$$= \lambda_A + \lambda_{BD} - \sum_{i=1}^{M} d_i \lambda_i$$

where:

- $\sum_{i=1}^{M}(1-d_i)\lambda_i$ is the failure intensity for the M modes after the corrective actions
- $(\lambda_{BD} - \sum_{i=1}^{M} \lambda_i)$ is the remaining failure intensity for all unseen BD modes

All M BD modes observed by test time T may not be fixed by time T so the actual failure intensity at time T may not be $r(T)$. However, $r(T)$ can be viewed as the achieved failure intensity at time T if all fixes were updated and incorporated into the system. All of the fixes for the BD modes found during the test are incorporated as delayed fixes at the end of the test phase. Therefore, the system failure intensity is constant at $r(0) = \lambda_A + \lambda_{BD}$ through the test phase and will then jump to a lower value $r(T)$ after the delayed fixes have been implemented. Let N_A and N_{BD} be the total number of A and BD failures observed during the test $(0, T)$ and let $N = N_A + N_{BD}$. In addition, there are M distinct BD modes observed during the test. After implementing the M fixes, the failure intensity for the system at time T (after the jump) is given by the function $r(T)$.

$r(0)$ is actually the demonstrated failure intensity, which is based on actual system performance of the hardware tested and not of some future configuration. A demonstrated reliability value should be determined at the end of each test phase. The demonstrated failure intensity is:

$$\hat{\lambda}_D(T) = r(0) = \frac{N_A + N_{BD}}{T}$$

The demonstrated MTBF is given by:

$$\widehat{MTBF}_D = [\hat{\lambda}_D(T)]^{-1}$$

The detailed procedure for estimating $r(T)$ is given in Crow [20] and is reviewed here.

Let $E[\cdot]$ denote the expected value:

$$E[r(T)] = \lambda_A + \sum_{i=1}^{M}(1-d_i)\lambda_i + \sum_{i=1}^{M} d_i\lambda_i e^{-\lambda_i T}$$

Under realistic assumptions, $E[r(T)]$ also may be expressed as:

$$E[r(T)] = \lambda_A + \sum_{i=1}^{M}(1-d_i)\lambda_i + \overline{d}h(T)$$

where $\overline{d}$ is the mean effectiveness factor and $h(T)$ is the instantaneous rate at which a new BD mode will occur at time T. $\overline{d}h(T)$ is the bias term (or sometimes called the 3rd term), such that:

$$B(T) = \overline{d}h(T)$$

The mean $\overline{d}$ is given by:

$$\overline{d} = \frac{1}{M}\sum_{i=1}^{M} d_i$$

Therefore, the projected failure intensity $r(T)$ is then estimated at the end of the test phase by:

$$\hat{r}(T) = \left(\frac{N_A}{T} + \sum_{i=1}^{M}(1-d_i)\frac{N_i}{T}\right) + \overline{d}h(T)$$

The projected MTBF is:

$$\widehat{MTBF}_P = [r(T)]^{-1}$$

Mathematical Formulation

As indicated previously, the failure intensity of the system at $t = 0$ is given by:

$$\begin{aligned}\lambda_{System} &= \lambda_A + \lambda_{BD} \\ &= \lambda_A + \lambda_{BD_{Seen}} + \lambda_{BD_{Unseen}}\end{aligned}$$

The estimate for the projected failure intensity, after the corrective actions have been implemented, is then:

$$\begin{aligned}\lambda_{\text{Projected}} &= \lambda_A + (1-d)\left(\lambda_{BD} - h(t)\right) + h(t) \\ &= \lambda_A + (1-d)\,\lambda_{BD} - (1-d)\,h(t) + h(t) \\ &= \lambda_A + (1-d)\,\lambda_{BD} - h(t) + dh(t) + h(t) \\ &= \lambda_A + (1-d)\,\lambda_{BD} + dh(t)\end{aligned}$$

This is the basic format of the equation for the projected failure intensity when the Crow Extended model is used with the test-find-test strategy.

h(t) Function

h(t) is defined as the unseen BD mode failure intensity. It is also defined as the rate at which new unique BD modes are being discovered. The maximum likelihood estimate of $h(t)$ is calculated using:

$$\hat{h}(t) = \hat{\lambda}_{BD}\overline{\beta}_{BD}t^{\overline{\beta}_{BD}-1}$$

where $\overline{\beta}_{BD}$ is the unbiased estimate. The unbiased estimate is always used when calculating $\hat{h}(t)$.

The parameters of the $h(t)$ function, β_{BD} and λ_{BD}, are calculated using the first occurrence of each BD mode. In order to have growth, β_{BD} must be less than one. If β_{BD} is close to one then it is possible that the program could be in trouble since this indicates that there are very few repeat occurrences. In this case, each failure tends to be a unique mode. If β_{BD} is less than one then the rate at which new unique BD modes are occurring is decreasing.

Let $X_1 < X_2 < \ldots < X_M < T$ denote the cumulative test times for the first occurrences of BD modes. Then, the maximum likelihood estimates of λ_{BD} and β_{BD} are:

$$\hat{\beta}_{BD} = \frac{M}{\sum^{M}_{i=1} \ln\left(\frac{T}{X_i}\right)}$$

The unbiased estimate of β_{BD} is:

$$\bar{\beta}_{BD} = \frac{M-1}{M}\hat{\beta}_{BD}$$

$$\hat{\lambda}_{BD} = \frac{M}{T^{\bar{\beta}_{BD}}}$$

In particular, the maximum likelihood estimate for the rate of occurrence for the distinct BD modes at the termination time, T, is:

$$\begin{aligned}\hat{h}(T) &= \hat{\lambda}_{BD}\overline{\beta}_{BD}T^{\overline{\beta}_{BD}-1} \\ &= \frac{M\overline{\beta}_{BD}}{T}\end{aligned}$$

The parameters associated with $h(t)$ can be viewed in Weibull++ by selecting **BD modes** from the dropdown to the right of **Results (All Modes)** in the Results Area on the control panel to the right of the data sheet. The displayed value for the failure intensity, **FI**, is $h(T)$.

Growth Potential

Growth Potential, when represented in terms of MTBF, is the maximum system reliability that can be attained for the system design with the current management strategy. The maximum MTBF will be attained when all K BD modes have been observed and fixed with EFs d_i . If the system is tested long enough then the failures that are observed in this case will be either repeat BD modes or A modes. In other words, there is not another unique BD mode to find within the system, and in most cases the growth potential is a value that may never actually be achieved. The growth potential can be thought of as an upper bound on the system's MTBF and ideally, should be about 30% above the system's requirement. As the system's MTBF gets closer to the growth potential, it becomes more difficult to increase the system's MTBF because it is taking more and more test time to propagate the next unique BD mode. The BD modes present opportunities for growth, but the rate of occurrence for unique BD modes goes down due to the decrease in $h(t)$. The growth potential is reached when $h(t)$ is equal to zero. In other words, the difference between the projected failure intensity and the growth potential is $h(t)$. At about 2/3 of the growth potential it becomes increasingly difficult to increase the system's MTBF as $h(t)$ function starts to flatten out.

The failure intensity $r(T)$ will depend on the management strategy that determines the classification of the A and BD failure modes. The engineering effort applied to the corrective actions determines the effectiveness factors. In addition, $r(T)$ depends on $h(t)$, which is the rate at which problem failure modes are being seen during testing. $h(t)$ drives the opportunity to take corrective actions based on the seen failure modes and it is an important factor in the overall reliability growth rate. The reliability growth potential is the limiting value of $r(T)$ as T increases. This limit is the maximum MTBF that can be attained with the current management strategy. In terms of failure intensity, the growth potential is expressed by the following equation:

$$r_{GP} = \lambda_A + \sum_{i=1}^{K} (1 - d_i)\lambda_i$$

In terms of the MTBF, the growth potential is given by:

$$MTBF_{GP} = 1/r_{GP}$$

The procedure for estimating the growth potential is as follows. Suppose that the system is tested for a period of time T and that N failures have been observed. According to the management strategy, N_A of these failures are A modes and N_{BD} of these failures are BD modes. For the BD modes, there will be M distinct fixes. As before, N_i is the total number of failures for the i^{th} BD mode and d_i is the corresponding assigned EF. From this data, the growth potential failure intensity is estimated by:

$$\hat{r}_{GP}(T) = \left(\frac{N_A}{T} + \sum_{i=1}^{M} (1 - d_i) \frac{N_i}{T} \right)$$

The growth potential MTBF is estimated by:

$$M\hat{T}BF_{GP} = [\hat{r}_{GP}]^{-1}$$

Example - Test-Find-Test Data

Consider the data in the first table below. A system was tested for $T = 400$ hours. There were a total of $N = 42$ failures and all corrective actions will be delayed until after the end of the 400 hour test. Each failure has been designated as either an A failure mode (the cause will not receive a corrective action) or a BD mode (the cause will receive a corrective action). There are $N_A = 10$ A mode failures and $N_{BD} = 32$ BD mode failures. In addition, there are $M = 16$ distinct BD failure modes, which means 16 distinct corrective actions will be incorporated into the system at the end of test. The total number of failures for the j^{th} observed distinct BD mode is denoted by N_j, and the total number of BD failures during the test is $N_{BD} = \sum_{j=1}^{M} N_j$. These values and effectiveness factors are given in the second table

Do the following:

1. Determine the projected MTBF and failure intensity.
2. Determine the growth potential MTBF and failure intensity.
3. Determine the demonstrated MTBF and failure intensity.

Test-Find-Test Data						
i	X_i	**Mode**		i	X_i	**Mode**
1	15	BD1		22	260.1	BD1
2	25.3	BD2		23	263.5	BD8
3	47.5	BD3		24	273.1	A

4	54	BD4		25	274.7	BD6
5	56.4	BD5		26	285	BD13
6	63.6	A		27	304	BD9
7	72.2	BD5		28	315.4	BD4
8	99.6	BD6		29	317.1	A
9	100.3	BD7		30	320.6	A
10	102.5	A		31	324.5	BD12
11	112	BD8		32	324.9	BD10
12	120.9	BD2		33	342	BD5
13	125.5	BD9		34	350.2	BD3
14	133.4	BD10		35	364.6	BD10
15	164.7	BD9		36	364.9	A
16	177.4	BD10		37	366.3	BD2
17	192.7	BD11		38	373	BD8
18	213	A		39	379.4	BD14
19	244.8	A		40	389	BD15
20	249	BD12		41	394.9	A
21	250.8	A		42	395.2	BD16

Effectiveness Factors for the Unique BD Modes			
BD Mode	**Number N_j**	**First Occurrence**	**EF d_i**
1	2	15.0	.67
2	3	25.3	.72
3	2	47.5	.77

4	2	54.0	.77
5	3	54.0	.87
6	2	99.6	.92
7	1	100.3	.50
8	3	112.0	.85
9	3	125.5	.89
10	4	133.4	.74
11	1	192.7	.70
12	2	249.0	.63
13	1	285.0	.64
14	1	379.4	.72
15	1	389.0	.69
16	1	395.2	.46

Solution

1. The maximum likelihood estimates of β_{BD} and λ_{BD} are determined to be:

$$\hat{\beta}_{BD} = \frac{M}{\sum^{M}{}_{i=1} \ln(\frac{T}{X_i})}$$
$$=0.7970$$
$$\hat{\lambda}_{BD} = 0.1350$$

The unbiased estimate of β is:

$$\overline{\beta}_{BD} = \frac{M-1}{M}\hat{\beta}_{BD}$$
$$=0.7472$$

Based on the test data, $\overline{d} = \frac{1}{M}\sum_{i=1}^{M} d_i = 0.72125$. Therefore, $B(T) = \overline{d}\frac{M\overline{\beta}_{BD}}{T} = 0.0215$. The projected failure intensity due to incorporating the 16 corrective actions is:

$$r(T) = \left(\frac{N_A}{T} + \sum_{i=1}^{M} (1 - d_i) \frac{N_i}{T} \right) + \bar{d} \left(\frac{M}{T} \bar{\beta}_{BD} \right)$$
$$= 0.0661$$

The projected MTBF is:

$$M\widehat{T}BF_P = [r(T)]^{-1} = 15.127$$

2. To estimate the maximum reliability that can be attained with this management strategy, use the following calculations.

$$N_A/T = 0.0250$$

$$\frac{1}{T} \sum_{i=1}^{16} (1 - d_i) N_i = 0.0196$$

The growth potential failure intensity is estimated by:

$$\hat{r}_{GP}(T) = \left(\frac{N_A}{T} + \sum_{i=1}^{M} (1 - d_i) \frac{N_i}{T} \right)$$
$$= 0.0250 + 0.0196$$
$$= 0.0446$$

The growth potential MTBF is:

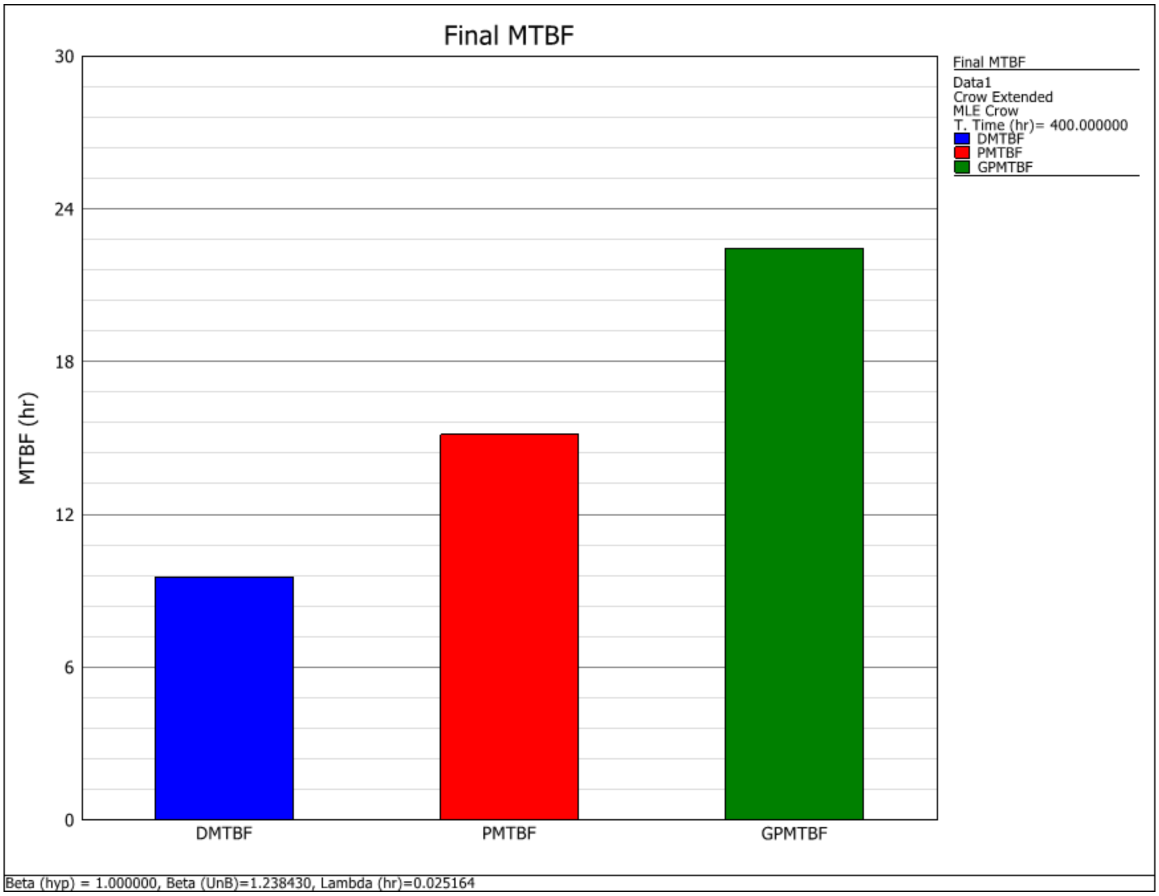
$$M\widehat{T}BF_{GP} = [\hat{r}_{GP}]^{-1} = 22.4467$$

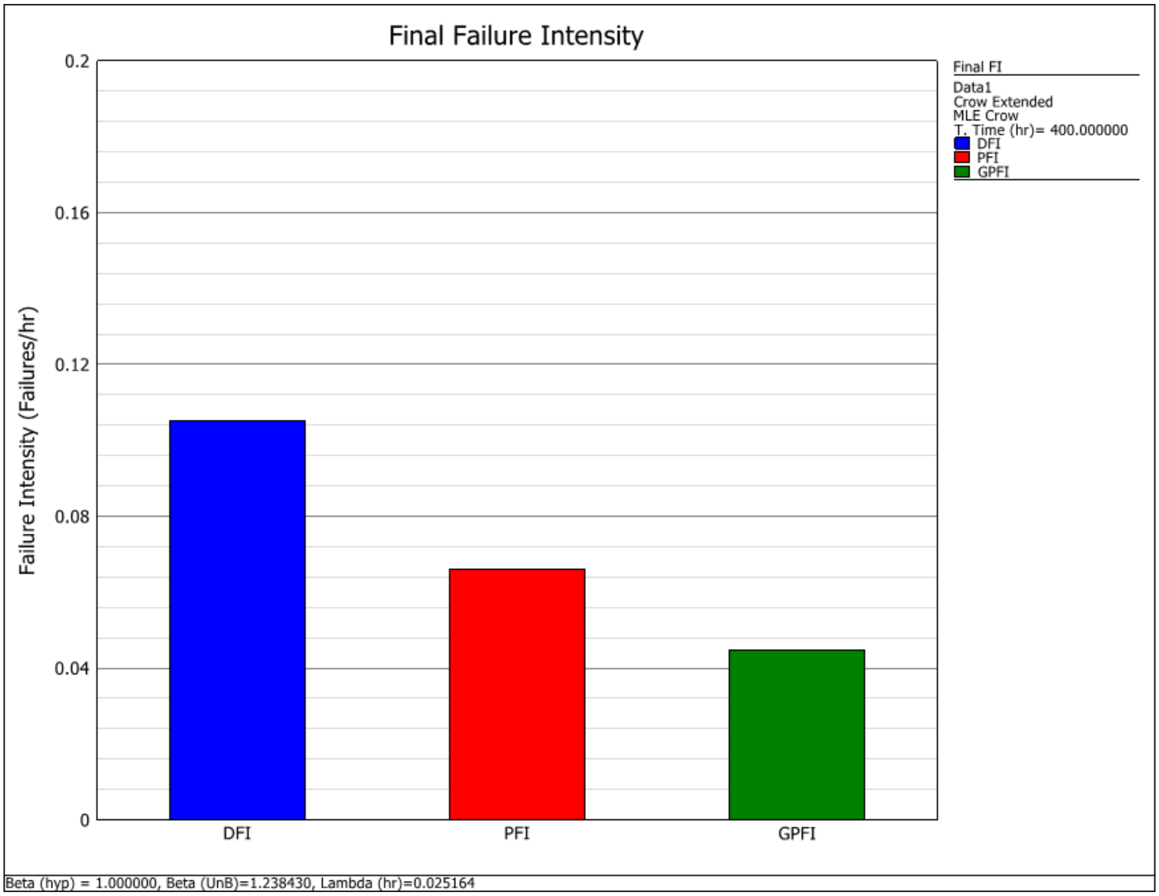
3. The demonstrated failure intensity and MTBF are estimated by:

$$\widehat{\lambda}_D(T) = \frac{N_A + N_{BD}}{T}$$
$$= \frac{42}{400}$$
$$= 0.1050$$

$$M\widehat{T}BF_D = [\widehat{\lambda}_D(T)]^{-1}$$
$$= 9.5238$$

The first chart below shows the demonstrated, projected and growth potential MTBF. The second shows the demonstrated, projected and growth potential failure intensity.


Final MTBF
30
24
18
12
6
0
MTBF (hr)
DMTBF
PMTBF
GPMTBF
Final MTBF
Data1
Crow Extended
MLE Crow
T. Time (hr)= 400.000000
DMTBF
PMTBF
GPMTBF
Beta (hyp) = 1.000000, Beta (UnB)=1.238430, Lambda (hr)=0.025164



Test-Fix-Find-Test

Traditional reliability growth models provide assessments for two types of testing and corrective action strategies: *test-fix-test* and *test-find-test*. In test-fix-test, failure modes are found during testing and corrective actions for these modes are incorporated during the test. Data from this type of test can be modeled appropriately with the Crow-AMSAA model. In test-find-test, modes are found during testing, but all of the corrective actions are delayed and incorporated after the completion of the test. Data from this type of test can be modeled appropriately with the Crow-AMSAA Projection model, which was described above in the Test-Find-Test section. However, a common strategy involves a combination of these two approaches, where some corrective actions are incorporated during the test and some corrective actions are delayed and incorporated at the end of the test. This strategy is referred to as *test-fix-find-test*. Data from this test can be modeled appropriately with the Crow Extended reliability growth model, which is described next.

Recall that B failure modes are all failure modes that will receive a corrective action. In order to provide the assessment and management metric structure for corrective actions during and after a test, two types of B modes are defined. BC failure modes are corrected during the test and BD failure modes are delayed until the end of the test. Type A failure modes are defined as before; (i.e., those failure modes that will not receive a corrective action, either during or at the end of the test).

Development of the Crow Extended Model

Let λ_{BD} denote the constant failure intensity for the BD failure modes, and let $h(t|BD)$ denote the first occurrence function for the BD failure modes. In addition, as before, let K be the number of BD failure modes, let d_i be the effectiveness factor for the i^{th} BD failure mode and let $\overline{d}$ be the average effectiveness factor.

The Crow Extended model projected failure intensity is given by:

$$\lambda_{EM} = \lambda_{CA} - \lambda_{BD} + \sum_{i=1}^{K}(1-d_i)\lambda_i + \overline{d}h(T|BD)$$

where $\lambda_{CA} = \lambda\beta T^{\beta-1}$ is the achieved failure intensity at time T.

The Crow Extended model projected MTBF is:

$$M_{EM} = 1/\lambda_{EM}$$

This is the MTBF after the delayed fixes have been implemented. Under the extended reliability growth model, the demonstrated failure intensity before the delayed fixes is the first term, λ_{CA}. The demonstrated MTBF at time T before the delayed fixes is given by:

$$M_{CA} = [\lambda_{CA}]^{-1}$$

If you assume that there are no delayed corrective actions (BD modes), then the model reduces to a special case of the Crow-AMSAA model where the achieved MTBF equals the projection, λ_{CA}. That is, there is no jump. If you assume that there are no corrective actions during the test (BC modes) then the model reduces to the test-find-test scenario described in the previous section.

Estimation of the Model

In the general estimation of the Crow Extended model, it is required that all failure times during the test are known. Furthermore, the ID of each A, BC and BD failure mode needs to be entered.

The estimate of the projected failure intensity for the Crow Extended model is given by:

$$\widehat{\lambda}_{EM} = \widehat{\lambda}_{CA} - \widehat{\lambda}_{BD} + \sum_{i=1}^{M}(1-d_i)\frac{N_i}{T} + \overline{d}\widehat{h}(T|BD)$$

where N_i is the total number of failures for the i^{th} BD mode and d_i is the corresponding assigned EF. In order to obtain the first term, $\widehat{\lambda}_{CA}$, fit all of the data (regardless of mode classification) to the Crow-AMSAA model to estimate $\widehat{\beta}$ and $\widehat{\lambda}$, thus:

$$\widehat{\lambda}_{CA} = \widehat{\lambda}\widehat{\beta}T^{\widehat{\beta}-1}$$

The remaining terms are analyzed with the Crow Extended model, which is applied only to the BD data.

$$\widehat{\lambda}_{BD} = \frac{N_{BD}}{T}$$

$$\widehat{h}(T|BD) = \widehat{\lambda}_{BD}\widehat{\beta}_{BD}T^{\widehat{\beta}_{BD}-1} = \frac{M\widehat{\beta}_{BD}}{T}$$

$\widehat{\beta}_{BD}$ is the unbiased estimated of β for the Crow-AMSAA model based on the first occurrence of M distinct BD modes.

The structure for the Crow Extended model includes the following special data analysis cases:

1. Test-fix-test with no failure modes known or with BC failure modes known. With this type of data, the Crow Extended model will take the form of the traditional Crow-AMSAA analysis.

2. Test-find-test with BD failure modes known. With this type of data, the Crow Extended model will take the form of the Crow-AMSAA Projection analysis described previously in the Test-Find-Test section.

3. Test-fix-find-test with BC and BD failure modes known. With this type of data, the full capabilities of the Crow Extended model will be applied, as described in the following sections.

Reliability Growth Potential and Maturity Metrics

The growth potential and some maturity metrics for the Crow Extended model are calculated as follows.

- Initial system MTBF and failure intensity are given by:

$$\widehat{M}_I = \frac{\Gamma\left(1+\frac{1}{\widehat{\beta}}\right)}{\widehat{\lambda}^{\frac{1}{\widehat{\beta}}}}$$

and:

$$\widehat{\lambda}_I = [\widehat{M}_I]^{-1}$$

where $\widehat{\beta}$ and $\widehat{\lambda}$ are the estimators of the Crow-AMSAA model for all data regardless of the failure mode classification (i.e., A, BC or BD).

- The A mode failure intensity and MTBF are given by:

$$\widehat{\lambda}_A = \frac{N_A}{T}$$

$$\widehat{M}_A = [\widehat{\lambda}_A]^{-1}$$

- The Initial BD mode failure intensity is given by:

$$\widehat{\lambda}_{BD} = \frac{N_{BD}}{T}$$

- The BC mode initial failure intensity and MTBF are given by:

$$\widehat{\lambda}_{I(BC)} = \widehat{\lambda}_I - \widehat{\lambda}_A - \widehat{\lambda}_{BD}$$

$$\widehat{M}_{I(BC)} = [\widehat{\lambda}_{I(BC)}]^{-1}$$

- Failure intensity $h(T|BC)$ and instantaneous MTBF $M(T|BC)$ for new BC failure modes at the end of test time T are given by:

$$\widehat{h}(T|BC) = \widehat{\lambda}\widehat{\beta}T^{\widehat{\beta}-1}$$

$$\widehat{M}(T|BC) = [\widehat{h}(T|BC)]^{-1}$$

where $\widehat{\beta}$ and $\widehat{\lambda}$ are the estimators of the Crow-AMSAA model for the first occurrence of distinct BC modes.

- Average effectiveness factor for BC failure modes is given by:

$$\hat{d}_{BC} = \frac{\left[\frac{N_{BC}^{\left(\frac{1}{\hat{\beta}_{BC}}\right)}}{\Gamma\left(1+\frac{1}{\hat{\beta}_{BC}}\right)}\right] - N_{BC}}{\left[\frac{N_{BC}^{\left(\frac{1}{\hat{\beta}_{BC}}\right)}}{\Gamma\left(1+\frac{1}{\hat{\beta}_{BC}}\right)}\right] - M_{BC}}$$

where N_{BC} is the total number of observed BC modes, M_{BC} is the number of unique BC modes and $\hat{\beta}_{BC}$ is the MLE for the first occurrence of distinct BC modes. If $\hat{\beta}_{BC} \geq 1$ then $\hat{d}_{BC}$ equals zero.

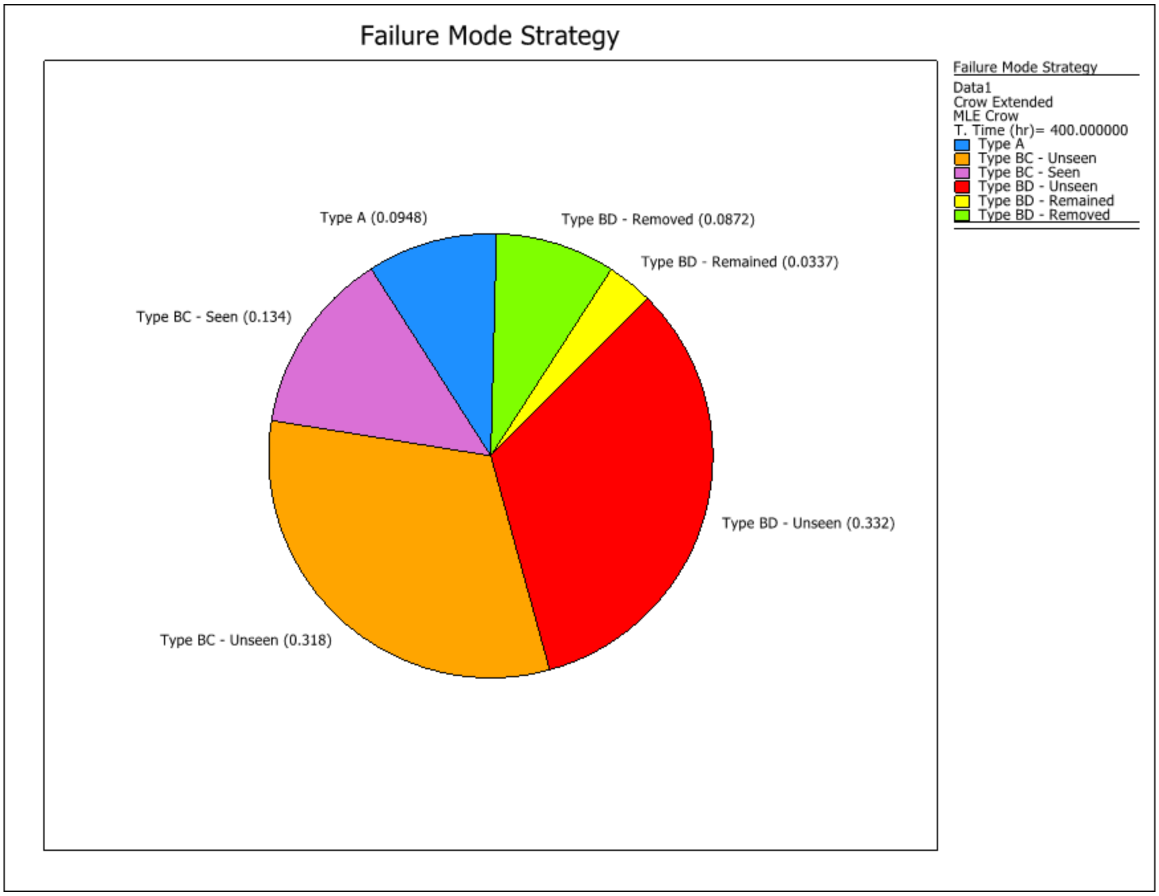
- Growth potential failure intensity and growth potential MTBF are given by:

$$\widehat{\lambda}_{GP} = \widehat{\lambda}_{CA} - \widehat{\lambda}_{BD} + \sum_{i=1}^{M} (1 - d_i) \frac{N_i}{T}$$

$$\widehat{M}_{GP} = [\widehat{\lambda}_{GP}]^{-1}$$

Failure Mode Management Strategy

Management controls the resources for corrective actions. Consequently, the effectiveness factors are part of the management strategy. For the BD mode failure intensity that has been seen during development testing, 100 d percent will be removed and 100 $(1 - d)$ percent will remain in the system. Therefore, after the corrective actions have been made, the current system instantaneous failure intensity consists of the failure intensity due to the A modes plus the failure intensity for the unseen BC modes, and plus the failure intensity for the unseen BD modes plus the failure intensity for the BD modes that have been seen. The following pie chart shows how the system's instantaneous failure intensity can be broken down into its individual pieces based on the current failure mode strategy.



Keep in mind that the individual components of the system's instantaneous failure intensity will depend on the classifications defined in the data. For example, if BC modes are not present within the data, then the BC mode MTBF will not be a part of the overall system MTBF. The individual pieces of the pie, as shown in the above figure, are calculated using the following equations.

Let:

$$\hat{r}(T) = \hat{\lambda}\hat{\beta}T^{\hat{\beta}-1}$$

where T is the test time and $\hat{\beta}$ and $\hat{\lambda}$ are the maximum likelihood estimates of the Crow-AMSAA model for all of the data. $\hat{\beta}$ is the biased estimate of β. Therefore:

$$\hat{\beta} = \frac{N}{\sum^{N}{}_{i=1} \ln\left(\frac{T}{X_i}\right)}$$

$$\hat{\lambda} = \frac{N}{T^{\hat{\beta}}}$$

where N is the total number of failures, and X_i is the i^{th} time-to-failure. Let the successive failures $0 < X_1 < X_2 < \ldots < X_3 < X_N$ be partitioned into the A mode failures (N_A), BC first occurrence failures (N_{BCF}), BC remaining failures (N_{BCR}), BD first occurrence failure (N_{BDF}) and the BD remaining failures (N_{BDR}). For continuous data, each portion of the pie chart, due to each of the modes, is calculated as follows:

- A modes

$$A = \left(\frac{T}{N^2}\right)\left[\sum_{i=1}^{N_A} \ln\left(\frac{T}{X_{Ai}}\right)\right]\hat{r}(T)$$

- BC modes unseen

$$BC_{unseen} = \left(\frac{T}{N^2}\right)\left[\sum_{i=1}^{N_{BCF}} \ln\left(\frac{T}{X_{BCFi}}\right)\right]\hat{r}(T)$$

- BC modes seen

$$BC_{seen} = \left(\frac{T}{N^2}\right)\left[\sum_{i=1}^{N_{BCR}} \ln\left(\frac{T}{X_{BCRi}}\right)\right]\hat{r}(T)$$

- BD modes unseen

$$BD_{unseen} = \left(\frac{T}{N^2}\right)\left[\sum_{i=1}^{N_{BDF}} \ln\left(\frac{T}{X_{BDFi}}\right)\right]\hat{r}(T)$$

- BD modes seen

$$BD_{seen} = \left(\frac{T}{N^2}\right)\left[\sum_{i=1}^{N_{BDR}} \ln\left(\frac{T}{X_{BDRi}}\right)\right]\hat{r}(T)$$

- BD modes remain

$$\begin{aligned} BD_{remain} &= \left(1 - \frac{1}{M}\sum_{i=1}^{M} d_i\right) \cdot BD_{seen} \\ &= \left(1 - \bar{d}\right) \cdot BD_{seen} \end{aligned}$$

- BD modes removed

$$\begin{aligned} BD_{removed} &= \frac{1}{M}\sum_{i=1}^{M} d_i \cdot BD_{seen} \\ &= \bar{d} \cdot BD_{seen} \end{aligned}$$

For grouped data, the maximum likelihood estimates of β and λ from the Crow-AMSAA (NHPP) model are calculated such that the following equations are satisfied:

$$\sum_{i=1}^{K} N_i \left[\frac{t_i^{\hat{\beta}} \ln(t_i) - t_{i-1}^{\hat{\beta}} \ln(t_{i-1})}{t_i^{\hat{\beta}} - t_{i-1}^{\hat{\beta}}} - \ln T\right] = 0$$

$$\hat{\lambda} = \frac{N}{T_K^{\hat{\beta}}}$$

where K is the number of groups and $N = \sum_{i=1}^{K} N_i$.

- A modes

$$A = \left(\frac{T}{N^2}\right)\left[N_A \ln(T) - \sum_{i=1}^{K} \frac{N_{Ai}}{\hat{\beta}}\left(\frac{t_i^{\hat{\beta}} \ln(t_i^{\hat{\beta}}) - t_{i-1}^{\hat{\beta}} \ln(t_{i-1}^{\hat{\beta}})}{t_i^{\hat{\beta}} - t_{i-1}^{\hat{\beta}}} - 1\right)\right] \hat{r}(T)$$

- BC modes unseen

$$BC_{unseen} = \left(\frac{T}{N^2}\right)\left[N_{BCF} \ln(T) - \sum_{i=1}^{K} \frac{N_{BCFi}}{\hat{\beta}}\left(\frac{t_i^{\hat{\beta}} \ln(t_i^{\hat{\beta}}) - t_{i-1}^{\hat{\beta}} \ln(t_{i-1}^{\hat{\beta}})}{t_i^{\hat{\beta}} - t_{i-1}^{\hat{\beta}}} - 1\right)\right] \hat{r}(T)$$

- BC modes seen

$$BC_{seen} = \left(\frac{T}{N^2}\right)\left[N_{BCR} \ln(T) - \sum_{i=1}^{K} \frac{N_{BCRi}}{\hat{\beta}}\left(\frac{t_i^{\hat{\beta}} \ln(t_i^{\hat{\beta}}) - t_{i-1}^{\hat{\beta}} \ln(t_{i-1}^{\hat{\beta}})}{t_i^{\hat{\beta}} - t_{i-1}^{\hat{\beta}}} - 1\right)\right] \hat{r}(T)$$

- BD modes unseen

$$BD_{unseen} = \left(\frac{T}{N^2}\right)\left[N_{BDF} \ln(T) - \sum_{i=1}^{K} \frac{N_{BDFi}}{\hat{\beta}}\left(\frac{t_i^{\hat{\beta}} \ln(t_i^{\hat{\beta}}) - t_{i-1}^{\hat{\beta}} \ln(t_{i-1}^{\hat{\beta}})}{t_i^{\hat{\beta}} - t_{i-1}^{\hat{\beta}}} - 1\right)\right] \hat{r}(T)$$

- BD modes seen

$$BD_{seen} = \left(\frac{T}{N^2}\right)\left[N_{BDR} \ln(T) - \sum_{i=1}^{K} \frac{N_{BDRi}}{\hat{\beta}}\left(\frac{t_i^{\hat{\beta}} \ln(t_i^{\hat{\beta}}) - t_{i-1}^{\hat{\beta}} \ln(t_{i-1}^{\hat{\beta}})}{t_i^{\hat{\beta}} - t_{i-1}^{\hat{\beta}}} - 1\right)\right] \hat{r}(T)$$

- BD modes remain

$$\begin{aligned} BD_{remain} &= \left(1 - \frac{1}{M}\sum_{i=1}^{M} d_i\right) \cdot BD_{seen} \\ &= \left(1 - \bar{d}\right) \cdot BD_{seen} \end{aligned}$$

- BD modes removed

$$BD_{removed} = \frac{1}{M}\sum_{i=1}^{M} d_i \cdot BD_{seen}$$
$$= \bar{d} \cdot BD_{seen}$$

Example - Test-Fix-Find-Test Data

Consider the data given in the first table below. There were 56 total failures and $T = 400$. The effectiveness factors of the unique BD modes are given in the second table. Determine the following:

1. Calculate the demonstrated MTBF and failure intensity.
2. Calculate the projected MTBF and failure intensity.
3. What is the rate at which unique BD modes are being generated during this test?
4. If the test continues for an additional 50 hours, what is the minimum number of new unique BD modes expected to be generated?

Test-Fix-Find-Test Data					
i	X_i	**Mode**	i	X_i	**Mode**
1	0.7	BC17	29	192.7	BD11
2	3.7	BC17	30	213	A
3	13.2	BC17	31	244.8	A
4	15	BD1	32	249	BD12
5	17.6	BC18	33	250.8	A
6	25.3	BD2	34	260.1	BD1
7	47.5	BD3	35	263.5	BD8
8	54	BD4	36	273.1	A
9	54.5	BC19	37	274.7	BD6
10	56.4	BD5	38	282.8	BC27
11	63.6	A	39	285	BD13

12	72.2	BD5	40	304	BD9
13	99.2	BC20	41	315.4	BD4
14	99.6	BD6	42	317.1	A
15	100.3	BD7	43	320.6	A
16	102.5	A	44	324.5	BD12
17	112	BD8	45	324.9	BD10
18	112.2	BC21	46	342	BD5
19	120.9	BD2	47	350.2	BD3
20	121.9	BC22	48	355.2	BC28
21	125.5	BD9	49	364.6	BD10
22	133.4	BD10	50	364.9	A
23	151	BC23	51	366.3	BD2
24	163	BC24	52	373	BD8
25	164.7	BD9	53	379.4	BD14
26	174.5	BC25	54	389	BD15
27	177.4	BD10	55	394.9	A
28	191.6	BC26	56	395.2	BD16

Effectiveness Factors for the Unique BD Modes	
BD Mode	**EF d_i**
1	.67
2	.72
3	.77
4	.77
5	.87

6	.92
7	.50
8	.85
9	.89
10	.74
11	.70
12	.63
13	.64
14	.72
15	.69
16	.46

Solution

1. In order to obtain $\widehat{\lambda}_{CA}$, use the traditional Crow-AMSAA model for test-fix-test to fit all 56 data points, regardless of the failure mode classification to get:

   $$\widehat{\beta} = 0.91026$$
   $$\widehat{\lambda} = 0.23969$$

   Thus the achieved or demonstrated failure intensity is estimated by:

   $$\begin{aligned}\widehat{\lambda}_{CA} &= \widehat{\lambda}\widehat{\beta}T^{\widehat{\beta}-1}\\ &= 0.23969 \times 0.91026 \times 400^{(0.91026-1)}\\ &= 0.12744\end{aligned}$$

   The achieved or demonstrated MTBF, M_{CA} , is the system reliability attained at the end of test, $T = 400$, and is estimated by:

   $$\widehat{M}_{CA} = [\widehat{\lambda}_{CA}]^{-1} = 7.84708$$

2. For this data set, $M = 16$ and $T = 400$.

   $$\widehat{\lambda}_{BD} = \frac{N_{BD}}{T} = \frac{32}{400} = 0.08$$

$$\overline{d}=\sum_{i=1}^{M}d_i/M=0.72125$$

$$\sum_{i=1}^{16}(1-d_i)N_i/T=0.01955$$

Calculate maximum likelihood estimates, $\hat{\beta}$ and $\hat{\lambda}$, of the BD modes:

$$\begin{aligned}\hat{\beta}_{BD} &= 0.74715\\ \hat{\lambda}_{BD} &= 0.18197\end{aligned}$$

Then:

$$\overline{d}\widehat{h}(T|BD)=0.0215$$

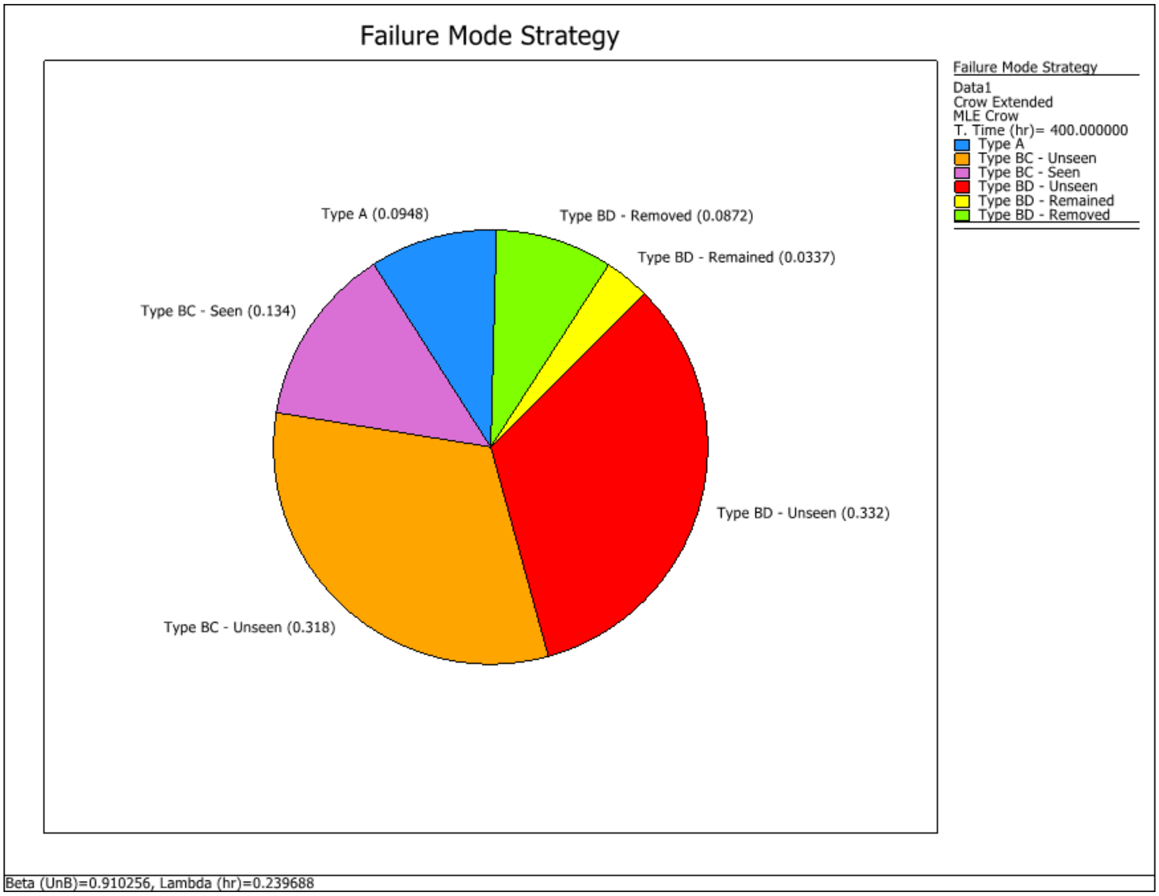
Therefore:

$$\begin{aligned}\widehat{\lambda}_{EM} &= \widehat{\lambda}_{CA}-\widehat{\lambda}_{BD}+\sum_{i=1}^{K}(1-d_i)\frac{N_i}{T}+\overline{d}\widehat{h}(T|BD)\\ &= 0.12744-0.08+0.0196+0.0215\\ &= 0.08854\end{aligned}$$

The Crow Extended model projected MTBF is:

$$\begin{aligned}\widehat{M}_{EM} &= [\widehat{\lambda}_{EM}]^{-1}\\ &= 11.29418\end{aligned}$$

Consequently, based on the Crow Extended model and the data shown in the tables above, the MTBF grew to 7.85 as a result of the corrective actions for the BC failure modes during the test. The MTBF then jumped to 11.29 after the test as a result of the delayed corrective actions for the BD failure modes. The management strategy can be summarized by the Failure Mode Strategy plot, as shown next.

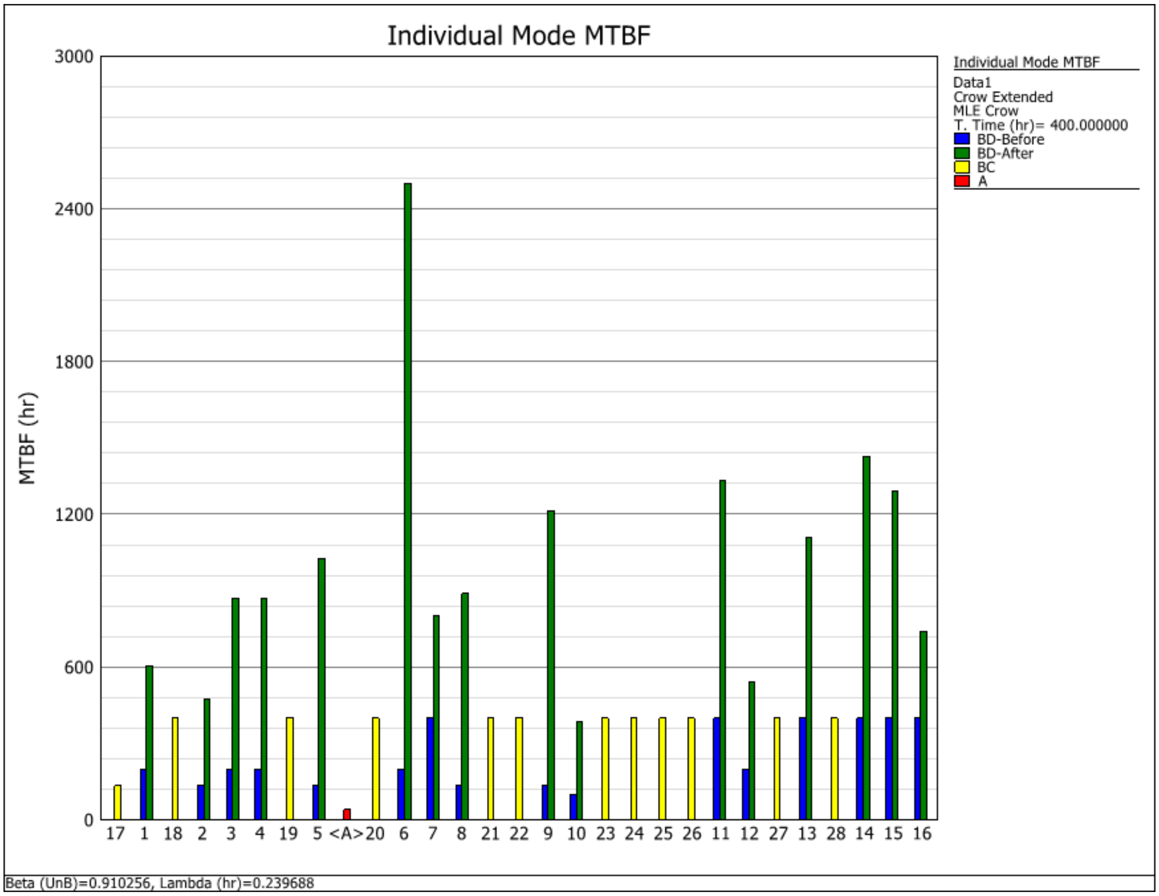


This pie chart shows that 9.48% of the system's failure intensity has been left in (A modes), 31.81% of the failure intensity due to the BC modes has not been seen yet and 13.40% was removed during the test (BC modes - seen). In addition, 33.23% of the failure intensity due to the BD modes has not been seen yet, 3.37% will remain in the system since the corrective actions will not be completely effective at eliminating the identified failure modes, and 8.72% will be removed after the delayed corrective actions.

3. The rate at which unique BD modes are being generated is equal to $h(T|BD)^{-1}$, where:

$$\begin{aligned} h(T|BD)^{-1} &= \frac{1}{\widehat{\lambda}_{BD}\widehat{\beta}_{BD}T^{\widehat{\beta}_{BD}-1}} \\ &= \frac{T}{M\widehat{\beta}_{BD}} \\ &= \mathbf{33.4605} \end{aligned}$$

4. Unique BD modes are being generated every 33.4605 hours. If the test continues for another 50 hours, then at least one new unique BD mode would be expected to be seen from this additional testing. As shown in the next figure, the MTBF of each individual failure mode can be plotted, and the failure modes with the lowest MTBF can be identified. These are the failure modes that cause the majority of the system failures.



Confidence Bounds

The Weibull++ software provides two methods to estimate the confidence bounds for the Crow Extended model when applied to developmental testing data. The Fisher Matrix approach is based on the Fisher Information Matrix and is commonly employed in the reliability field. The Crow bounds were developed by Dr. Larry Crow.

See the Crow Extended Confidence Bounds chapter for details on how these confidence bounds are calculated.

Confidence Bounds Example

Calculate the 2-sided 90% confidence bounds on the demonstrated, projected and growth potential failure intensity for the Test-Find-Test data given above.

Solution

The estimated demonstrated failure intensity is $\widehat{\lambda}_D(T) = \frac{N_A+N_B}{T} = 0.1050$. Based on this value, the Fisher Matrix confidence bounds for the demonstrated failure intensity at the 90% confidence level are:

$$\begin{aligned}[\lambda_D(T)]_L =& \hat{\lambda}_D(T) + \frac{C^2}{2} - \sqrt{\hat{\lambda}_D(T)C^2 + \frac{C^4}{4}} \\ =& 0.08152\end{aligned}$$

$$\begin{aligned}[\lambda_D(T)]_U =& \hat{\lambda}_D(T) + \frac{C^2}{2} + \sqrt{\hat{\lambda}_D(T)C^2 + \frac{C^4}{4}} \\ =& 0.13525\end{aligned}$$

The Crow confidence bounds for the demonstrated failure intensity at the 90% confidence level are:

$$\begin{aligned}[\lambda_D(T)]_L =& \widehat{\lambda}_D(T)\frac{\chi^2_{(2N,1-\alpha/2)}}{2N} \\ =& 0.07985 \\ [\lambda_D(T)]_U =& \widehat{\lambda}_D(T)\frac{\chi^2_{(2N,\alpha/2)}}{2N} \\ =& 0.13299\end{aligned}$$

The projected failure intensity is:

$$\begin{aligned}\hat{\lambda_p} &= \frac{N_i}{T} + \sum_{i=1}^{M}(1-d_i)\frac{N}{T} + \bar{d}\left(\frac{M}{T}\bar{\beta}\right) \\ &= 0.06611\end{aligned}$$

Based on this value, the Fisher Matrix confidence bounds at the 90% confidence level for the projected failure intensity are:

$$\begin{aligned}[\hat{\lambda}_P(T)]_L =& \hat{\lambda}_P(T)e^{z_\alpha\sqrt{Var(\hat{\lambda}_P(T))}/\hat{\lambda}_P(T)} \\ =& 0.04902\end{aligned}$$

$$\begin{aligned}[\hat{\lambda}_P(T)]_U =& \hat{\lambda}_P(T)e^{-z_\alpha\sqrt{Var(\hat{\lambda}_P(T))}/\hat{\lambda}_P(T)} \\ =& 0.08915\end{aligned}$$

The Crow confidence bounds for the projected failure intensity are:

$$[\lambda_P(T)]_L = \hat{\lambda}_P(T) + \frac{C^2}{2} - \sqrt{\hat{\lambda}_P(T) \cdot C^2 + \frac{C^4}{4}}$$
$$= 0.04807$$
$$[\lambda_P(T)]_U = \hat{\lambda}_P(T) + \frac{C^2}{2} + \sqrt{\hat{\lambda}_P(T) \cdot \ C^2 + \frac{C^4}{4}}$$
$$= 0.09090$$

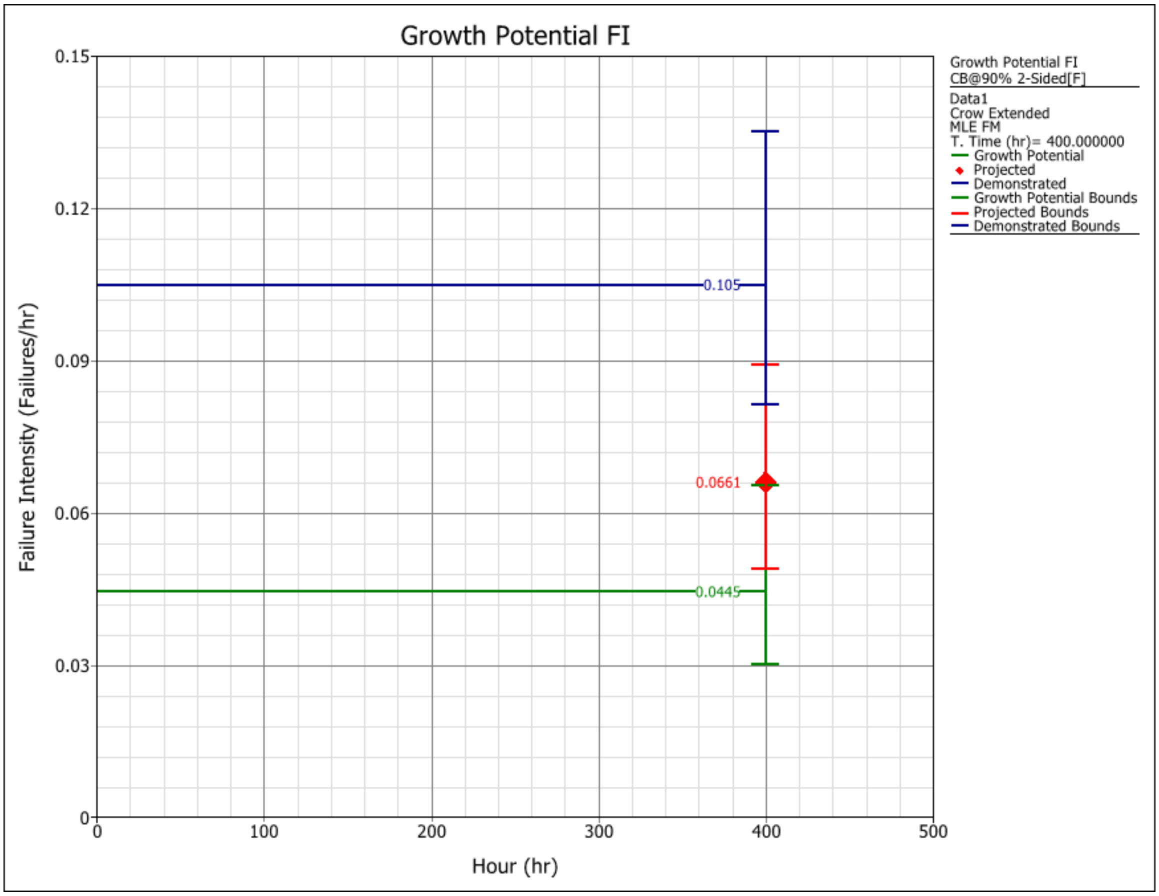
The growth potential failure intensity is:

$$\hat{r}_{GP}(T) = \left(\frac{N_A}{T} + \sum_{i=1}^{M} (1 - d_i) \frac{N_i}{T} \right) = 0.04455.$$

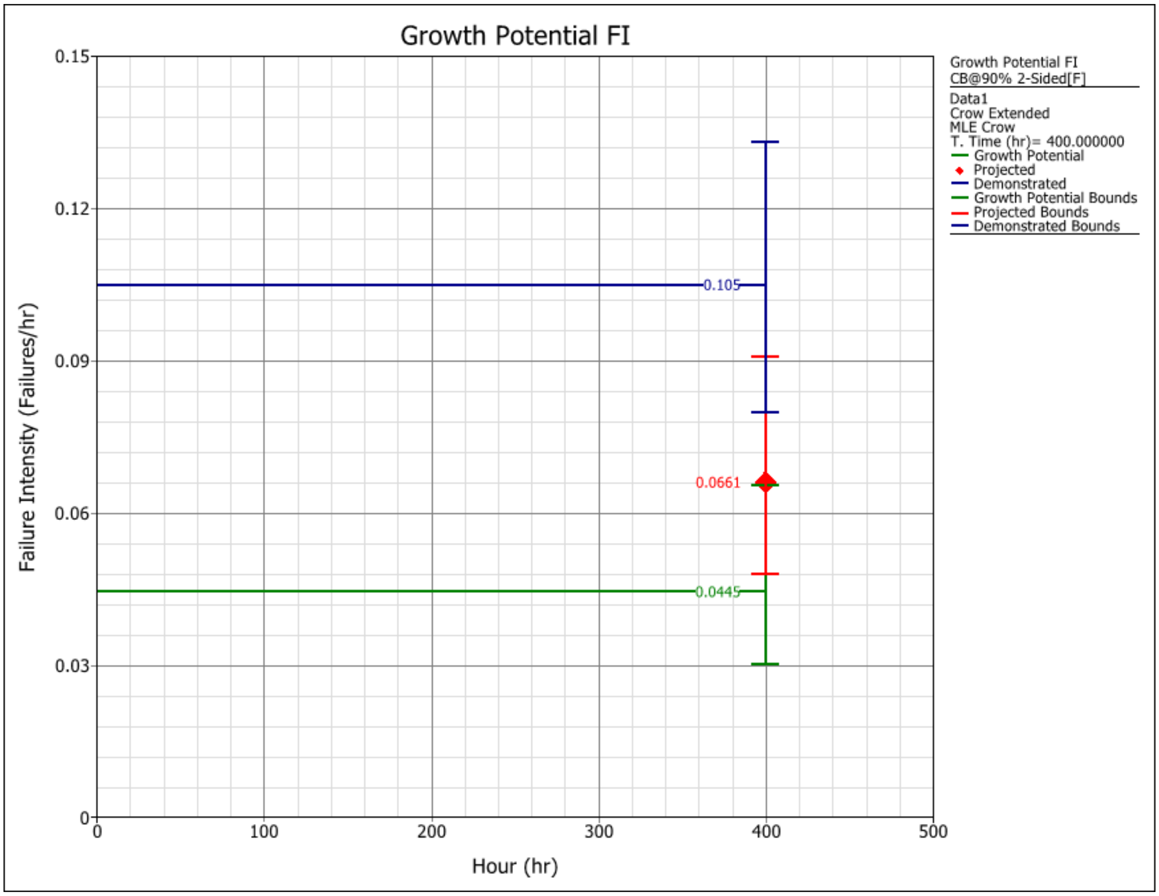
Based on this value, the Fisher Matrix and Crow confidence bounds at the 90% confidence level for the growth potential failure intensity are:

$$r_L = \hat{r}_{GP} + \frac{C^2}{2} - \sqrt{\hat{r}_{GP} C^2 + \frac{C^4}{4}}$$
$$= 0.03020$$
$$r_U = \hat{r}_{GP} + \frac{C^2}{2} + \sqrt{\hat{r}_{GP} C^2 + \frac{C^4}{4}}$$
$$= 0.0656$$

The figure below shows the Fisher Matrix confidence bounds at the 90% confidence level for the demonstrated, projected and growth potential failure intensity.



The following figure shows these bounds based on the Crow method.



Another Confidence Bounds Example

Calculate the 2-sided confidence bounds at the 90% confidence level on the demonstrated, projected and growth potential MTBF for the Test-Fix-Find-Test data given above.

Solution

For this example, there are A, BC and BD failure modes, so the estimated demonstrated failure intensity, $\hat{\lambda}_D(T)$, is simply the Crow-AMSAA model applied to all A, BC, and BD data.

$$\hat{\lambda}_D(T) = \widehat{\lambda}_{CA} = \widehat{\lambda}\widehat{\beta}T^{\widehat{\beta}-1} = 0.12744$$

Therefore, the demonstrated MTBF is:

$$MTBF_D = [\hat{\lambda}_D(T)]^{-1} = 7.84708$$

Based on this value, the Fisher Matrix confidence bounds for the demonstrated failure intensity at the 90% confidence level are:

$$
\begin{aligned}
[\lambda_D(T)]_L &= \hat{\lambda}_{CA}(T)e^{z_\alpha\sqrt{Var(\hat{\lambda}_{CA}(T))}/\hat{\lambda}_{CA}(T)} \\
&= 0.09339
\end{aligned}
$$

$$
\begin{aligned}
[\lambda_D(T)]_U &= \hat{\lambda}_{CA}(T)e^{-z_\alpha\sqrt{Var(\hat{\lambda}_{CA}(T))}/\hat{\lambda}_{CA}(T)} \\
&= 0.17390
\end{aligned}
$$

The Fisher Matrix confidence bounds for the demonstrated MTBF at the 90% confidence level are:

$$
\begin{aligned}
MTBF_{D_L} &= \frac{1}{[\lambda_D(T)]_U} \\
&= 5.75054 \\
MTBF_{D_U} &= \frac{1}{[\lambda_D(T)]_L} \\
&= 10.70799
\end{aligned}
$$

The Crow confidence bounds for the demonstrated MTBF at the 90% confidence level are:

$$
\begin{aligned}
MTBF_{D_L} &= \frac{1}{[\lambda_D(T)]_U} \\
&= \frac{1}{\widehat{\lambda}_D(T)\frac{\chi^2(2N,\alpha/2)}{2N}} \\
&= 5.6325 \\
MTBF_{D_U} &= \frac{1}{[\lambda_D(T)]_L} \\
&= \frac{1}{\widehat{\lambda}_D(T)\frac{\chi^2(2N,1-\alpha/2)}{2N}} \\
&= 10.8779
\end{aligned}
$$

The projected failure intensity is:

$$
\begin{aligned}
\hat{\lambda}_P(T) &= \widehat{\lambda}_{CA} - \widehat{\lambda}_{BD} + \sum_{i=1}^{M}(1-d_i)\frac{N_i}{T} + \bar{d}\widehat{h}(T|BD) \\
&= 0.0885
\end{aligned}
$$

Based on this value, the Fisher Matrix confidence bounds at the 90% confidence level for the projected failure intensity are:

$$
\begin{aligned}
[\lambda_P(T)]_L &= \hat{\lambda}_P(T)e^{z_\alpha\sqrt{Var(\hat{\lambda}_P(T))}/\hat{\lambda}_P(T)} \\
&= 0.0681
\end{aligned}
$$

$$
\begin{aligned}
[\lambda_P(T)]_U &= \hat{\lambda}_P(T)e^{-z_\alpha\sqrt{Var(\hat{\lambda}_P(T))}/\hat{\lambda}_P(T)} \\
&= 0.1152
\end{aligned}
$$

The Fisher Matrix confidence bounds for the projected MTBF at the 90% confidence level are:

$$\begin{aligned} MTBF_{P_L} &= \frac{1}{[\lambda_P(T)]_U} \\ &= 8.6818 \\ MTBF_{P_U} &= \frac{1}{[\lambda_P(T)]_L} \\ &= 14.6926 \end{aligned}$$

The Crow confidence bounds for the projected failure intensity are:

$$\begin{aligned} [\lambda_P(T)]_L &= \hat{\lambda}_P(T) + \frac{C^2}{2} - \sqrt{\hat{\lambda}_P(T) \cdot \ C^2 + \frac{C^4}{4}} \\ &= 0.0672 \\ [\lambda_P(T)]_U &= \hat{\lambda}_P(T) + \frac{C^2}{2} + \sqrt{\hat{\lambda}_P(T) \cdot C^2 + \frac{C^4}{4}} \\ &= 0.1166 \end{aligned}$$

The Crow confidence bounds for the projected MTBF at the 90% confidence level are:

$$\begin{aligned} MTBF_{P_L} &= \frac{1}{[\widehat{\lambda}_P(T)]_U} \\ &= 8.5743 \\ MTBF_{P_U} &= \frac{1}{[\widehat{\lambda}_P(T)]_L} \\ &= 14.8769 \end{aligned}$$

The growth potential failure intensity is:

$$\widehat{\lambda}_{GP} = \widehat{\lambda}_{CA} - \widehat{\lambda}_{BD} + \sum_{i=1}^{M}(1 - d_i)\frac{N_i}{T} = 0.0670$$

Based on this value, the Fisher Matrix and Crow confidence bounds at the 90% confidence level for the growth potential failure intensity are:

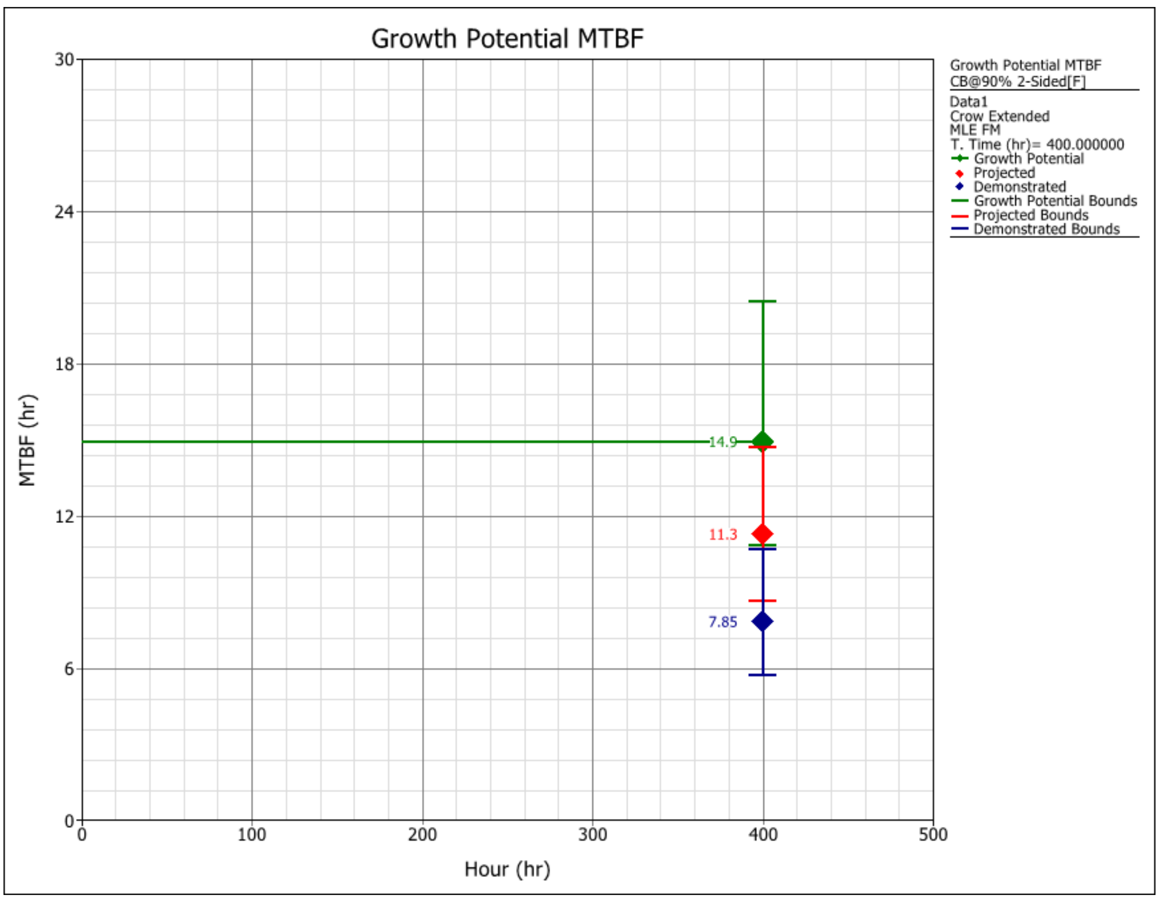
$$\begin{aligned} r_L &= \hat{r}_{GP} + \frac{C^2}{2} - \sqrt{\hat{r}_{GP}C^2 + \frac{C^4}{4}} \\ &= 0.0488 \\ r_U &= \hat{r}_{GP} + \frac{C^2}{2} + \sqrt{\hat{r}_{GP}C^2 + \frac{C^4}{4}} \\ &= 0.0919 \end{aligned}$$

The Fisher Matrix and Crow confidence bounds for the growth potential MTBF at the 90% confidence level are:

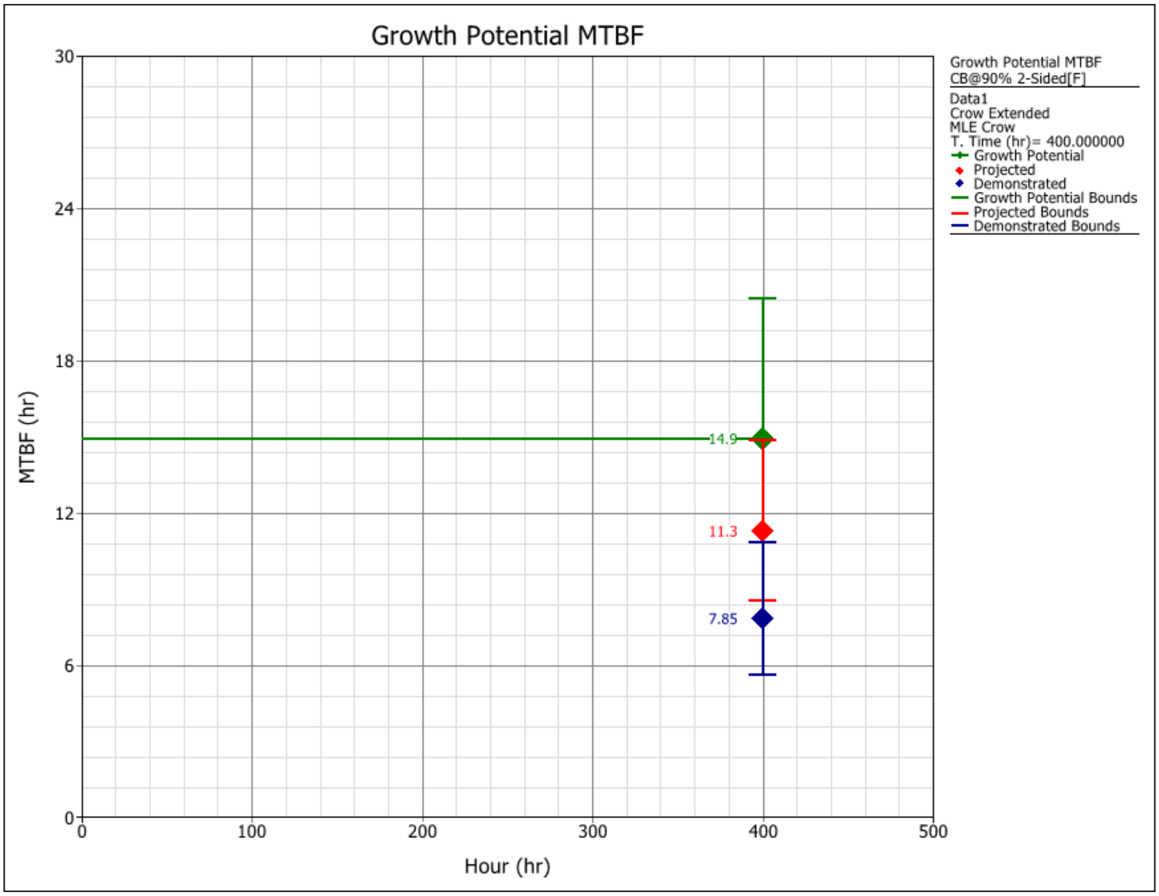
$$MTBF_{GP_L} = \frac{1}{r_U} = 10.8790$$

$$MTBF_{GP_U} = \frac{1}{r_L} = 20.4855$$

The figure below shows the Fisher Matrix confidence bounds at the 90% confidence level for the demonstrated, projected and growth potential MTBF.



The next figure shows these bounds based on the Crow method.



Grouped Data

Parameter estimation for grouped data using the Crow Extended model is the same as the procedure used for the traditional Crow-AMSAA (NHPP) model. The equations used to estimate the parameters of the Crow Extended model are presented next. For test-find-test data, the maximum likelihood estimates of λ_{BD} and β_{BD} are calculated using the first occurrences of the BD modes such that:

$$\sum_{i=1}^{k} n_i \left[\frac{T_i^{\widehat{\beta}} \ln T_i - T_{i-1}^{\widehat{\beta}} \ln T_{i-1}}{T_i^{\widehat{\beta}} - T_{i-1}^{\widehat{\beta}}} - \ln T_k \right] = 0$$

$$\widehat{\lambda} = \frac{n}{T_k^{\widehat{\beta}}}$$

where n_i is the number of distinct BD modes within the i^{th} interval. For test-fix-find-test data, the maximum likelihood estimates of λ_{BC} and β_{BC} are estimated in the same manner using the first occurrences of the BC modes.

Confidence Bounds for Grouped Data

- **Parameters:** The confidence bounds on the parameters for the Crow Extended model for grouped data are calculated using the same procedure presented in the Crow-AMSAA Confidence Bounds chapter.
- **Failure Intensity and MTBF:**
  - If there are no BC modes, the confidence bounds on the demonstrated failure intensity and MTBF, projected failure intensity and MTBF and growth potential failure intensity and MTBF are the same as the procedure presented for non-grouped data.
  - If there are BC modes, then the confidence bounds on the demonstrated failure intensity and MTBF are the same as the procedure presented in the Crow-AMSAA Confidence Bounds chapter, and the confidence bounds on the projected failure intensity and MTBF and growth potential failure intensity and MTBF are the same as for non-grouped data.
- **Time:** The confidence bounds on time are the same as the procedure presented in the Crow-AMSAA Confidence Bounds chapter.

Mixed Data

The Crow Extended model can also be applied to discrete data from one-shot (success/failure) testing. In the Weibull++ software, the **Discrete Data > Mixed Data** option creates a data sheet that can accommodate data from tests where a single unit is tested for each successive configuration (individual trial-by-trial), where multiple units are tested for each successive configuration (configurations in groups) or a combination of both. This data sheet can be analyzed with either the Crow-AMSAA (NHPP) model or the Crow Extended model.

For discrete data, corrective actions cannot take place at the time of failure. With that in mind, the mixed data type does not allow for BC modes. For discrete data there are only A or BD modes. In terms of practical applications, think of a growth test for missile systems. Because missiles are one-shot items, any corrective actions applied to the failure modes are delayed until at least the next trial.

Note that for calculation purposes, it is required to have at least three failures in the first interval. If that is not the case, then the data set needs to be grouped before calculating. The Weibull++ software performs this operation in the background.

Example

A one-shot system underwent reliability growth testing for a total of 20 trials. The test was performed as a combination of groups of units with the same configuration and individual trials.

The following table shows the data set. The **Failures in Interval** column specifies the number of failures that occurred in each interval and the **Cumulative Trials** column specifies the cumulative number of trials at the end of that interval. In other words, the first three rows contain the data from the first trial, in which 8 units with the same configuration were tested and 3 failures (with different failure modes) were observed. The next row contains data from the second trial, in which 2 units with the same configuration were tested and no failures occurred. And so on.

Mixed Data			
Failures in Interval	**Cumulative Trials**	**Classification**	**Mode**
1	8	BD	1
1	8	BD	2
1	8	BD	3
0	10		
0	11		
0	12		
1	13	BD	2
0	14		
0	15		
1	16	BD	4
0	17		
0	18		
0	19		
1	20	BD	5

The table also gives the classifications of the failure modes. There are 5 BD modes. The average effectiveness factor for the BD modes is 0.7. Do the following:

1. Calculate the demonstrated reliability at the end of the test.
2. Calculate the growth potential reliability.

Solution

1. Based on the equations presented in Crow-AMSAA (NHPP), the parameters of the Crow-AMSAA (NHPP) model are estimated as follows:

$$\widehat{\beta}=0.8572$$

and:

$$\widehat{\lambda}=0.4602$$

However, because there are only A or BD modes for mixed data, there is no growth during the test. In other words, the hypothesis for the $\hat{\beta}$ parameter is that $\hat{\beta}=1$. From the Crow-AMSAA (NHPP) model, we know that:

$$\hat{\lambda}=\frac{n}{T_n^{\hat{\beta}}}$$

or, if $\hat{\beta}=1$, this becomes:

$$\begin{aligned}\hat{\lambda}=&\frac{n}{T_n}\\=&\frac{6}{20}\\=&0.3\end{aligned}$$

As we have seen, the Crow-AMSAA instantaneous failure intensity, $\lambda_i(T)$, is defined as:

$$\lambda_i(T)=\lambda\beta T^{\beta-1}, \text{with } T>0,\ \lambda>0 \text{ and } \beta>0$$

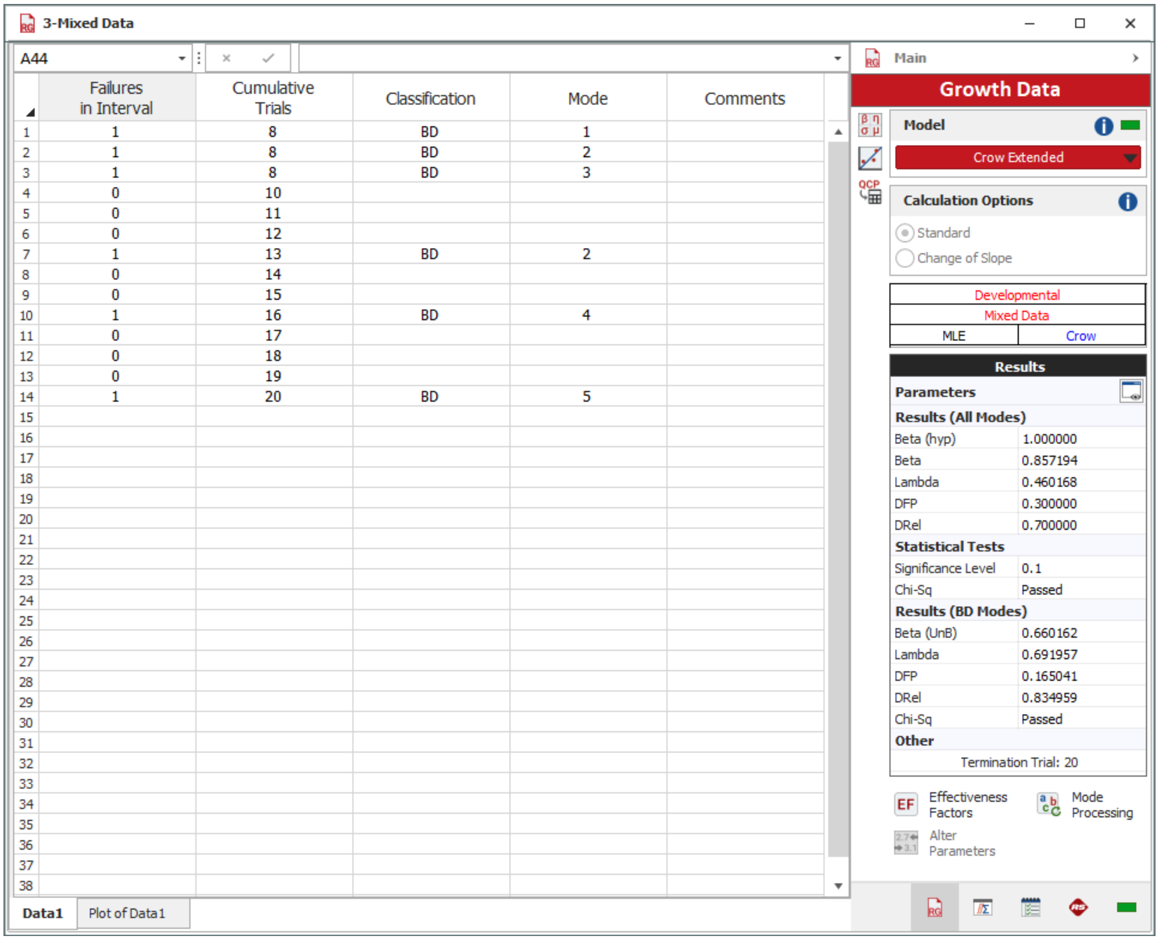
Using trials instead of time, and accommodating for $\hat{\beta}=1$, we can calculate the instantaneous failure probability at the end of the test, or $T=20$:

$$Q_i(20)=\widehat{\lambda}=0.3$$

So the instantaneous reliability at the end of the test, or demonstrated reliability, is:

$$\begin{aligned}R_i(20)=&1-Q_i(20)\\=&1-0.3\\=&0.7\end{aligned}$$

The next figure shows the data sheet as calculated in the Weibull++ software.



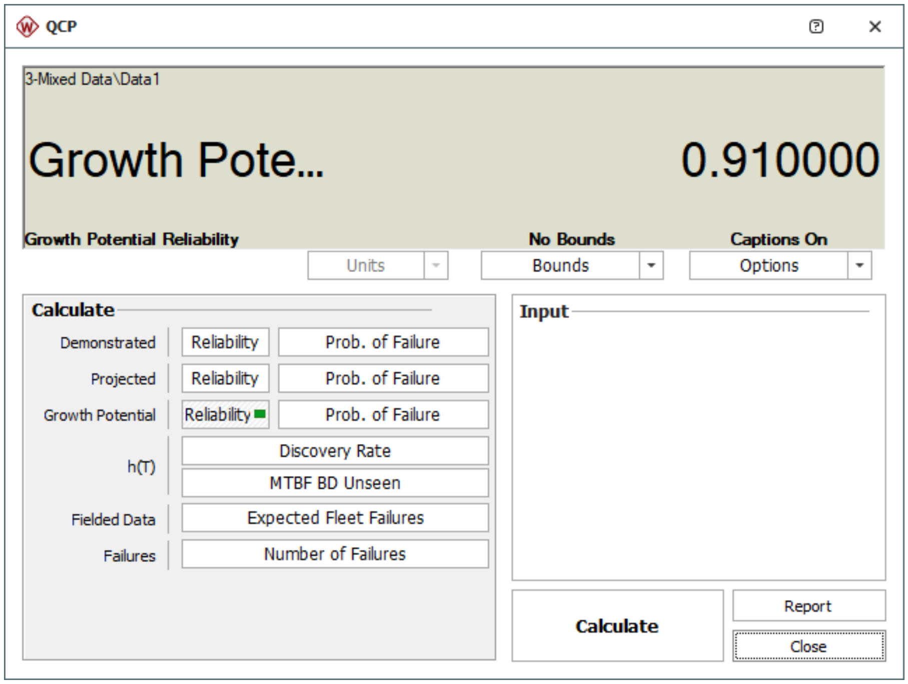
2. The growth potential unreliability is:

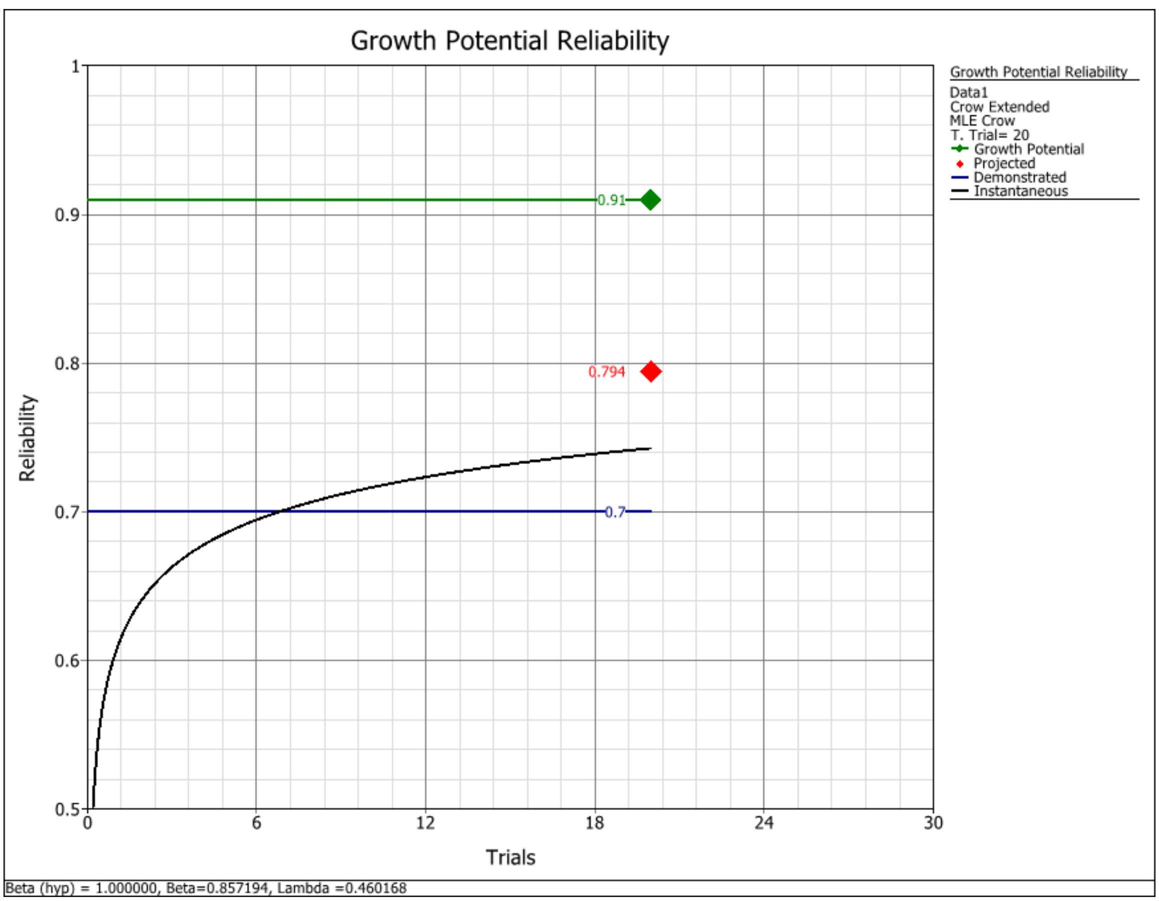
$$\begin{aligned}\widehat{Q}_{GP}(T) &= \left(\frac{N_A}{T} + \sum_{i=1}^{M}(1-d_i)\frac{N_i}{T}\right)\\ &= \sum_{i=1}^{M}(1-0.7)\frac{N_i}{T}\\ &= 0.3 * (\frac{1+1+1+1+1+1)}{20}\\ &= 0.09\end{aligned}$$

So the growth potential reliability is:

$$\begin{aligned}\widehat{R}_{GP}(T) &= 1-\widehat{Q}_{GP}(T)\\ &= 1-0.09\\ &= 0.91\end{aligned}$$

The figures below show the calculation of the growth potential reliability for the mixed data using the Weibull++ software's QCP, followed by the growth potential plot.


QCP
3-Mixed Data\Data1
Growth Pote...
0.910000
Growth Potential Reliability
No Bounds
Captions On
Units
Bounds
Options
Calculate
Demonstrated
Projected
Growth Potential
h(T)
Fielded Data
Failures
Reliability
Prob. of Failure
Discovery Rate
MTBF BD Unseen
Expected Fleet Failures
Number of Failures
Input
Calculate
Report
Close


Growth Potential Reliability
Reliability
Trials
0.91
0.794
0.7
Growth Potential Reliability
Data1
Crow Extended
MLE Crow
T. Trial= 20
Growth Potential
Projected
Demonstrated
Instantaneous
Beta (hyp) = 1.000000, Beta=0.857194, Lambda =0.460168

Multiple Systems with Event Codes

The Multiple Systems with Event Codes data type is used to analyze the failure data from a reliability growth test in which a number of systems are tested concurrently and the implemented fixes are tracked during the test phase. With this data type, all of the systems under test are assumed to have the same system hours at any given time. The Crow Extended model is used for this data type, so all the underlying assumptions regarding the Crow Extended model apply. As such, this data type is applicable only to data from within a single test phase.

As previously presented, the failure mode classifications for the Crow Extended model are defined as follows:

- **A** indicates that no corrective action was performed or will be performed (management chooses not to address for technical, financial or other reasons).
- **BC** indicates that the corrective action was implemented during the test. The analysis assumes that the effect of the corrective action was experienced during the test (as with other test-fix-test reliability growth analyses).
- **BD** indicates that the corrective action will be delayed until after the completion of the current test.

Therefore, implemented fixes can be applied only to BC modes since all BD modes are assumed to be delayed until the end of the test. For each BC mode, there must be a separate entry in the data set that records the time when the fix was implemented during the test.

Event Codes

A Multiple Systems with Event Codes data sheet that is analyzed with the Crow Extended model has an **Event** column that allows you to indicate the types of events that occurred during a test phase. The possible event codes that can be used in the analysis are:

- ***I***: denotes that a certain BC failure mode has been corrected at the specific time; in other words, a fix has been implemented. For this data type, each BC mode must have an associated *I* event. The *I* event is essentially a timestamp for when the fix was implemented during the test.
- ***Q***: indicates that the failure was due to a quality issue. An example of this might be a failure caused by a bolt not being tightened down properly. You have the option to decide whether or not to include quality issues in the analysis. This option can be specified by checking or clearing the **Include Q Events** check box under Event Code Options on the Analysis tab.

- ***P***: indicates that the failure was due to a performance issue. You can determine whether or not to include performance issues in the analysis. This option can be specified by checking or clearing the **Include P Events** check box under Event Code Options on the Analysis tab.
- ***X***: indicates that you wish to exclude the data point from the analysis. An *X* can be placed in front of any existing event code (e.g., XF to exclude a particular failure time) or entered by itself. The row of data with the *X* will not be included in the analysis.
- ***S***: indicates the system start time. This event code is only selectable in the Normal View.
- ***F***: indicates a failure time.
- ***E***: indicates the system end time. This event code is only selectable in the Normal View.

The analysis is based on the equivalent system that combines the operating hours of all the systems.

Equivalent Single System

In order to analyze a Multiple Systems with Event Codes data sheet, the data are converted into a Crow Extended equivalent single system. The implemented fixes (*I* events) are taken into account when building the equivalent single system from the data for multiple systems.

The basic assumptions and constraints for the use of this data type are listed below:

- Failure modes are assumed to be independent of each other and with respect to the system configuration. The same applies to their related implemented fixes (*I* events). As such, each mode and its related implemented fixes (*I* events) are examined separately in terms of their impact to the system configuration.
- If there are BC modes in the data set, there must be at least 3 unique BC modes to analyze the data (together with implemented fixes for each one of them).
- If there are BD modes in the data set, there must be at least 3 unique BD modes to analyze the data.
- To be consistent with the definition of BC modes in the Crow Extended model, every BC mode must have at least one implemented fix (*I* event) on at least one system.
- Implemented fixes (*I* events) cannot be delayed to a later phase, because the Crow Extended model applies to a single phase only.

The following list shows the basic rules for calculating the equivalent single system on which the Crow Extended model is applied. Note that the list is not exhaustive since there is an infinite number of scenarios that can occur. These rules cover the most common scenarios. The main concept is to add the time that each system was tested under the same configuration.

1. To get to the equivalent single system, each failure time for A modes and BD modes is calculated by adding the time that each system was tested under the same configuration. In practice this means multiplying the failure time in the system by the number of total systems under test. For example, if we have 4 total systems, and system 2 has a BD1 mode at time 30, the BD1 mode failure time in the equivalent single system would be $30*4=120$. If system 3 had another BD1 mode at time 40, then that would yield another BD1 mode in the equivalent single system at time $40*4=160$. These calculations are done assuming that the start time for the systems are at time zero. If the start time is different than zero, then that time would have to be subtracted from the failure time on each system. For example, if system 1 started at time S=10, and there was a failure at time 30, the equivalent system time would be $(30-10)*4=80$.

2. Each failure time for a BC mode that occurred before an implemented fix (*I* event) for that mode is also calculated by multiplying the failure time in the system by the number of total systems in test, as described above.

3. The implemented fix (*I* event) time in the equivalent single system is calculated by adding the test time invested in each system before that *I* event takes place. It is the total time that the system has spent at the same configuration in terms of that specific mode.

4. If the same BC mode occurs in another system after a fix (*I* event) has been implemented in one or more systems, the failure time in the equivalent single system is calculated by adding the test time for that BC mode, and one of the following for each of the other systems:

    - If a BC mode occurs in a system that has already seen an *I* event for that mode, then you add the time up to the *I* event.

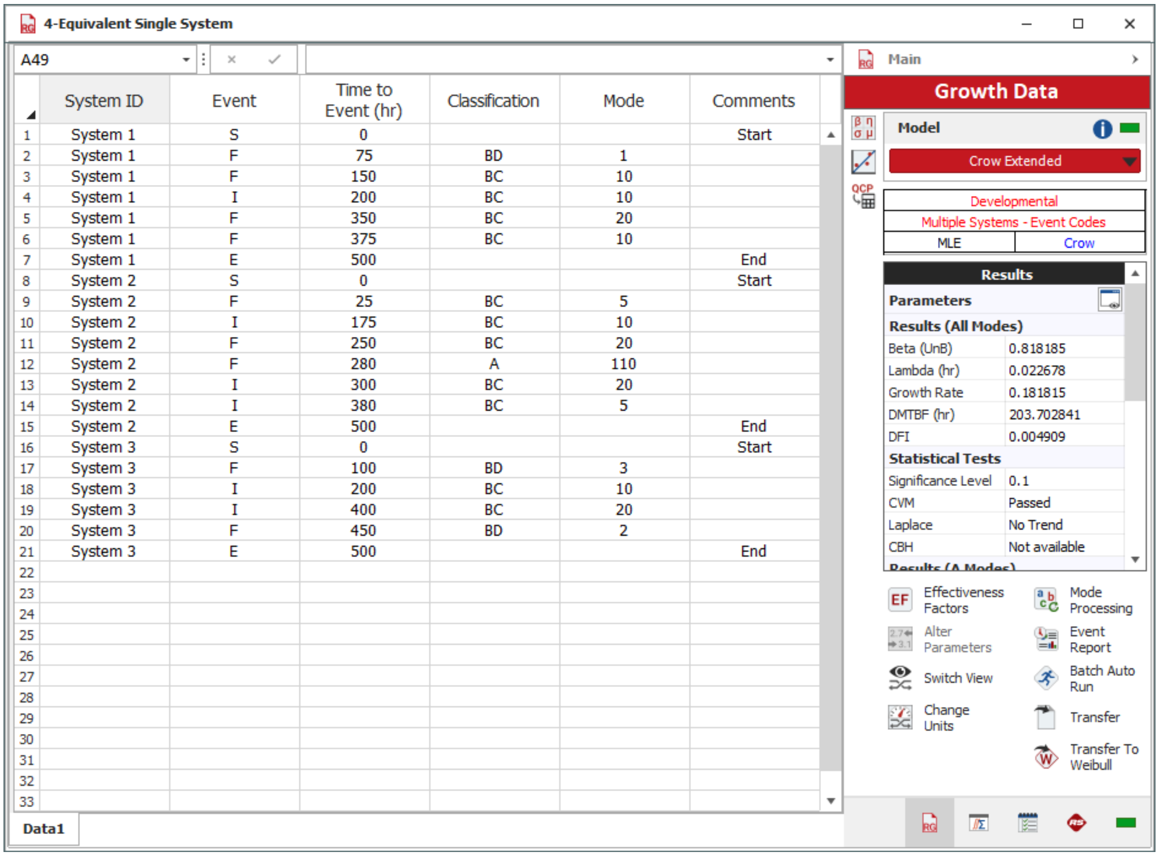
      or

    - If the *I* events occurred later than the BC failure time or those systems did not have any *I* events for that mode, then you add the time of the BC failure.

5. If the same BC mode occurs in the same system after a fix (*I* event) has been implemented in one or more systems, the failure time in the equivalent single system is calculated by adding the test time of each system after that *I* event was implemented to the *I* event time in the equivalent single system, or zero if an *I* event was not present in that system.

EXAMPLE: EQUIVALENT SINGLE SYSTEM

Consider the data set shown in the following figure.

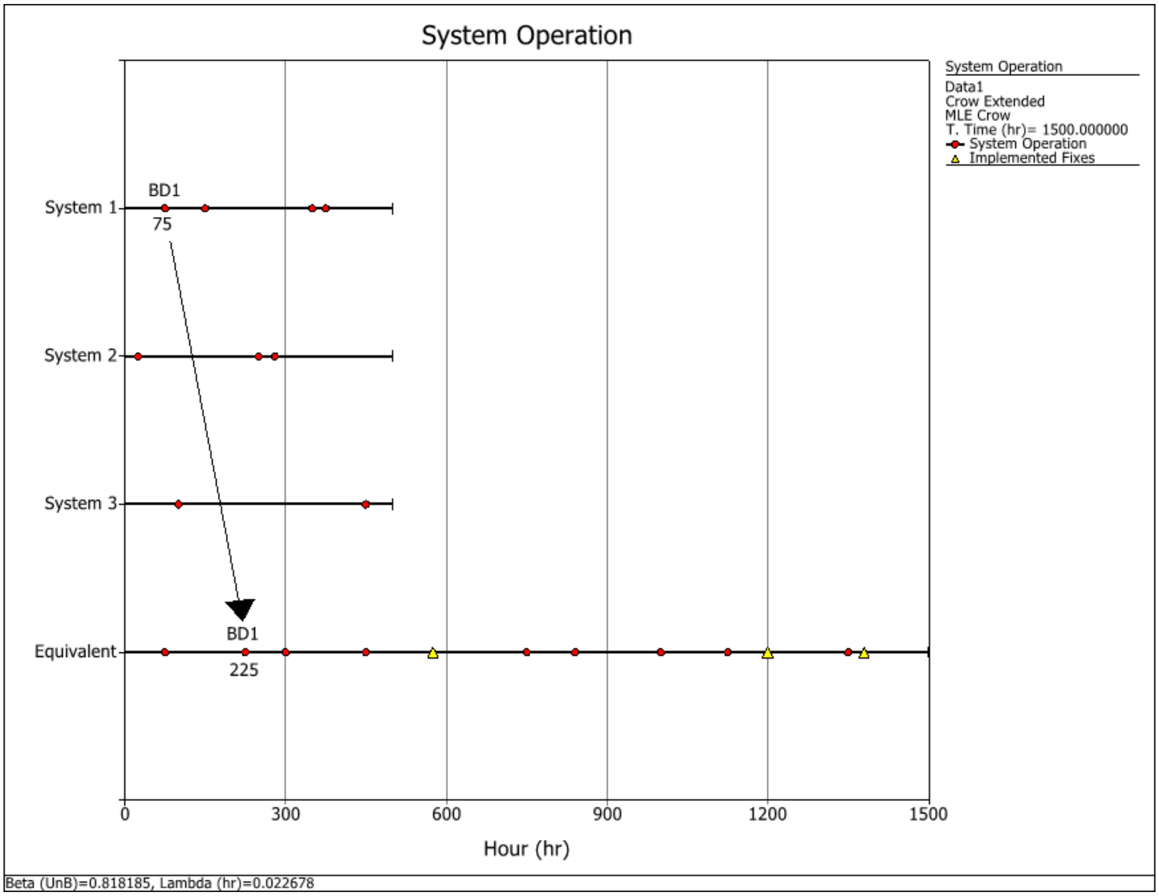


	System ID	Event	Time to Event (hr)	Classification	Mode	Comments
1	System 1	S	0			Start
2	System 1	F	75	BD	1	
3	System 1	F	150	BC	10	
4	System 1	I	200	BC	10	
5	System 1	F	350	BC	20	
6	System 1	F	375	BC	10	
7	System 1	E	500			End
8	System 2	S	0			Start
9	System 2	F	25	BC	5	
10	System 2	I	175	BC	10	
11	System 2	F	250	BC	20	
12	System 2	F	280	A	110	
13	System 2	I	300	BC	20	
14	System 2	I	380	BC	5	
15	System 2	E	500			End
16	System 3	S	0			Start
17	System 3	F	100	BD	3	
18	System 3	I	200	BC	10	
19	System 3	I	400	BC	20	
20	System 3	F	450	BD	2	
21	System 3	E	500			End

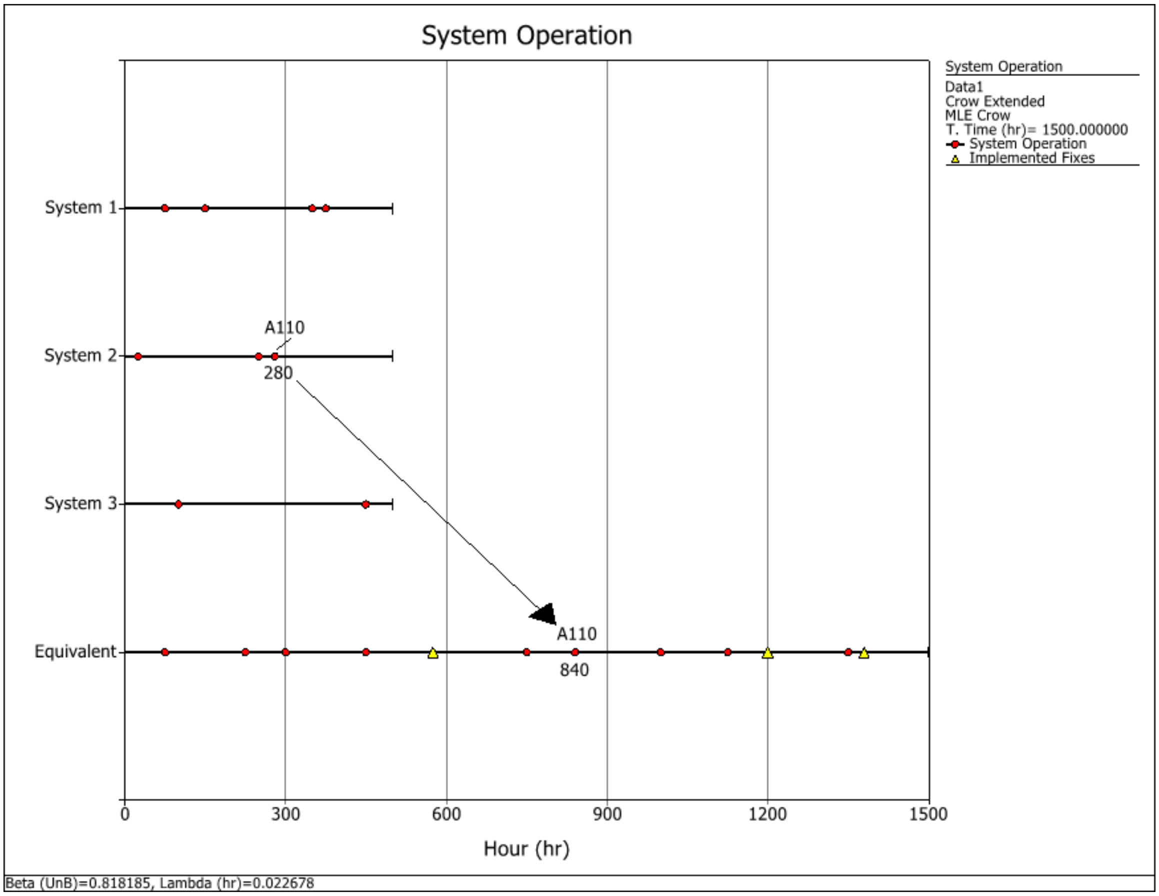
Solution

The first step to creating the equivalent system is to isolate each failure mode and its implemented fixes independently from each other. The numbered steps follow the five rules and are presented in the same numbering sequence.

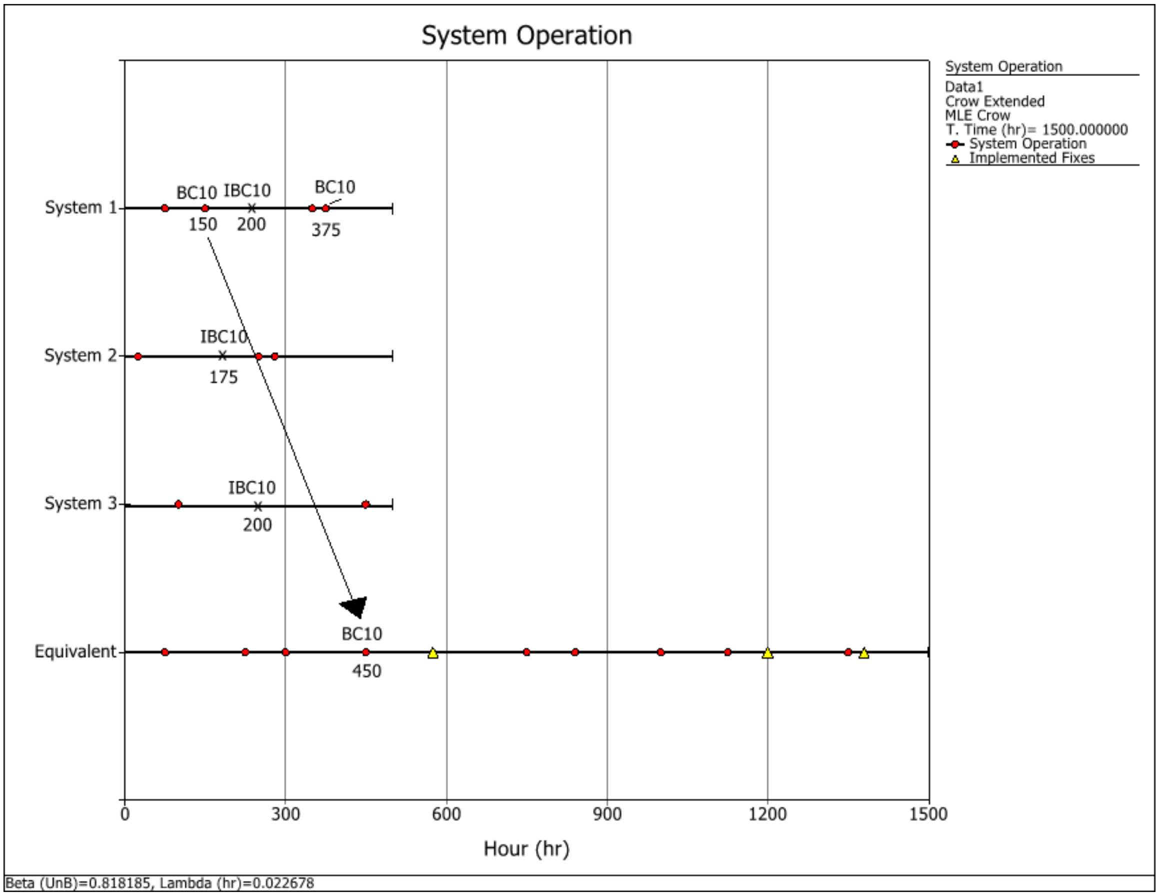
1. The next figure illustrates the application of rule #1 for mode BD1. The mode in the equivalent single system is calculated as $(75 + 75 + 75) = 225$ or $75 * 3 = 225$.



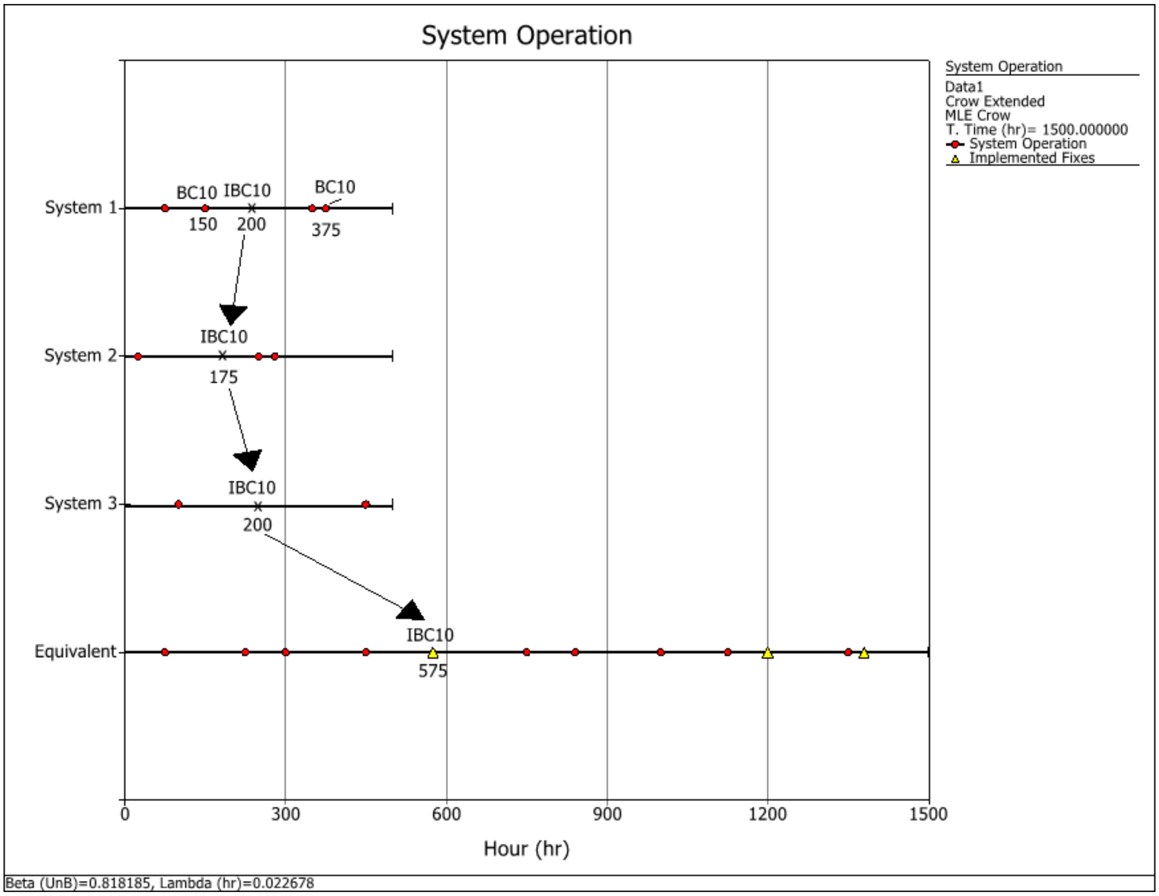
The next figure illustrates the application of rule #1 for mode A110. The mode in the equivalent single system is calculated as $(280+280+280)=840$ or $280*3=840$.



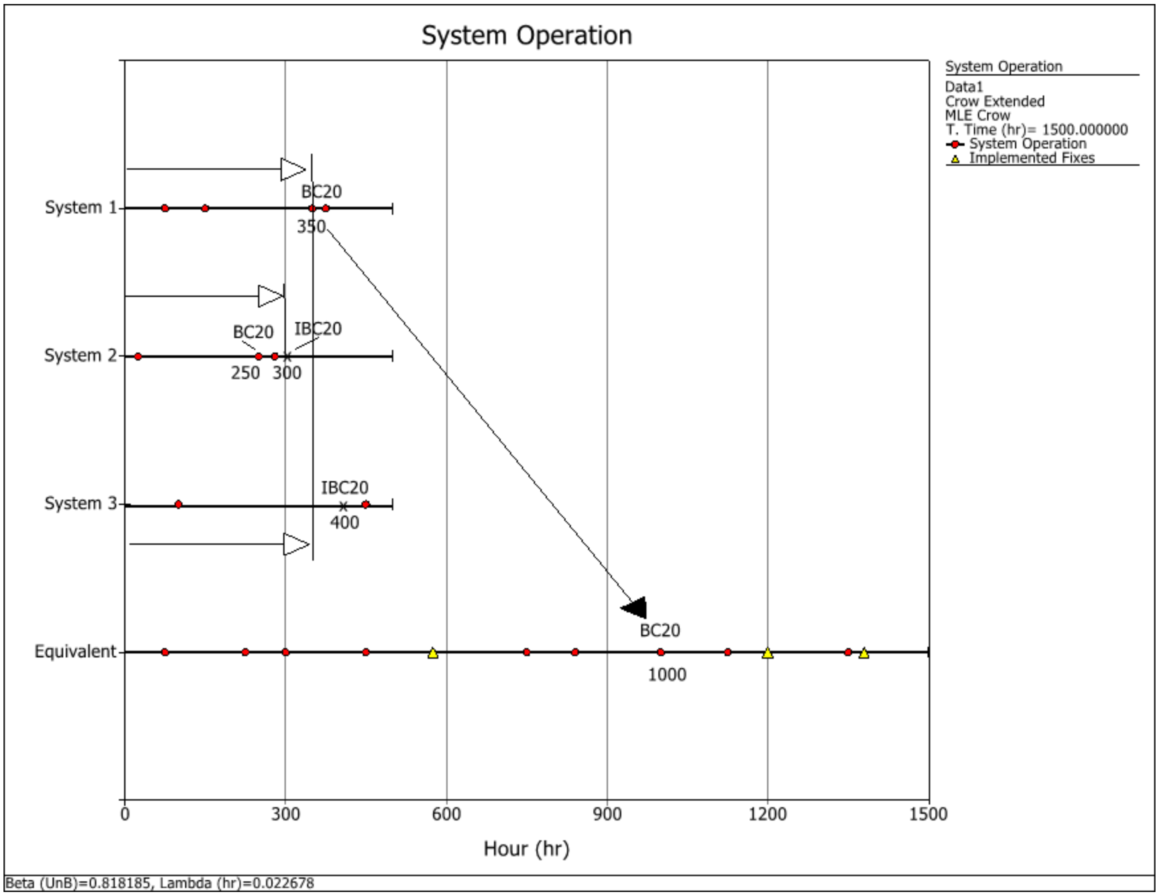
2. The next figure illustrates the application of rule #2 for the first occurrence of the mode BC10 in system 1. The mode in the equivalent single system is calculated as $(150+150+150)=450$ or $150*3=450$.



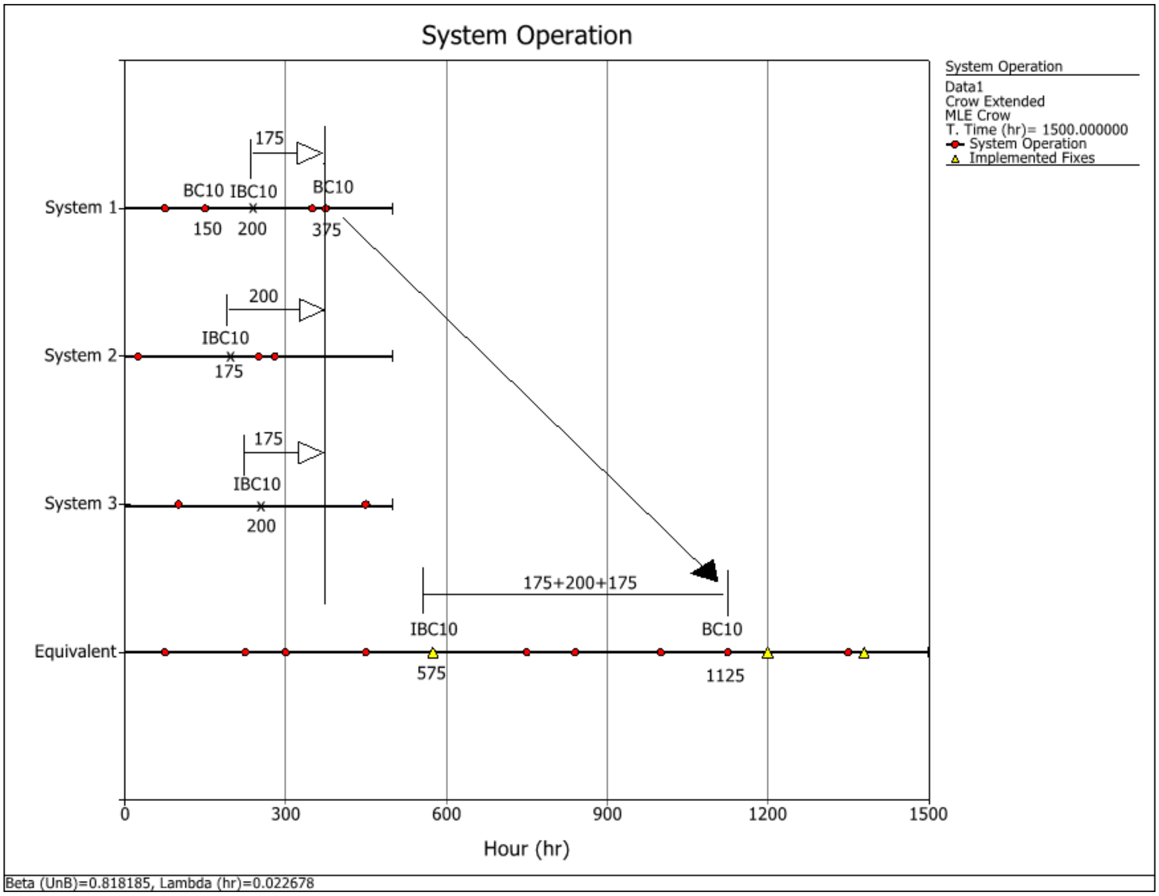
3. The next figure illustrates the application of rule #3 for implemented fixes (*I* events) of the mode BC10. In the graph, the *I* events are symbolized by having the letter "I" before the naming of the mode. In this case, IBC10 is for the implemented fix of mode BC10. The IBC10 in the equivalent single system is calculated as $(200 + 175 + 200) = 575$.



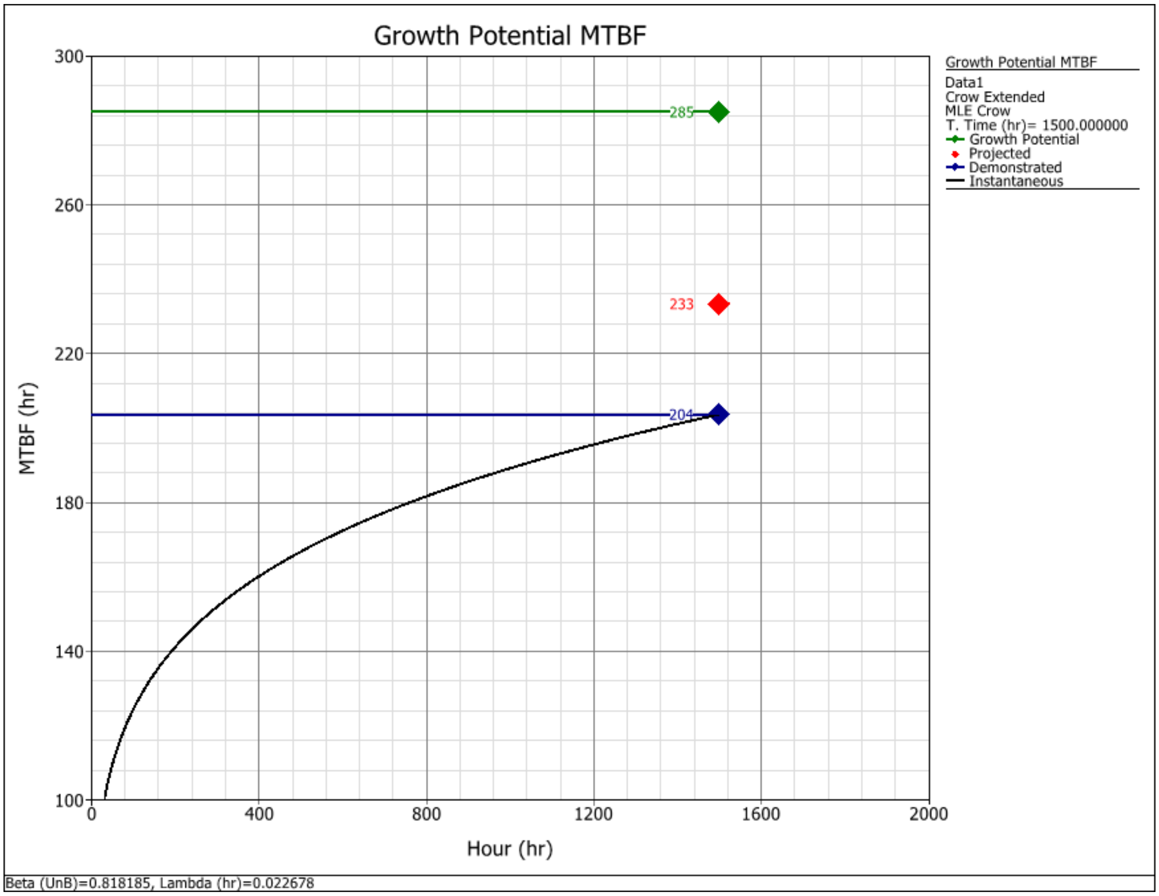
4. The next figure illustrates the application of rule #4 for the mode BC20 in system 1, which occurs after a fix for the same mode was implemented in system 2. The mode in the equivalent single system is calculated as $(350 + 300 + 350) = 1000$.



5. The next figure illustrates the application of rule #5 for the second occurrence of the mode BC10 in system 1, which occurs after an implemented fix (*I* event) has occurred for that mode in the same system. The mode in the equivalent single system is calculated as $575 + (175 + 200 + 175) = 1125$.

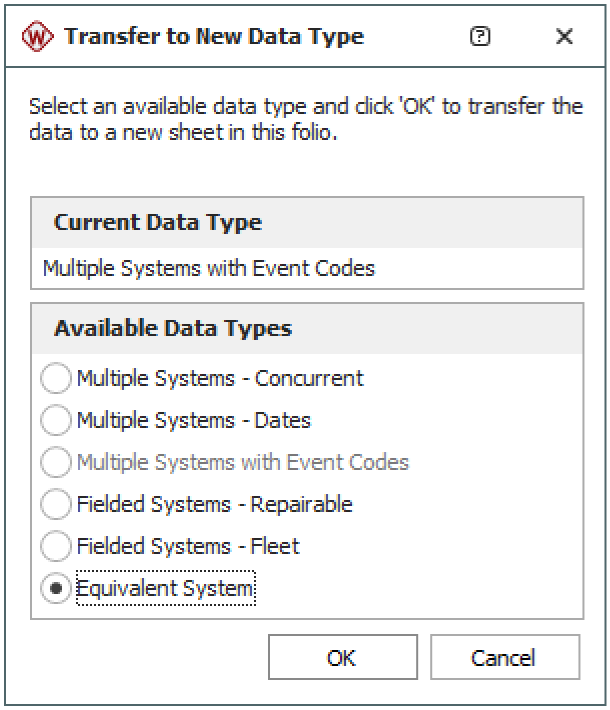


After having transferred the data set to the Crow Extended equivalent system, the data set is analyzed using the Crow Extended model. The last figure shows the growth potential MTBF plot.



Transferring Data to an Equivalent Single System

Weibull++ provides the capability to transfer a Multiple Systems with Event Codes data sheet to various other data types. The following picture shows the available data types that the data sheet can be converted into. When selecting to transfer to an equivalent single system, the data sheet is converted to a Crow Extended - Continuous Evaluation data sheet.

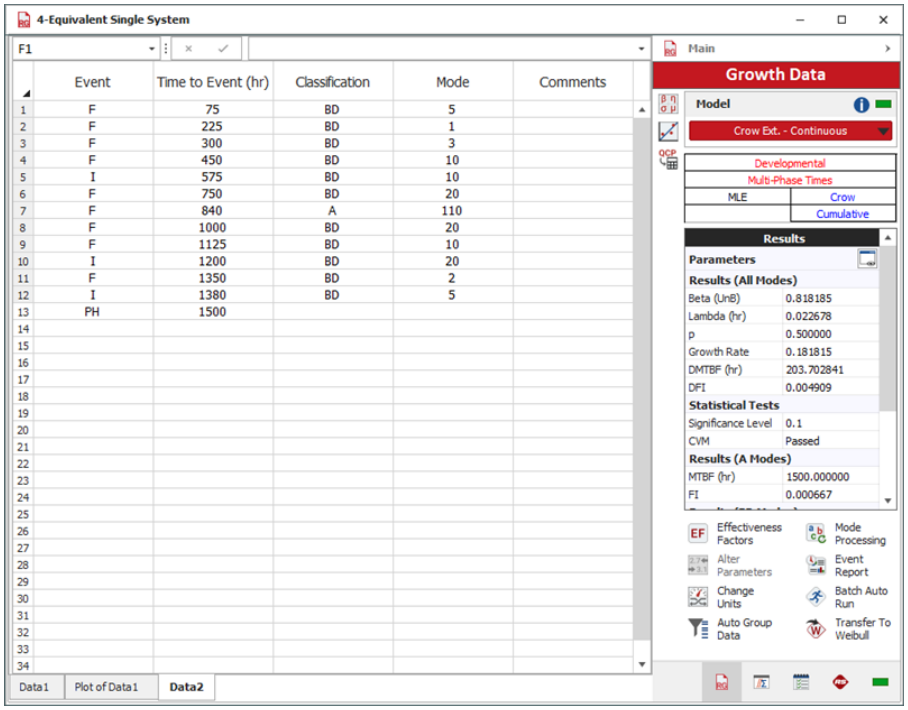


The Crow Extended - Continuous Evaluation model is designed for analyzing data across multiple test phases, while considering the data for all phases as one data set. Familiarity with this model is necessary for the discussion presented in this section.

When using the Crow Extended - Continuous Evaluation model to transfer the data sheet from Multiple Systems with Event Codes to an equivalent single system, the following rules are used (in addition to the five basic rules presented earlier for calculating the equivalent single system):

- BD modes in the Crow Extended data sheet become BD modes in the equivalent single system of the Crow Extended - Continuous Evaluation data sheet.
- BC modes in the Crow Extended data sheet become BD modes in the equivalent single system of the Crow Extended - Continuous Evaluation data sheet. These BD modes will have associated implemented fixes (*I* events). Implemented fixes (*I* events) for BC modes in the Crow Extended data sheet become implemented fixes (*I* events) for the converted BD modes in the equivalent single system of the Crow Extended - Continuous Evaluation data sheet.
- If an implemented fix (*I* event) occurred at the same time as the failure, and was implemented at that exact time across all systems, then this becomes a BC mode in the equivalent single system. If the fixes (*I* events) were not all implemented at the same time or if the fix was not implemented on all systems at the failure time, then this becomes a BD mode in the equivalent single system.

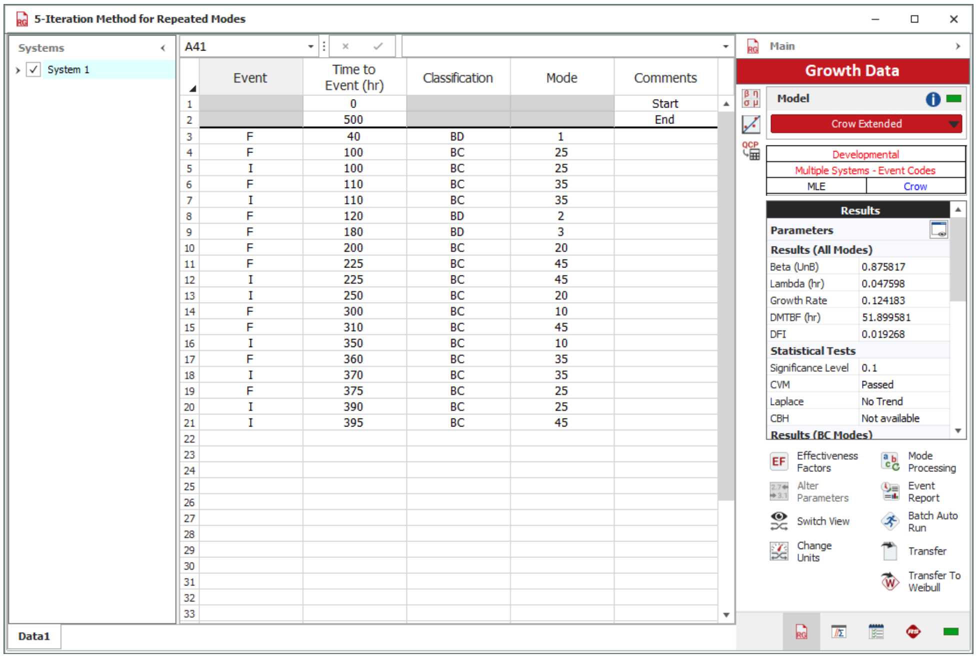
The next figure shows the transferred equivalent single system Crow Extended - Continuous Evaluation data sheet from the Multiple Systems with Event Codes data sheet for the data from the Equivalent Single System example given above.



Iteration Method for Naming Repeated Modes

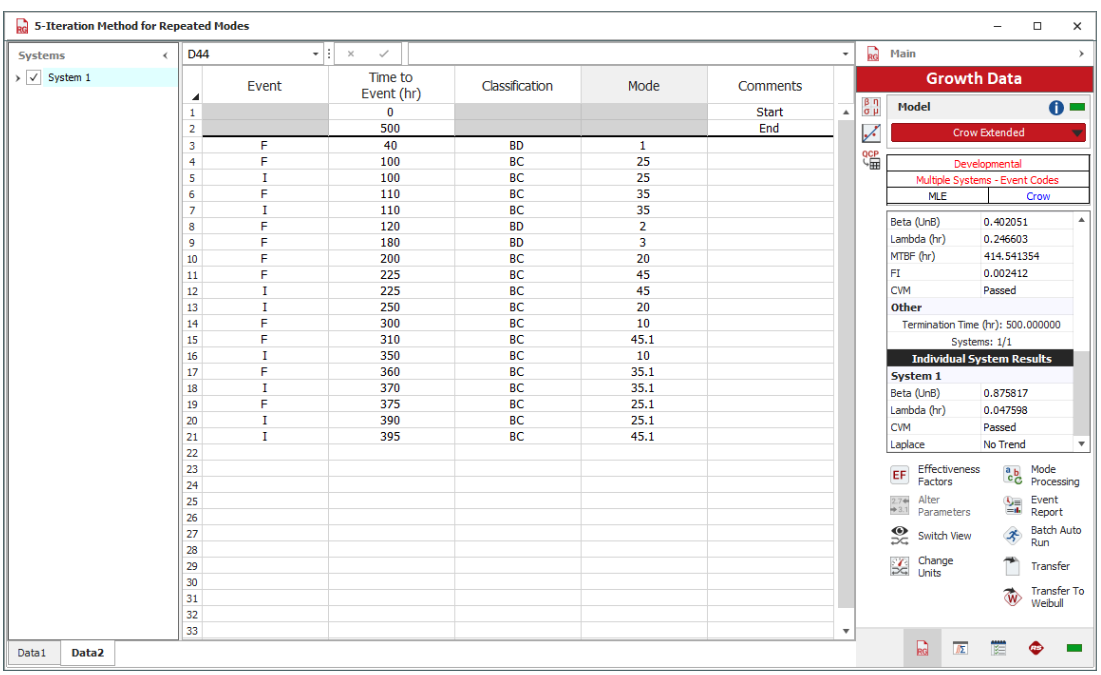
When recording modes for transfer from the Multiple Systems with Event Codes to a Crow Extended -Continuous Evaluation equivalent single system, it is recommended to consider using an iteration method to name subsequent recurrences of the same mode. This will help alleviate any issues with the conversion of the definitions of the modes from the Crow Extended model to the Crow Extended - Continuous Evaluation model. For example, if the first occurrence of a mode is BC25, then the second occurrence is suggested to be named as BC25.1. The reasoning behind this recommendation is that in the case that BC25 in the Multiple Systems with Event Codes data sheet has received implemented fixes (*I* events) at the same time that the failure occurred in all systems, then this mode will be translated as a BC mode in the Crow Extended - Continuous Evaluation equivalent single system. The next recurring failure would also be treated as a BC mode, but in reality it did not have an implemented fix (*I* event) at the time of failure.

For example, consider the data set shown in the following figure, which represents one system only for simplicity. Notice that the modes BC25, BC35 and BC45 received implemented fixes at the time of failure. Based on that, when they get transferred to the Crow Extended - Continuous Evaluation equivalent single system, they will be considered as BC modes. The subsequent failures of the modes 25, 35 and 45 will also be converted to BC modes, when in reality they had implemented fixes (*I* events) at a later time.

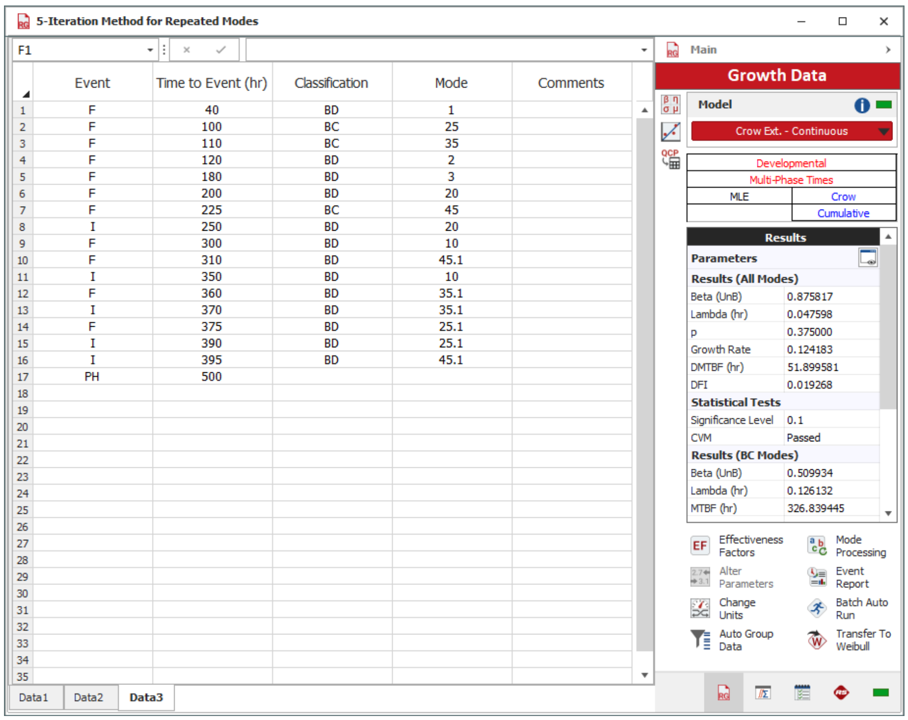


The Weibull++ software will display a warning if you try to convert this data sheet without using iterations.

The next figure shows the same data sheet with the use of iterations for the modes 25, 35 and 45. The subsequent failures are named as BC25.1, BC35.1 and BC45.1.



This way, the conversion to the Crow Extended - Continuous Evaluation model occurs in a valid fashion, because although the original BC modes are converted to BC25, BC35 and BC45, the subsequent failures are converted to BD25.1, BD35.1 and BD45.1 together with their respective implemented fixes (*I* events). This is shown in the next figure below. Note that the use of iterations is recommended only when transferring to the Crow Extended - Continuous Evaluation equivalent single system; it is not necessary when using the Multiple Systems with Event Codes data sheet that is calculated with the Crow Extended model.



More Examples

Adjusting the Failure Mode Management Strategy

Three systems were subjected to a reliability growth test to evaluate the prototype of a new product. Based on a failure analysis on the results of the test, the proposed management strategy is to delay corrective actions until after the test. The tables shown next display the data set and the associated effectiveness factors for the unique BD modes.

Multiple Systems (Concurrent Operating Times) Data			
	System 1	**System 2**	**System 3**
Start Time	0	0	0

End Time	541	454	436
Times-to-Failure	83 BD37	26 BD25	23 BD30
	83 BD43	26 BD43	46 BD49
	83 BD46	57 BD37	127 BD47
	169 A45	64 BD19	166 A2
	213 A18	169 A45	169 BD23
	299 A42	213 A32	213 BD7
	375 A1	231 BD8	213 BD29
	431 BD16	231 BD25	255 BD26
		231 BD27	369 A33
		231 A28	374 BD29
		304 BD24	380 BD22
		383 BD40	415 BD7

Effectiveness factors	
BD Mode	**Effectiveness Factor**
30	0.75
43	0.5
25	0.5
49	0.75
37	0.9
19	0.75
46	0.75
47	0.25

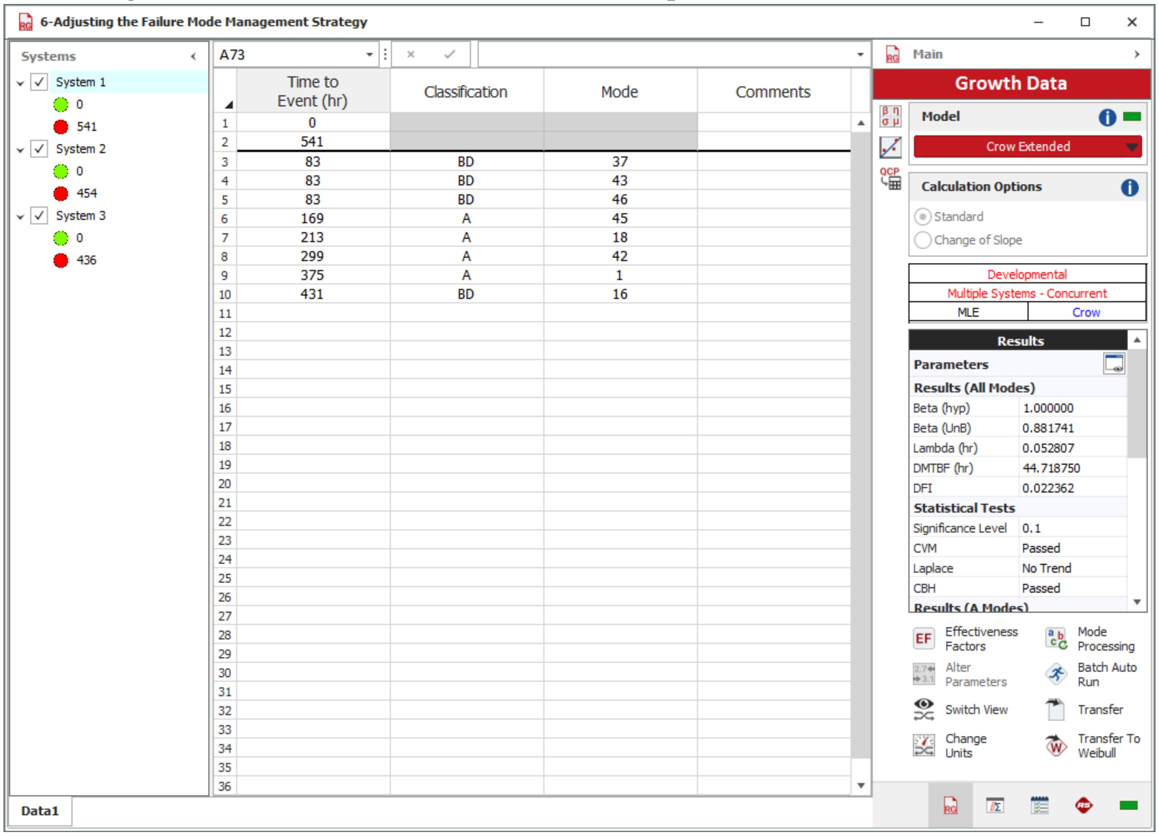
23	0.5
7	0.25
29	0.25
8	0.5
27	0.5
26	0.75
24	0.5
22	0.5
40	0.75
16	0.75

The prototype is required to meet a projected MTBF goal of 55 hours. Do the following:

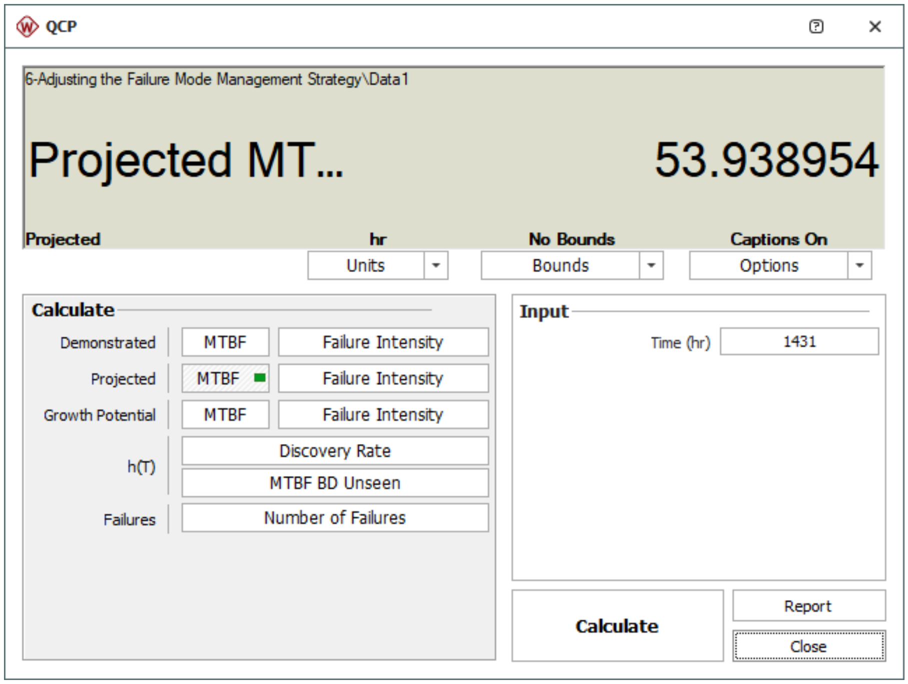
1. Estimate the parameters of the Crow Extended model.
2. Based on the current management strategy what is the projected MTBF?
3. If the projected MTBF goal is not met, alter the current management strategy to meet this requirement with as little adjustment as possible and without changing the EFs of the existing BD modes. Assume an EF = 0.7 for any newly assigned BD modes.

Solution

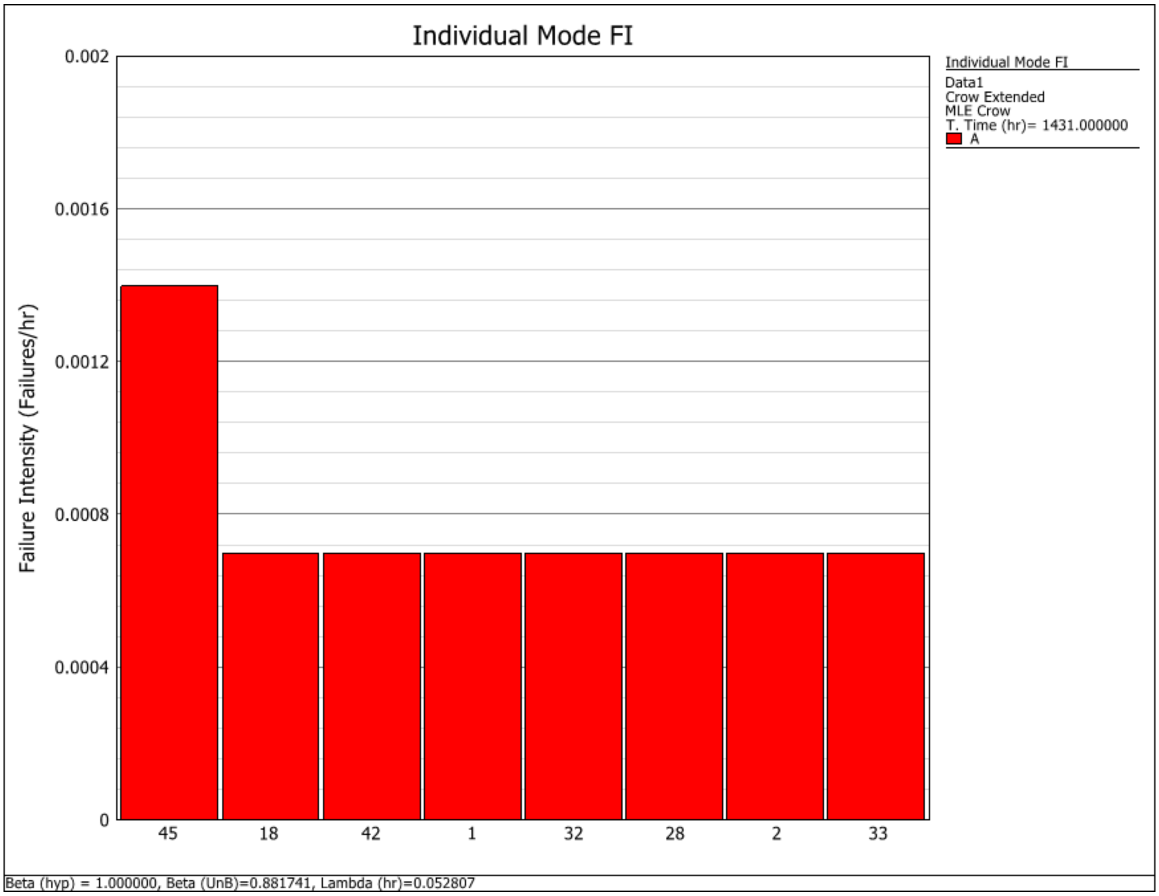
1. The next figure shows the estimated Crow Extended parameters.



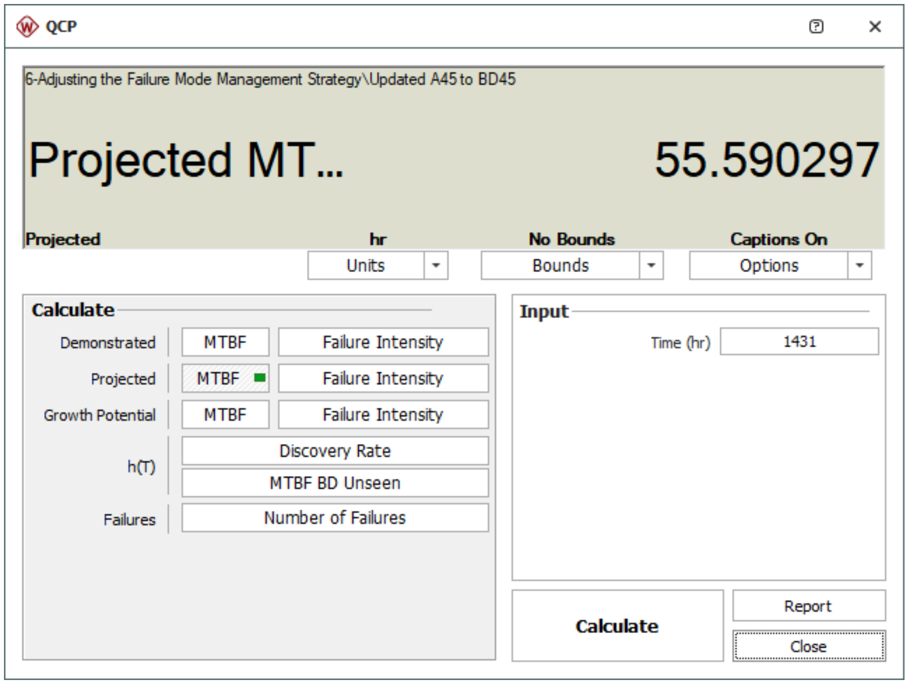
2. There are a couple of ways to calculate the projected MTBF, but the easiest is via the Quick Calculation Pad (QCP). The following result shows that the projected MTBF is estimated to be 53.9390 hours, which is below the goal of 55 hours.



3. To reach our goal, or to see if we can even get there, the management strategy must be changed. The effectiveness factors for the existing BD modes cannot be changed; however, it is possible to change an A mode to a BD mode, but which A mode(s) should be changed? To answer this question, create an Individual Mode Failure Intensity plot with just the A modes displayed, as shown next. As you can see from the plot, failure mode A45 has the highest failure intensity. Therefore, among the A modes, this particular failure mode has the greatest negative effect on the system MTBF.



Change A45 to BD45. Be sure to change all instances of A45 to a BD mode. Enter an effectiveness factor of 0.7 for BD45, and then recalculate the parameters of the Crow Extended model. Now go back to the QCP to calculate the projected MTBF, as shown below. The projected MTBF is now estimated to be 55.5903 hours. Based on the change in the management strategy, the projected MTBF goal is now expected to be met.



Estimating the Failure Intensity Remaining After Fixes

A reliability growth test was conducted for 200 hours. Some of the corrective actions were applied during the test while others were delayed until after the test was completed. The tables below give the data set and the effectiveness factors for the BD modes. Do the following:

1. Estimate the parameters of the Crow Extended model.
2. Determine the average effectiveness factor of the BC modes using the Function Wizard.
3. What percent of the failure intensity will be left in the system due to the BD modes after implementing the delayed fixes?

Grouped Failure Times Data			
Number at Event	**Time to Event**	**Classification**	**Mode**
3	25	BC	1
1	25	BD	9
1	25	BC	2
1	50	BD	10
1	50	BD	11

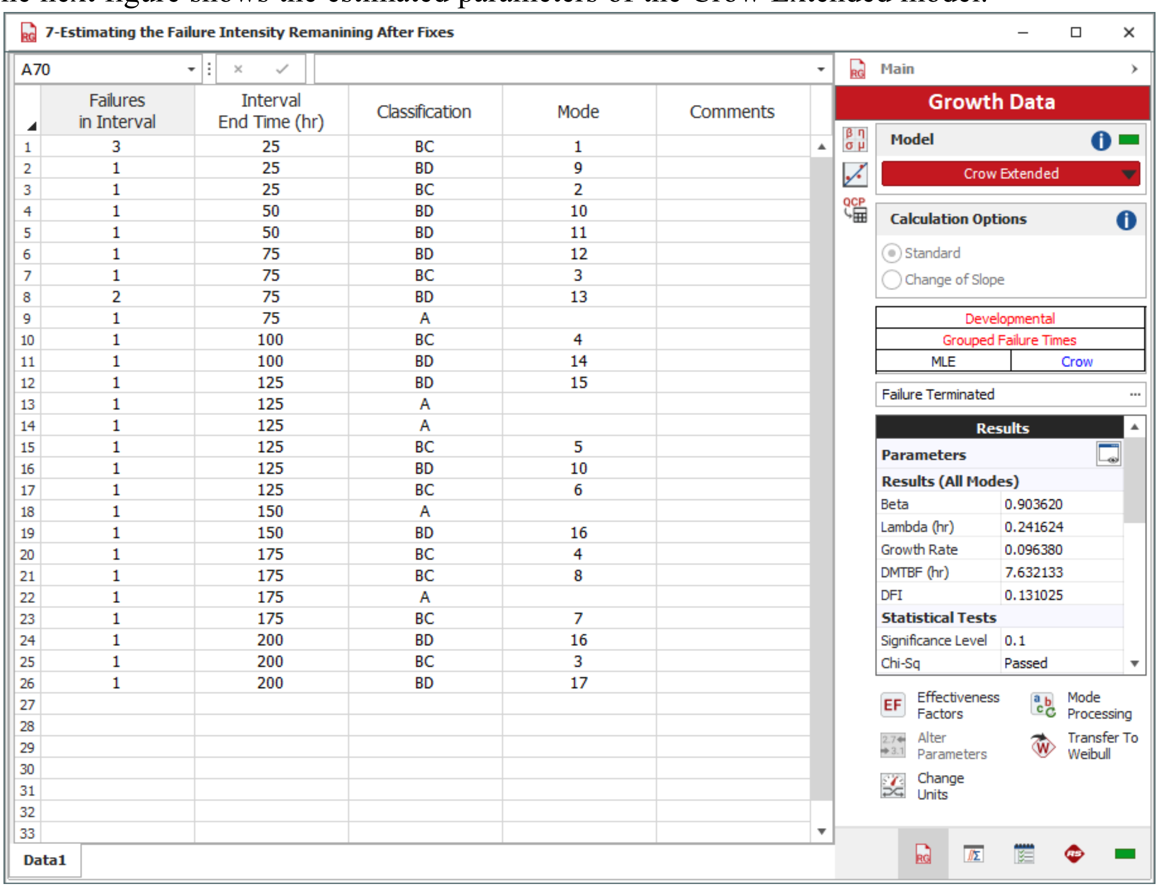
1	75	BD	12
1	75	BC	3
2	75	BD	13
1	75	A	
1	100	BC	4
1	100	BD	14
1	125	BD	15
1	125	A	
1	125	A	
1	125	BC	5
1	125	BD	10
1	125	BC	6
1	150	A	
1	150	BD	16
1	175	BC	4
1	175	BC	8
1	175	A	
1	175	BC	7
1	200	BD	16
1	200	BC	3
1	200	BD	17

Effectiveness Factors	
BD Mode	**Effectiveness Factor**

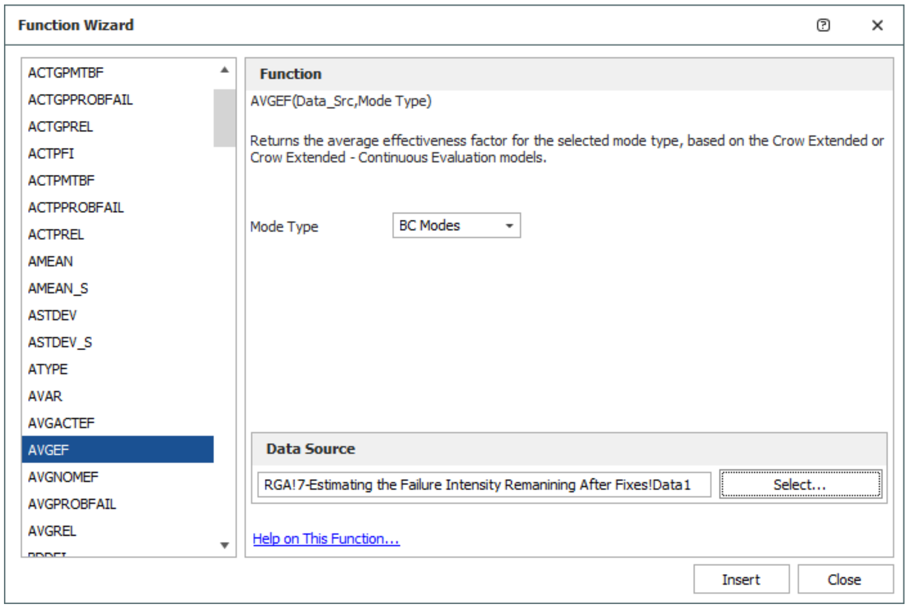
9	0.75
10	0.5
11	0.9
12	0.6
13	0.8
14	0.8
15	0.25
16	0.75
17	0.8

Solution

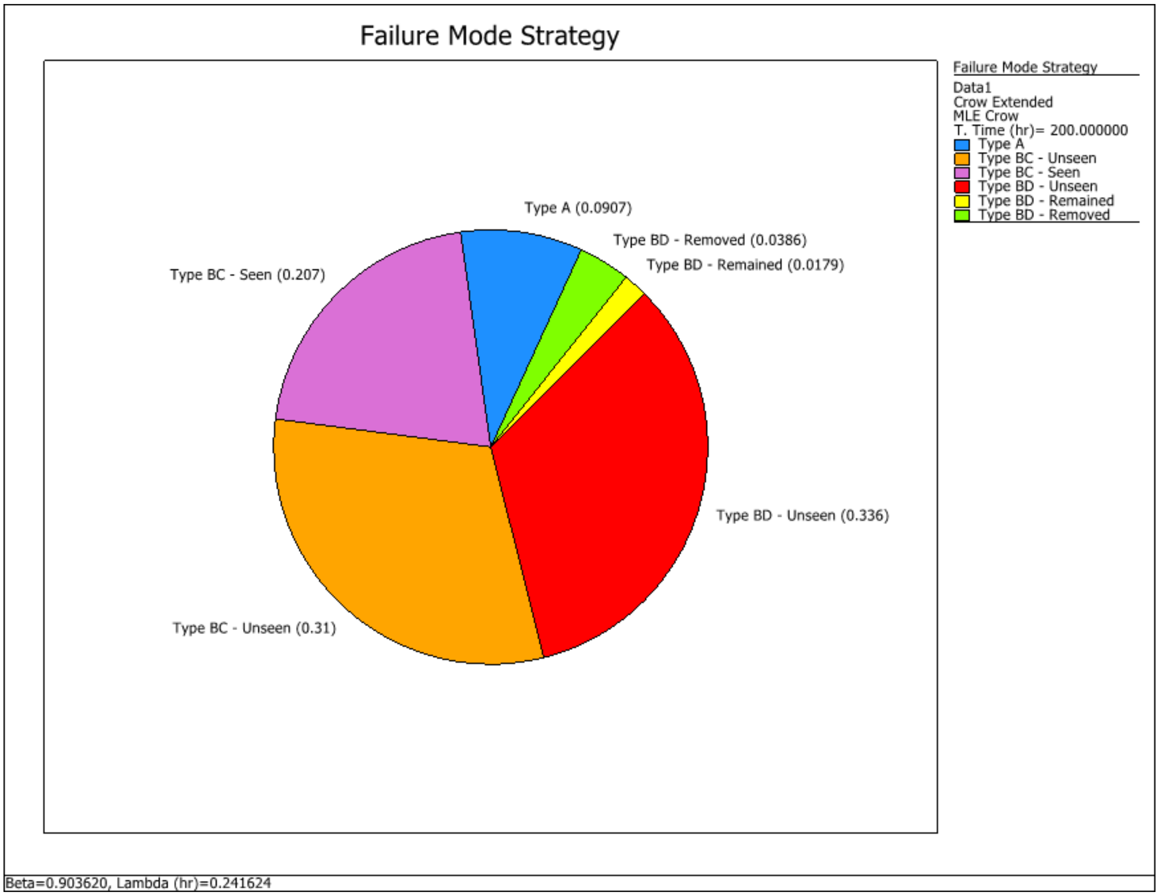
1. The next figure shows the estimated parameters of the Crow Extended model.



2. Insert a general spreadsheet into the folio, and then access the Function Wizard. In the Function Wizard, select **Average Effectiveness Factor** from the list of available functions and under Avg. Eff. Factor select BC modes, as shown next. Click **Insert** and the result will be inserted into the general spreadsheet. The average effectiveness factor for the BC modes is 0.6983.



3. The failure intensity left in the system due to the BD modes can be determined using the Failure Mode Strategy plot, as shown next. Therefore, the failure intensity left in the system due to the BD modes is 1.79%.



Determining if Design Will Meet MTBF Goal

Two prototypes of a new design are tested simultaneously. Whenever a failure is observed for one unit, the current operating time of the other unit is also recorded. The test is terminated after 300 hours. All of the design changes for the prototypes were delayed until after completing the test. The data set is given in the table below. Assume a fixed effectiveness factor equal to 0.7. The MTBF goal for the new design is 30 hours. Do the following:

1. Estimate the parameters of the Crow Extended model.
2. What is the projected MTBF and growth potential?
3. Under the current management strategy, is it even possible to reach the MTBF goal of 30 hours?

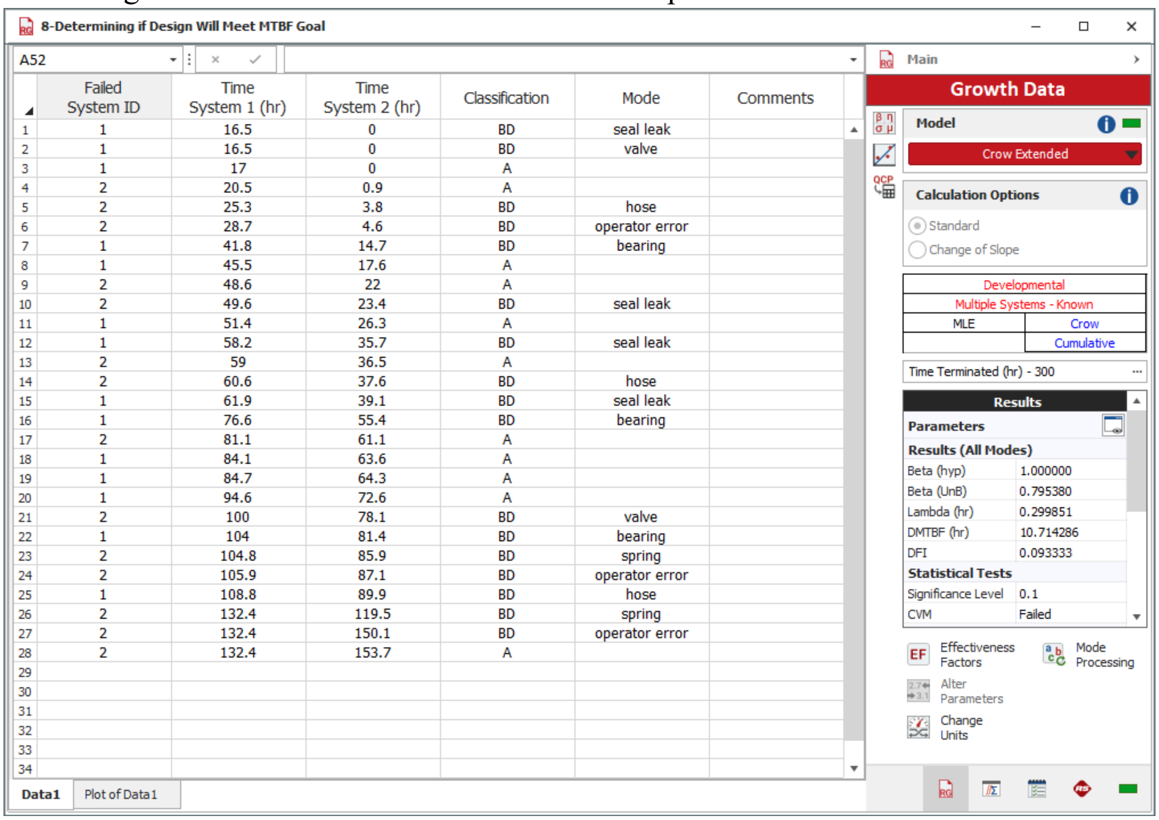
Multiple Systems (Known Operating Times) Data				
Failed Unit ID	**Time Unit 1**	**Time Unit 2**	**Classification**	**Mode**

1	16.5	0	BD	seal leak
1	16.5	0	BD	valve
1	17	0	A	
2	20.5	0.9	A	
2	25.3	3.8	BD	hose
2	28.7	4.6	BD	operator error
1	41.8	14.7	BD	bearing
1	45.5	17.6	A	
2	48.6	22	A	
2	49.6	23.4	BD	seal leak
1	51.4	26.3	A	
1	58.2	35.7	BD	seal leak
2	59	36.5	A	
2	60.6	37.6	BD	hose
1	61.9	39.1	BD	seal leak
1	76.6	55.4	BD	bearing
2	81.1	61.1	A	
1	84.1	63.6	A	
1	84.7	64.3	A	
1	94.6	72.6	A	
2	100	78.1	BD	valve
1	104	81.4	BD	bearing
2	104.8	85.9	BD	spring
2	105.9	87.1	BD	operator error

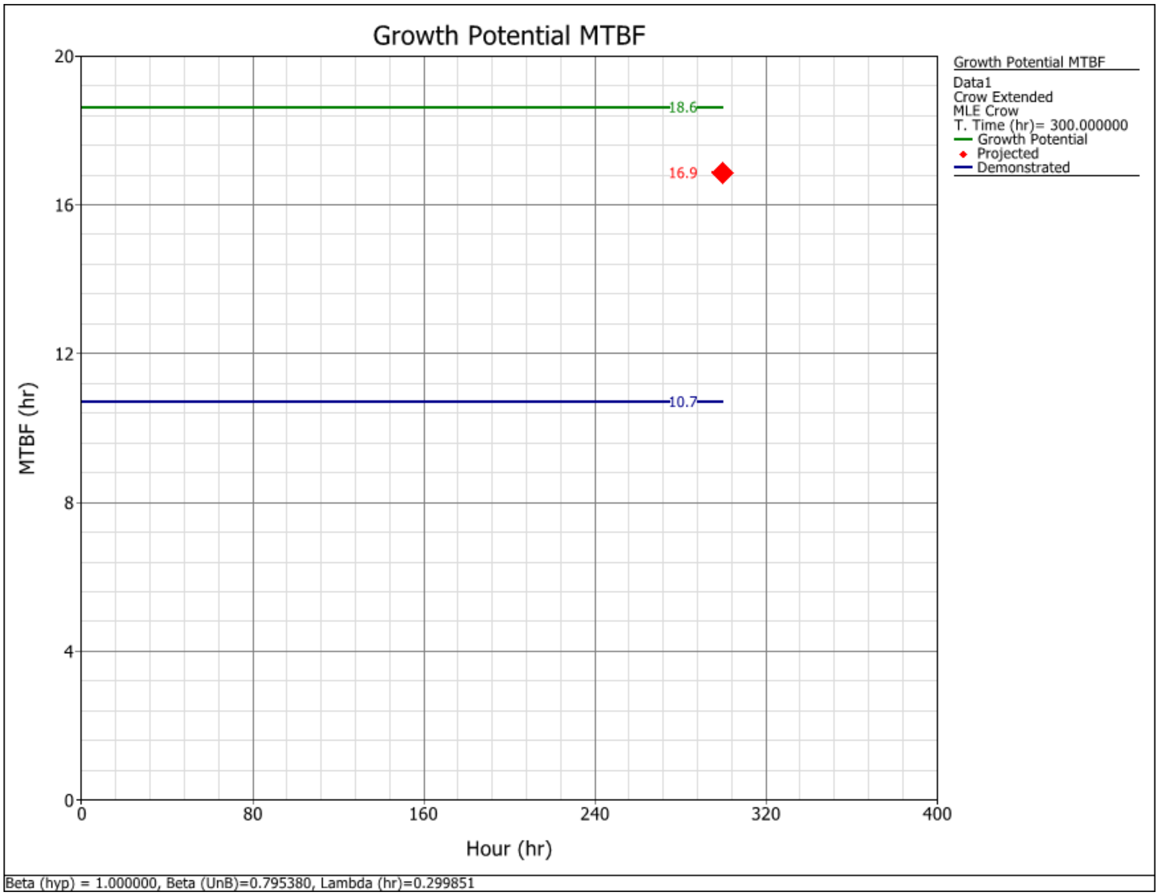
1	108.8	89.9	BD	hose
2	132.4	119.5	BD	spring
2	132.4	150.1	BD	operator error
2	132.4	153.7	A	

Solution

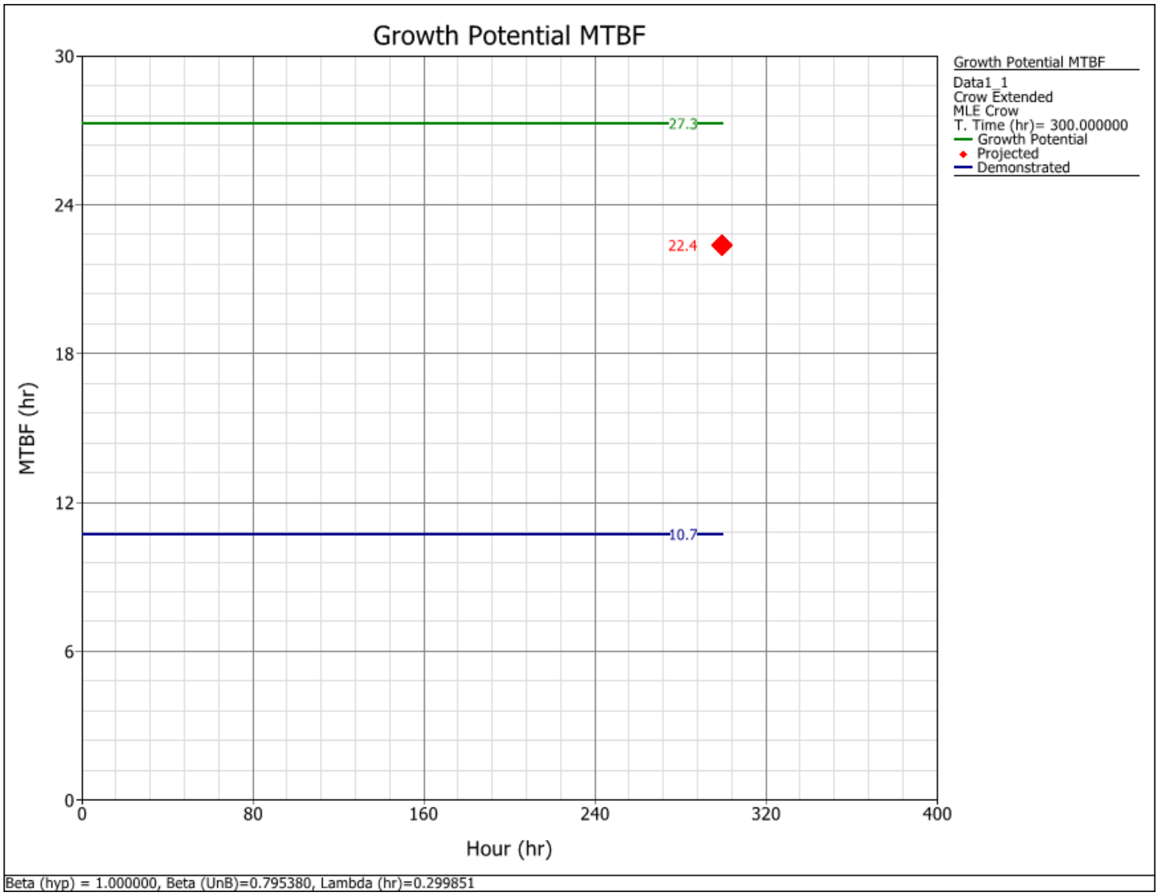
1. The next figure shows the estimated Crow Extended parameters.



2. One possible method for calculating the projected MTBF and growth potential is to use the Quick Calculation Pad, but you can also view these two values at the same time by viewing the Growth Potential MTBF plot, as shown next. From the plot, the projected MTBF is equal to 16.87 hours and the growth potential is equal to 18.63 hours.



3. The current projected MTBF and growth potential MTBF are both well below the required goal of 30 hours. To check if this goal can even be reached, you can set the effectiveness factor equal to 1. In other words, if all of the corrective actions were to remove the failure modes completely, what would be the projected and growth potential MTBF? Change the fixed effectiveness factor to 1, then recalculate the parameters and refresh the Growth Potential plot, as shown next. Even if you assume an effectiveness factor equal to 1, the growth potential is still only 27.27 hours. Based on the current design process, it will not be possible to reach the MTBF goal of 30 hours. Therefore, you have two options: start a new design stage or reduce the required MTBF goal.



Crow Extended - Continuous Evaluation

The Crow Extended model is designed for a single test phase. However, in most cases, testing for a system will be conducted in multiple phases. The Crow Extended - Continuous Evaluation model is designed for analyzing data across multiple test phases, while considering the data for all phases as one data set. To perform the analysis in the Weibull++ software, you would use the Multi-Phase data sheet.

The Crow Extended - Continuous Evaluation (3-parameter) model is an extension of the Crow Extended model, and is designed for practical testing situations where we need the flexibility to apply corrective actions at the time of failure or at a later time during the test, at the end of the current test or during a subsequent test phase. This three-parameter model is free of the constraint where the test must be stopped and all BD modes must be corrected at the end of the test, as in the Crow Extended model. The failure modes can be corrected during the test or when the testing is stopped for the incorporation of the fixes, or even not corrected at all at the end of the test phase. Based on this flexibility, the end time of the test is also not predefined, and it can be continuously updated with new test data. This is the reason behind the name, "continuous evaluation."

Definitions

Classifications

Under the Crow Extended - Continuous Evaluation model, corrective actions can be fixed at the time of failure or delayed to a later time (before time T, at time T or after time T, where T indicates the test's end time). The definition of *delayed* is expanded to include all type B failure modes corrected after the time of failure. This will include most, if not all, design-related failure modes requiring a root cause failure analysis. Failure modes that are corrected at the time of failure are typically related to human factors, such as manufacturing, operator error, etc.

For the Crow Extended - Continuous Evaluation model, the classifications are defined as follows:

- **A** indicates that a corrective action will not be performed (management chooses not to address these modes for technical, financial or other reasons).
- **BC** is defined as a failure mode that receives a corrective action at the time of failure and before the testing resumes. Typically, a type BC failure mode does not require extensive root cause failure analysis, and therefore can be corrected quickly. Type BC modes are generally easy to fix, and are usually related to issues such as quality, manufacturing, operator, etc.
- **BD** is defined as a failure mode that receives a corrective action at a test time after the first occurrence of the failure mode. Therefore, a fix is considered delayed if it is not implemented at the time of failure. A delayed fix can then occur at a later time during the test, at the end of the test (before the next phase) or during another test phase. Type BD failure modes typically require failure analysis and time to fabricate the corrective action. During the period $(0, T)$ into the test, there may be BD modes with corrective actions incorporated into the systems and other BD modes that have been seen, but not yet fixed.

The following table shows a comparison between the definitions of the classifications in the Crow Extended and Crow Extended - Continuous Evaluation models:

Comparison of Classification Definitions

Classification	Crow Extended	Crow Extended - Continuous Evaluation

A	Corrective action will not be performed	Same as Crow Extended
BC	Corrective action during the test	Corrective action at the time of failure
BD	Corrective action delayed until after the completion of the test	Corrective action delayed to a test time after the first occurrence of the failure mode

Reliability growth is achieved by decreasing the failure intensity. The failure intensity for the A failure modes will not change. Therefore, reliability growth can be achieved only by decreasing the BC and BD mode failure intensities. In general, the only part of the BD mode failure intensity that can be decreased is the part that has been seen during testing. The BC failure modes and fixed BD modes (delayed fixes implemented during the test) are corrected during testing, and their failure intensities will not change any more at the end of test.

Event Codes

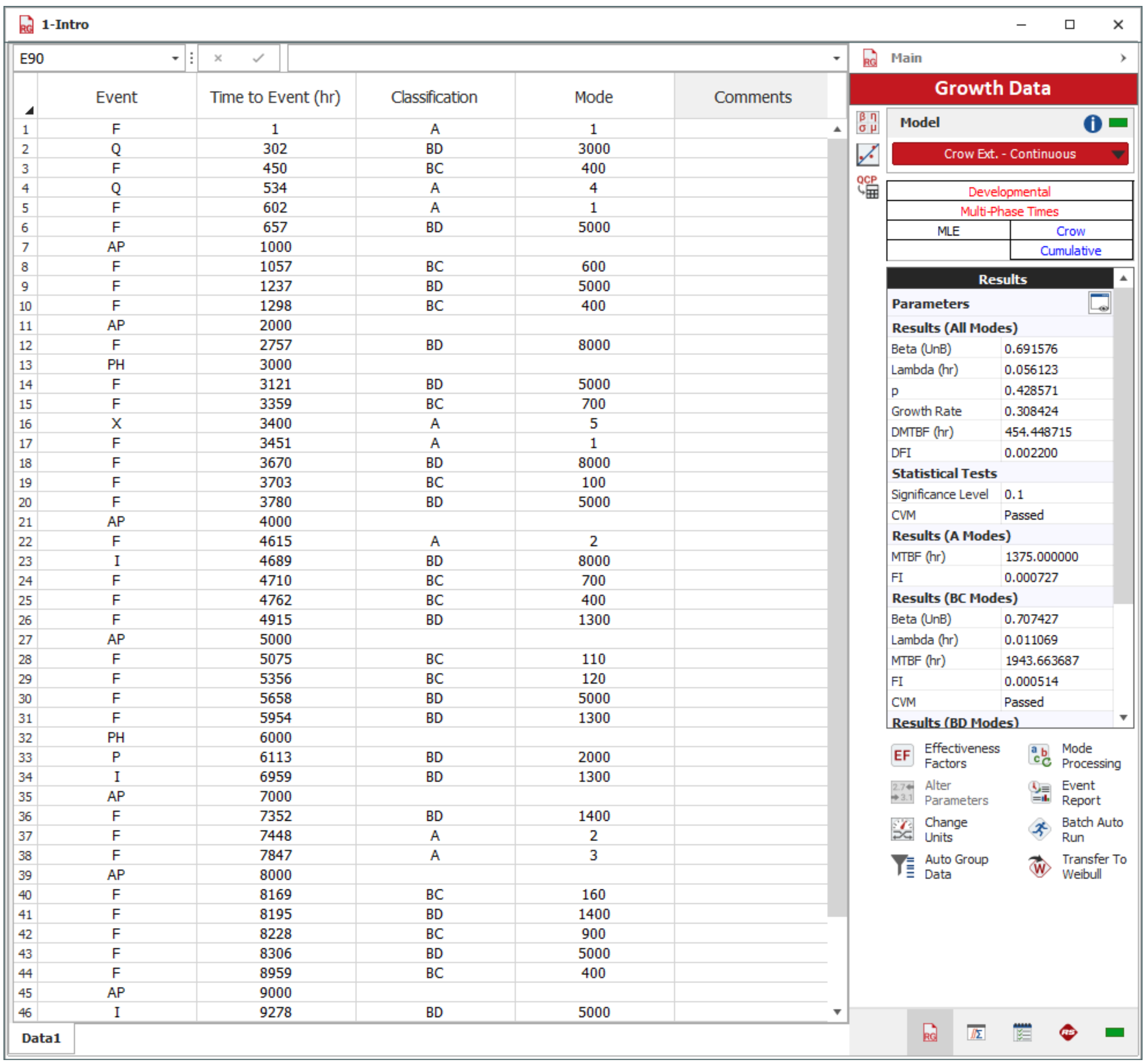
A Multi-Phase data sheet that is analyzed with the Crow Extended - Continuous Evaluation model has an **Event** column that allows you to indicate the types of events that occurred during testing. The possible event codes that can be used in the analysis are:

- ***F:*** indicates a failure time.
- ***I:*** denotes that a fix has been implemented for a BD failure mode at the specific time. BD modes that have not received a corrective action by time T would not have an associated *I* event in the data set.
- ***Q:*** indicates that the failure was due to a quality issue. An example of this might be a failure caused by a bolt not being tightened down properly. You have the option to decide whether or not to include quality issues in the analysis. This option can be specified by checking or clearing the **Include Q Events** check box under Continuous Options on the Analysis tab.
- ***P:*** indicates that the failure was due to a performance issue. You can determine whether or not to include performance issues in the analysis. This option can be specified by checking or clearing the *Include P Events* check box under Continuous Options on the Analysis tab.
- ***AP:*** indicates an analysis point. Analysis points can be shown in a multi-phase plot to track overall project progress and can be compared to an idealized growth curve.

- ***PH:*** indicates the end of a test phase. Test phases can be shown in a multi-phase plot to track overall project progress and can be compared to planned growth phases. The Weibull++ software allows for a maximum of seven test phases to be defined.

- ***X:*** indicates that the data point will be excluded from the analysis. An *X* can be placed in front of any existing event code or entered by itself. The row of data with the the *X* will not be included in the analysis.

The next figure shows an example of a Crow Extended - Continuous Evaluation folio with all of the possible event codes. As you can see, each failure is indicated with A, BC or BD in the **Classification** column. In addition, any text can be used to specify the mode. In this figure, failure codes were used in the **Mode** column for simplicity, but you could just as easily use Seal Leak, or whatever designation you deem appropriate for the failure mode.



	Event	Time to Event (hr)	Classification	Mode	Comments
1	F	1	A	1	
2	Q	302	BD	3000	
3	F	450	BC	400	
4	Q	534	A	4	
5	F	602	A	1	
6	F	657	BD	5000	
7	AP	1000			
8	F	1057	BC	600	
9	F	1237	BD	5000	
10	F	1298	BC	400	
11	AP	2000			
12	F	2757	BD	8000	
13	PH	3000			
14	F	3121	BD	5000	
15	F	3359	BC	700	
16	X	3400	A	5	
17	F	3451	A	1	
18	F	3670	BD	8000	
19	F	3703	BC	100	
20	F	3780	BD	5000	
21	AP	4000			
22	F	4615	A	2	
23	I	4689	BD	8000	
24	F	4710	BC	700	
25	F	4762	BC	400	
26	F	4915	BD	1300	
27	AP	5000			
28	F	5075	BC	110	
29	F	5356	BC	120	
30	F	5658	BD	5000	
31	F	5954	BD	1300	
32	PH	6000			
33	P	6113	BD	2000	
34	I	6959	BD	1300	
35	AP	7000			
36	F	7352	BD	1400	
37	F	7448	A	2	
38	F	7847	A	3	
39	AP	8000			
40	F	8169	BC	160	
41	F	8195	BD	1400	
42	F	8228	BC	900	
43	F	8306	BD	5000	
44	F	8959	BC	400	
45	AP	9000			
46	I	9278	BD	5000	

p Ratio

In the Crow Extended - Continuous evaluation, there is a certain probability of not incorporating a corrective action by time T. This is the additional (third parameter), as compared to the Crow Extended model. We define p as:

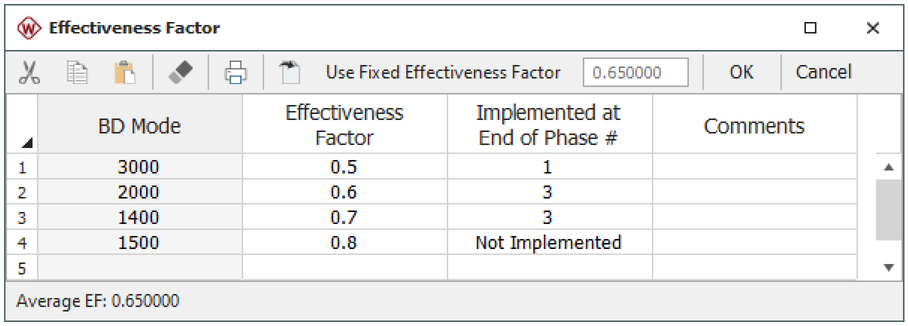
$$p = \frac{\text{All modes such that, if seen, will be corrected at time } T\text{, or later}}{\text{All failure modes such that, if seen, the corrective action will be delayed}}$$

It is assumed that the ratio p remains fixed over $(0,T)$. This implies that each time a distinct failure mode is seen, the probability that the corrective action is delayed until time T or later is p. In other words, each time a new BD mode is seen over $(0,T)$ there is a probability p that the corrective action for that mode will not have been incorporated into the system by time T. Under the Crow Extended two-parameter model, this probability is always equal to 1 at time T.

Effectiveness Factors

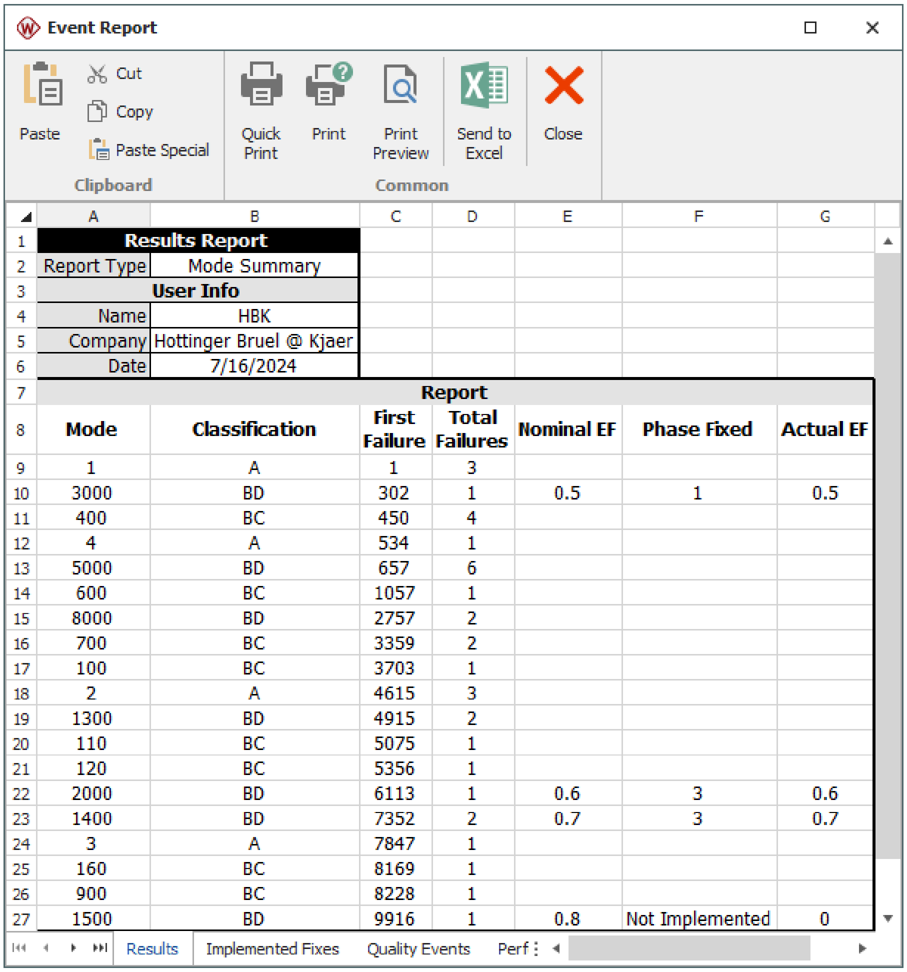
It is very important to note that failure modes are rarely totally eliminated by a corrective action. After failure modes have been found and fixed, a certain percentage of the failure intensity will be removed, but a certain percentage of the failure intensity will also remain. For each BD mode, an *effectiveness factor* (EF) is required to estimate how effective the corrective action will be in eliminating the failure intensity due to the failure mode. The EF is the fractional decrease in a mode's failure intensity after a corrective action has been made, and it must be a value between 0 and 1. A study on EFs showed that an average EF, d, is about 0.7. Therefore, about 30%, (i.e., $100(1-d)\%$) of the BD mode failure intensity will typically remain in the system after all of the corrective actions have been implemented. However, individual EFs for the failure modes may be larger or smaller than the average.

Similar to the Crow Extended model, each BD mode has an effectiveness factor that represents the decrease in failure intensity for that mode once the corrective action has been incorporated into the system. In addition, a delayed fix can be incorporated any time after the first occurrence of the failure mode. Therefore, delayed fixes can be incorporated before the end of the test phase, at the end of the test phase (just like the Crow Extended delayed fixes) or not incorporated at all at the end of the current test phase but postponed for a subsequent test phase. For calculation purposes, any delayed fixes that are incorporated during the test (those with an *I* event code) do not need to have an effectiveness factor specified, since the fix is already incorporated in the system. The next figure shows how effectiveness factors are defined in the Crow Extended - Continuous Evaluation model in the Weibull++ software. The figure also shows that you can specify whether the delayed fix was actually implemented at the end of the current test phase, at a later phase or not implemented at all.



For a Type BD failure mode that is not yet corrected but still deferred, the Actual Effectiveness Factor for that mode is zero. The Actual Effectiveness Factor for a deferred Type BD failure mode will stay at zero until the point when the corrective action will be incorporated is reached. At that time, the Actual Effectiveness Factor is changed to equal the Assigned Effectiveness Factor for that mode. At this point, the Nominal Effectiveness Factor (the effectiveness factor, assuming that fixes are implemented at the end of the specific phase) and the Actual Effectiveness Factor are the same. In other words, if a fix is not incorporated for a BD mode, its actual effectiveness for reducing failure intensity is zero, and the assigned effectiveness factor will be used only for projecting the MTBF (or failure intensity). This topic will be discussed in the Growth Potential and Projections section.

The next figure shows an event report in the Weibull++ software. At the end of the test phase, depending on whether the BD mode was specified as fixed or not fixed, the actual EF is zero (e.g., mode BD3000) or equal to the nominal EF (e.g., mode BD1500).



Results Report	
Report Type	Mode Summary
User Info	
Name	HBK
Company	Hottinger Bruel @ Kjaer
Date	7/16/2024

Mode	Classification	First Failure	Total Failures	Nominal EF	Phase Fixed	Actual EF
1	A	1	3			
3000	BD	302	1	0.5	1	0.5
400	BC	450	4			
4	A	534	1			
5000	BD	657	6			
600	BC	1057	1			
8000	BD	2757	2			
700	BC	3359	2			
100	BC	3703	1			
2	A	4615	3			
1300	BD	4915	2			
110	BC	5075	1			
120	BC	5356	1			
2000	BD	6113	1	0.6	3	0.6
1400	BD	7352	2	0.7	3	0.7
3	A	7847	1			
160	BC	8169	1			
900	BC	8228	1			
1500	BD	9916	1	0.8	Not Implemented	0

The Average Nominal EF is:

$$d_N = \frac{\sum^{M}{}_{i=1} d_{Ni}}{M}$$

where M is the total number of open and distinct BD modes at time T_J, and d_{N_i} is the Nominal Effectiveness Factor as specified for each of the BD mode.

The Average Actual EF is:

$$d_A = \frac{\sum^{M}{}_{i=1} d_{Ai}}{M}$$

where M is the total number of open and distinct BD modes at time T_J, and d_{A_i} is the Actual Effectiveness Factor at time T_j for each BD mode.

Growth Potential and Projections

The failure intensity that remains in the system will depend on the management strategy that determines the classification of the A, BC and BD failure modes. The engineering effort applied to the corrective actions determines the effectiveness factors. In addition, the failure intensity depends on $h(t)$, which is the rate at which unique BD failure modes are being discovered during testing. The rate of discovery drives the opportunity to take corrective actions based on the seen failure modes and it is an important factor in the overall reliability growth rate. The reliability growth potential is the limiting value of the failure intensity as time T increases. This limit is the maximum MTBF that can be attained with the current management strategy. The maximum MTBF will be attained when all BD modes have been observed and fixed.

If all seen BD modes are corrected by time T, that is, no deferred corrective actions at time T, then the growth potential is the maximum attainable based on the type BD designation of the failure modes, the corresponding assigned effectiveness factors and the remaining A modes in the system. This is called the *nominal growth potential*.

If some seen BD modes are not corrected at the end of the current test phase, then the prevailing growth potential is below the maximum attainable with the type BD designation of the failure modes and the corresponding assigned effectiveness factors.

The Crow-AMSAA (NHPP) model is used to estimate the current demonstrated MTBF or $MTBF_D$. The demonstrated MTBF does not take into account any type of projected improvements. Refer to the Crow-AMSAA (NHPP) chapter for more details.

The corresponding current demonstrated failure intensity is:

$$\lambda_D = \lambda\beta T^{\beta-1}$$

or:

$$\lambda_D = \frac{1}{MTBF_D}$$

The nominal growth potential factor is:

$$\lambda_{NGPFactor} = \sum_{i=1}^{M} (1 - d_{Ni}) \frac{N_i}{T}$$

where:

- M is the total number of distinct unfixed BD modes at time T_j .
- d_{Ni} is the assigned (nominal) EF for the i^{th} unfixed BD mode at time T_j.
- N_i is the total number of failures over (0, T_j) for the distinct unfixed BD mode i .

The nominal growth potential factor signifies the failure intensity of the M modes after corrective actions have been implemented for them, using the nominal values for the effectiveness factors.

Similarly, the actual growth potential factor is:

$$\lambda_{AGPFactor} = \sum_{i=1}^{M} (1 - d_{Ai}) \frac{N_i}{T}$$

where d_{Ai} is the actual EF for the i^{th} unfixed BD mode at time T_j.

The actual growth potential factor signifies the failure intensity of the M modes after corrective actions have been implemented for them, using the actual values for the effectiveness factors.

Based on the definition of BD modes for the Crow Extended - Continuous Evaluation model, the estimate of p at time T_j is calculated as follows:

$$p = \frac{\text{Total number of distinct unfixed BD modes at time } T_j}{\text{Total number of distinct BD modes at time } T_j \text{ (both fixed and unfixed)}}$$

The unfixed BD mode failure intensity at time T_j is:

$$\lambda_{BDunfixed} = \frac{\text{Total number of unfixed BD failures at time} T_j}{T_j}$$

Similar to the Crow Extended model, the discovery function at time T for the Crow Extended - Continuous Evaluation model is calculated using all the first occurrences of the all the BD modes, both fixed and unfixed. $h(t)$ is the unseen BD mode failure intensity and is also the rate at which new unique BD modes are being discovered.

$$\begin{aligned} \widehat{h}(T|BD) &= \widehat{\lambda}_{BD}\widehat{\beta}_{BD}T^{\widehat{\beta}_{BD}-1} \\ &= \frac{M\widehat{\beta}_{BD}}{T} \end{aligned}$$

where:

- $\widehat{\beta}_{BD}$ is the unbiased estimated of β for the Crow-AMSAA (NHPP) model based on the first occurrence of M distinct BD modes.
- $\widehat{\lambda}_{BD}$ is the unbiased estimated of λ for the Crow-AMSAA (NHPP) model based on the first occurrence of M distinct BD modes.

The parameters $\widehat{\beta}_{BD}$ and $\widehat{\lambda}_{BD}$ are also known as the Rate of Discovery Parameters.

The nominal growth potential failure intensity is:

$$\lambda_{NGP} = \lambda_D - \lambda_{BD\text{unfixed}} + \lambda_{NGP\text{Factor}} + d_N \cdot p \cdot h(T) - d_N h(T)$$

and the nominal growth potential MTBF is:

$$MTBF_{NGP} = \frac{1}{\lambda_{NGP}}$$

The nominal projected failure intensity at time T is:

$$\lambda_{NP} = \lambda_{NGP} + d_N h(T)$$

and the nominal projected MTBF at time T is:

$$MTBF_{NP} = \frac{1}{\lambda_{NP}}$$

The actual growth potential failure intensity is:

$$\lambda AGP = \lambda_D - \lambda_{BD\text{unfixed}} + \lambda AGP\text{Factor} + d_A \cdot p \cdot h(T) - d_A h(T)$$

and the actual growth potential MTBF is:

$$MTBF_{AGP} = \frac{1}{\lambda_{AGP}}$$

The actual projected failure intensity at time T is:

$$\lambda_{AP} = \lambda_{AGP} + d_A \cdot h\,(T)$$

and the actual projected MTBF at time T is:

$$MTBF_{AP} = \frac{1}{\lambda_{AP}}$$

In terms of confidence intervals and goodness-of-fit tests, the calculations are the same as for the Crow Extended model.

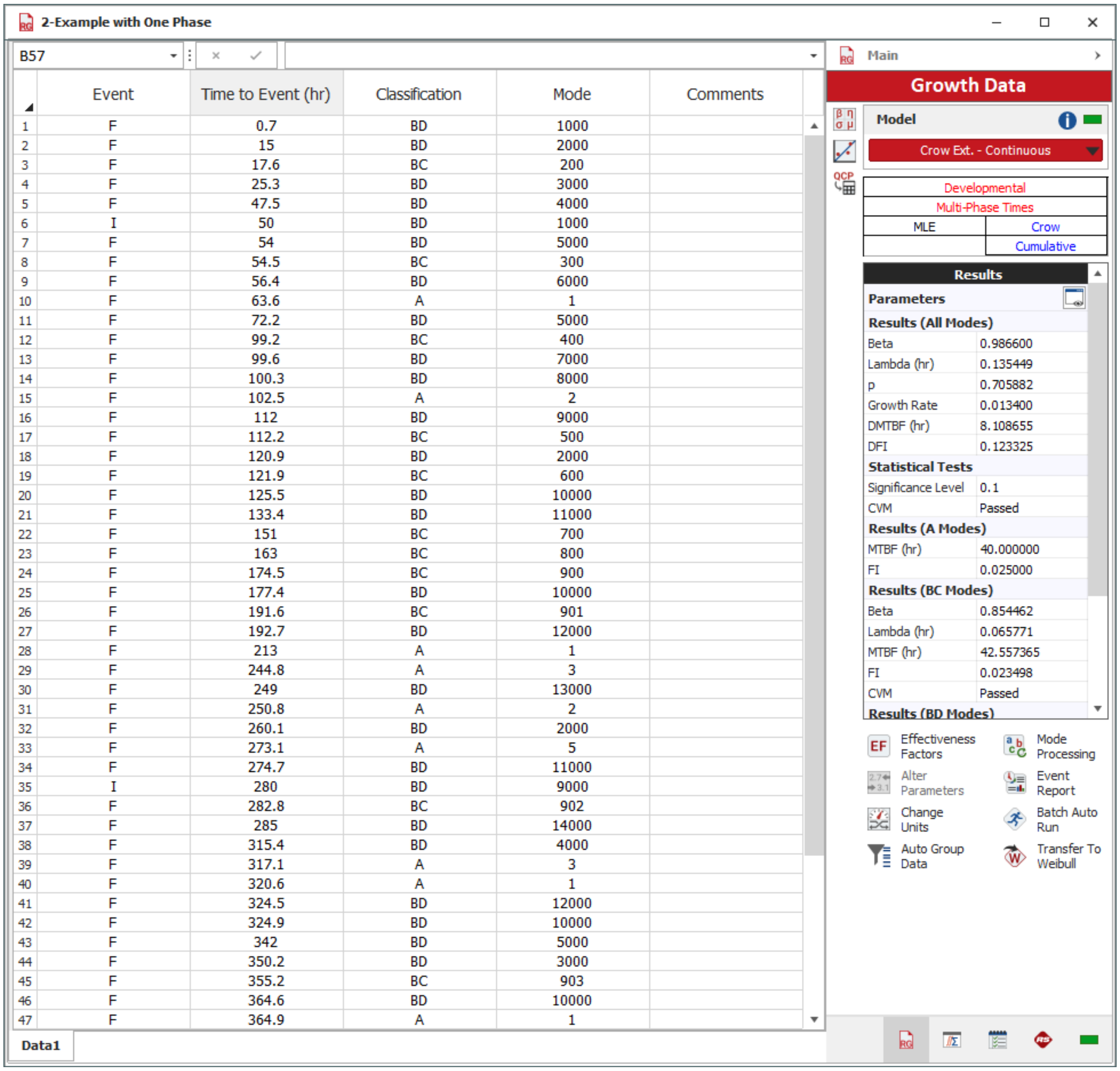
Example - Single Phase

The following table shows a date set with failure and fix implementation events.

Multi-Phase Data for a Time Terminated Test at $T = 400$								
Event	**Time to Event**	**Classification**	**Mode**		**Event**	**Time to Event**	**Classification**	**Mode**
F	0.7	BD	1000		F	244.8	A	3
F	15	BD	2000		F	249	BD	13000
F	17.6	BC	200		F	250.8	A	2
F	25.3	BD	3000		F	260.1	BD	2000
F	47.5	BD	4000		F	273.1	A	5
I	50	BD	1000		F	274.7	BD	11000
F	54	BD	5000		I	280	BD	9000
F	54.5	BC	300		F	282.8	BC	902
F	56.4	BD	6000		F	285	BD	14000
F	63.6	A	1		F	315.4	BD	4000
F	72.2	BD	5000		F	317.1	A	3
F	99.2	BC	400		F	320.6	A	1
F	99.6	BD	7000		F	324.5	BD	12000
F	100.3	BD	8000		F	324.9	BD	10000
F	102.5	A	2		F	342	BD	5000
F	112	BD	9000		F	350.2	BD	3000
F	112.2	BC	500		F	355.2	BC	903
F	120.9	BD	2000		F	364.6	BD	10000
F	121.9	BC	600		F	364.9	A	1

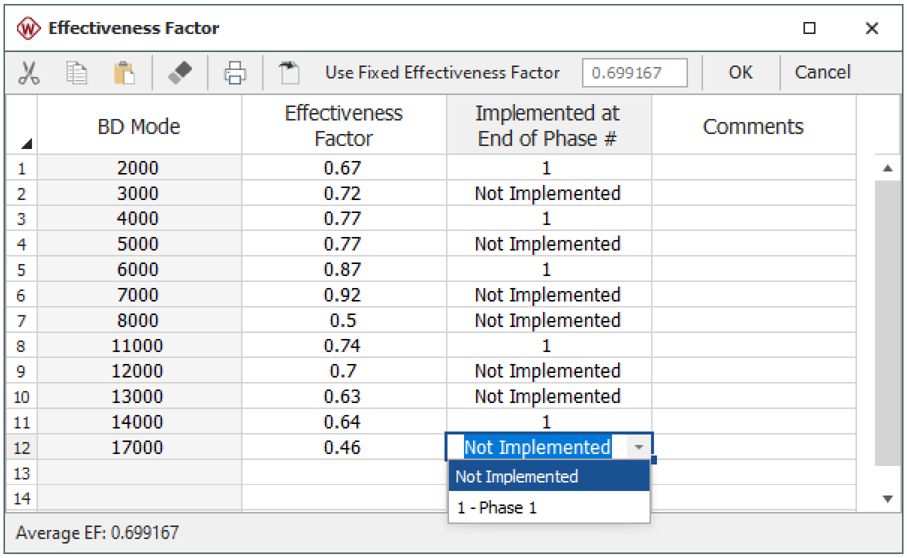
F	125.5	BD	10000		I	365	BD	10000
F	133.4	BD	11000		F	366.3	BD	2000
F	151	BC	700		F	379.4	BD	15000
F	163	BC	800		F	389	BD	16000
F	174.5	BC	900		I	390	BD	15000
F	177.4	BD	10000		I	393	BD	16000
F	191.6	BC	901		F	394.9	A	3
F	192.7	BD	12000		F	395.2	BD	17000
F	213	A	1					

The following figure shows the data set entered in the Weibull++ software's Multi-Phase data sheet. Note that because this is a time terminated test with a single phase ending at $T = 400$, the last event entry is a phase (PH) with time to event = 400.



	Event	Time to Event (hr)	Classification	Mode	Comments
1	F	0.7	BD	1000	
2	F	15	BD	2000	
3	F	17.6	BC	200	
4	F	25.3	BD	3000	
5	F	47.5	BD	4000	
6	I	50	BD	1000	
7	F	54	BD	5000	
8	F	54.5	BC	300	
9	F	56.4	BD	6000	
10	F	63.6	A	1	
11	F	72.2	BD	5000	
12	F	99.2	BC	400	
13	F	99.6	BD	7000	
14	F	100.3	BD	8000	
15	F	102.5	A	2	
16	F	112	BD	9000	
17	F	112.2	BC	500	
18	F	120.9	BD	2000	
19	F	121.9	BC	600	
20	F	125.5	BD	10000	
21	F	133.4	BD	11000	
22	F	151	BC	700	
23	F	163	BC	800	
24	F	174.5	BC	900	
25	F	177.4	BD	10000	
26	F	191.6	BC	901	
27	F	192.7	BD	12000	
28	F	213	A	1	
29	F	244.8	A	3	
30	F	249	BD	13000	
31	F	250.8	A	2	
32	F	260.1	BD	2000	
33	F	273.1	A	5	
34	F	274.7	BD	11000	
35	I	280	BD	9000	
36	F	282.8	BC	902	
37	F	285	BD	14000	
38	F	315.4	BD	4000	
39	F	317.1	A	3	
40	F	320.6	A	1	
41	F	324.5	BD	12000	
42	F	324.9	BD	10000	
43	F	342	BD	5000	
44	F	350.2	BD	3000	
45	F	355.2	BC	903	
46	F	364.6	BD	10000	
47	F	364.9	A	1	

The next figure shows the effectiveness factors for the unfixed BD modes, and information concerning whether the fix will be implemented. Since we have only one test phase for this example, the notation "1" indicates that the fix will be implemented at the end of the first (and only) phase.



	BD Mode	Effectiveness Factor	Implemented at End of Phase #	Comments
1	2000	0.67	1	
2	3000	0.72	Not Implemented	
3	4000	0.77	1	
4	5000	0.77	Not Implemented	
5	6000	0.87	1	
6	7000	0.92	Not Implemented	
7	8000	0.5	Not Implemented	
8	11000	0.74	1	
9	12000	0.7	Not Implemented	
10	13000	0.63	Not Implemented	
11	14000	0.64	1	
12	17000	0.46	Not Implemented	
13				
14				

Do the following:

1. Determine the current demonstrated MTBF and failure intensity at time T.
2. Determine the nominal and actual average effectiveness factor at time T.
3. Determine the p ratio.
4. Determine the nominal and actual growth potential factor.
5. Determine the unfixed BD mode failure intensity at time T.
6. Determine the rate of discovery parameters and the rate of discovery function at time T.
7. Determine the nominal growth potential failure intensity and MTBF at time T.
8. Determine the nominal projected failure intensity and MTBF at time T.
9. Determine the actual growth potential failure intensity and MTBF at time T.
10. Determine the actual projected failure intensity and MTBF at time T.

Solution

1. As described in the Crow-AMSAA (NHPP) chapter, for a time terminated test, β is estimated by the following equation:

$$\widehat{\beta} = \frac{n}{n \ln T^{*} - \sum_{i=1}^{n} \ln T_i}$$

where T^{*} is the termination time and n is the total number of failures. In this example, $T^{*} = 400$ and $n = 50$. Note that there are 5 fix implementation events and 1 event that marks the end of the phase. These should not be counted as failures. So in this case, we find that $\beta = 0.9866$. We calculated the biased estimate of β, but note that we could have used the unbiased estimate as presented in the Crow-AMSAA (NHPP) chapter. The choice of calculating the biased or unbiased estimate of β can be configured in the Application Setup window in the Weibull++ software. Solve for λ, based on the Crow-AMSAA (NHPP) equation explained in the Crow AMSAA (NHPP) chapter:

$$\begin{aligned}\widehat{\lambda} &= \frac{n}{T^{*\beta}} \\ &= \frac{50}{400^{0.9866}} \\ &= 0.1354\end{aligned}$$

The demonstrated MTBF of the system at time $T = 400$ is:

$$\begin{aligned}MTBF_D &= \frac{1}{\lambda\beta T^{\beta-1}} \\ &= 8.1087\end{aligned}$$

The corresponding current demonstrated failure intensity is:

$$\begin{aligned}\lambda_D &= \frac{1}{MTBF_D} \\ &= 0.1233\end{aligned}$$

2. The average nominal effectiveness factor at time T is:

$$\begin{aligned}d_N &= \frac{\sum^{M}{}_{i=1} d_{Ni}}{M} \\ &= \frac{0.67+0.72+0.77+0.77+0.87+0.92+0.5+0.74+0.7+0.63+0.64+0.46}{12} \\ &= 0.6992\end{aligned}$$

The average actual effectiveness factor at time T is given by:

$$\begin{aligned}d_A &= \frac{\sum^{M}{}_{i=1} d_{Ai}}{M} \\ &= \frac{0.67+0+0.77+0+0.87+0+0+0.74+0+0+0.64+0}{12} \\ &= 0.3075\end{aligned}$$

3. The p ratio is calculated by:

$$\begin{aligned}p &= \frac{\text{Total number of distinct unfixed BD modes at time 400}}{\text{Total number of distinct BD modes at time 400 (both fixed and unfixed)}} \\ &= \frac{12}{12+5} \\ &= 0.7059\end{aligned}$$

4. The nominal growth potential factor is:

$$\lambda_{NGPFactor} = \sum_{i=1}^{M} (1 - d_{Ni}) \frac{N_i}{T}$$

The total number M of distinct unfixed BD modes at time 400 is $M = 12$.

d_{Ni} is the assigned (nominal) EF for the i^{th} unfixed BD mode at time T_j, (which is shown in the picture of the Effectiveness Factors window given above).

N_i is the total number of failures over (0, 400) for the distinct unfixed BD mode i. This is summarized in the following table.

Number of Failures for Unfixed BD Modes		
Classification	**Mode**	**Number of Failures**
BD	2000	4
BD	3000	2
BD	4000	2
BD	5000	3
BD	6000	1
BD	7000	1
BD	8000	1
BD	11000	2
BD	12000	2
BD	13000	1
BD	14000	1
BD	17000	1
		Sum = 21

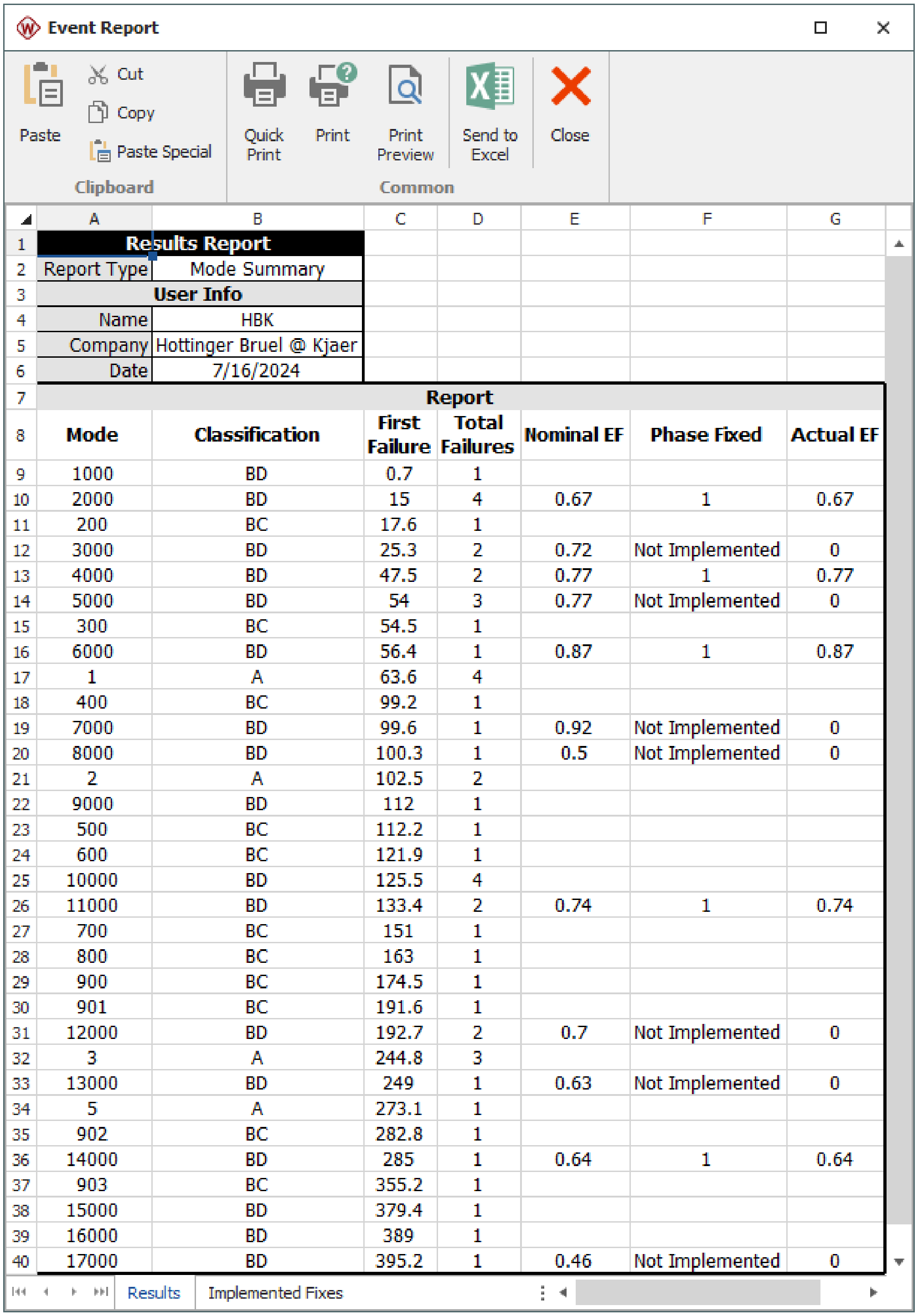
Based on the information given above, the nominal growth potential factor is calculated as:

$$\lambda_{NGPFactor} = 0.0153$$

The actual growth potential factor is:

$$\lambda_{AGPFactor} = \sum_{i=1}^{M} (1 - d_{Ai}) \frac{N_i}{T}$$

where d_{Ai} is the actual EF for the i^{th} unfixed BD mode at time 400, depending on whether a fix was implemented at time 400 or not. The next figure shows an event report from the Weibull++ software where the actual EF is zero if a fix was not implemented at 400, or equal to the nominal EF if the fix was implemented at 400.



Results Report	
Report Type	Mode Summary
User Info	
Name	HBK
Company	Hottinger Bruel @ Kjaer
Date	7/16/2024

Report

Mode	Classification	First Failure	Total Failures	Nominal EF	Phase Fixed	Actual EF
1000	BD	0.7	1			
2000	BD	15	4	0.67	1	0.67
200	BC	17.6	1			
3000	BD	25.3	2	0.72	Not Implemented	0
4000	BD	47.5	2	0.77	1	0.77
5000	BD	54	3	0.77	Not Implemented	0
300	BC	54.5	1			
6000	BD	56.4	1	0.87	1	0.87
1	A	63.6	4			
400	BC	99.2	1			
7000	BD	99.6	1	0.92	Not Implemented	0
8000	BD	100.3	1	0.5	Not Implemented	0
2	A	102.5	2			
9000	BD	112	1			
500	BC	112.2	1			
600	BC	121.9	1			
10000	BD	125.5	4			
11000	BD	133.4	2	0.74	1	0.74
700	BC	151	1			
800	BC	163	1			
900	BC	174.5	1			
901	BC	191.6	1			
12000	BD	192.7	2	0.7	Not Implemented	0
3	A	244.8	3			
13000	BD	249	1	0.63	Not Implemented	0
5	A	273.1	1			
902	BC	282.8	1			
14000	BD	285	1	0.64	1	0.64
903	BC	355.2	1			
15000	BD	379.4	1			
16000	BD	389	1			
17000	BD	395.2	1	0.46	Not Implemented	0

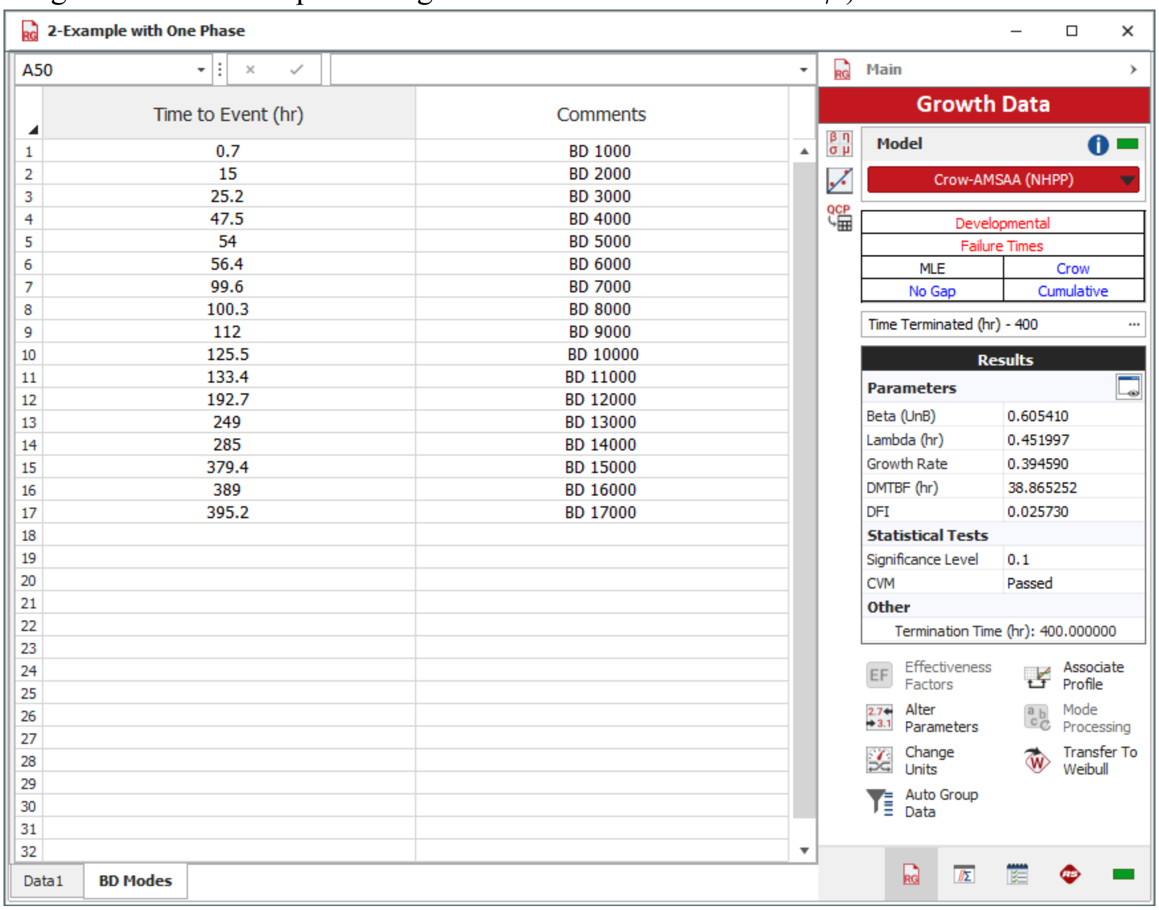
Based on the information given above, the actual growth potential factor is calculated as:

$$\lambda_{AGPFactor} = 0.0344$$

5. The total number of unfixed BD modes listed in the data table is 21. The unfixed BD mode failure intensity at time 400 is:

$$\lambda_{BDunfixed} = \frac{\text{Total number of unfixed BD failure at time 400}}{400}$$
$$= \frac{21}{400}$$
$$= 0.0525$$

6. The discovery rate parameters at time 400 are calculated by using all the first occurrences of all the BD modes, both fixed and unfixed. $\hat{\beta}_{BD}$ is the unbiased estimated of β for the Crow-AMSAA (NHPP) model based on the first occurrence of the 17 distinct BD modes in our example. $\hat{\lambda}_{BD}$ is the unbiased estimate of λ for the Crow-AMSAA (NHPP) model based on the first occurrence of the 17 distinct BD modes. The next figure shows the first time to failure for each of the 17 distinct modes and the results of the analysis using the Crow-AMSAA (NHPP) model in the Weibull++ software (note that in this case, the calculation settings in the User Setup is configured to calculate the unbiased β).



So we have:

$$\widehat{\beta}_{BD} = 0.6055$$

and:

$$\widehat{\lambda}_{BD} = 0.4518$$

The equations used to determine these parameters have been explained in question 1 of this example and are also presented in detail in the Crow-AMSA (NHPP) chapter.

The discovery rate function at time 400 is:

$$\begin{aligned}\widehat{h}(T|BD) =& \widehat{\lambda}_{BD}\widehat{\beta}_{BD}T^{\widehat{\beta}_{BD}-1}\\ =&0.4518 \cdot 0.6055 \cdot 400^{0.6055-1}\\ =&0.0257\end{aligned}$$

This is the failure intensity of the unseen BD modes at time 400. In this case, it means that 0.0257 new BD modes are discovered per hour, or one new BD mode is discovered every 38.9 hours.

7. The nominal growth potential failure intensity is:

$$\begin{aligned}\lambda_{NGP} =& \lambda_D - \lambda_{BDunfixed} + \lambda_{NGPFactor} + d_N \cdot p \cdot h(400) - d_N h(400)\\ =&0.1233 - 0.0525 + 0.0153 + 0.6992 \cdot 0.7059 \cdot 0.0257 - 0.6992 \cdot 0.0257\\ =&0.080\end{aligned}$$

This is the minimum attainable failure intensity if all delayed corrective actions are implemented for the modes that have been seen and delayed corrective actions are also implemented for the unseen BD modes, assuming testing would continue until all unseen BD modes are revealed. The nominal growth potential MTBF is:

$$\begin{aligned}MTBF_{NGP} =& \frac{1}{\lambda_{NGP}}\\ =&\frac{1}{0.080}\\ =&12.37\end{aligned}$$

This is the maximum attainable MTBF if all delayed corrective actions are implemented for the modes that have been seen and delayed corrective actions are also implemented for the unseen BD modes, assuming testing would continue until all unseen BD modes are revealed.

8. The nominal projected failure intensity at time 400 is:

$$\begin{aligned}\lambda_{NP} &= \lambda_{NGP} + d_N h(400) \\ &= 0.080 + 0.6992 \cdot 0.0257 \\ &= 0.0988\end{aligned}$$

This is the projected failure intensity assuming all delayed fixes have been implemented for the modes that have been seen. The nominal projected MTBF at time 400 is:

$$\begin{aligned}MTBF_{NP} &= \frac{1}{\lambda_{NP}} \\ &= \frac{1}{0.0988} \\ &= 10.11\end{aligned}$$

This is the projected MTBF assuming all delayed fixes have been implemented for the modes that have been seen.

9. The actual growth potential failure intensity is:

$$\begin{aligned}\lambda_{AGP} &= \lambda_D - \lambda_{BDunfixed} + \lambda_{AGPFactor} + d_A \cdot p \cdot h(400) - d_A h(400) \\ &= 0.1233 - 0.0525 + 0.0344 + 0.3075 \cdot 0.7059 \cdot 0.0257 - 0.3075 \cdot 0.0257 \\ &= 0.1029\end{aligned}$$

This is the minimum attainable failure intensity based on the current management strategy. The actual growth potential MTBF is:

$$\begin{aligned}MTBF_{AGP} &= \frac{1}{\lambda_{AGP}} \\ &= \frac{1}{0.1029} \\ &= 9.71\end{aligned}$$

This is the maximum attainable MTBF based on the current management strategy.

10. The actual projected failure intensity at time 400 is:

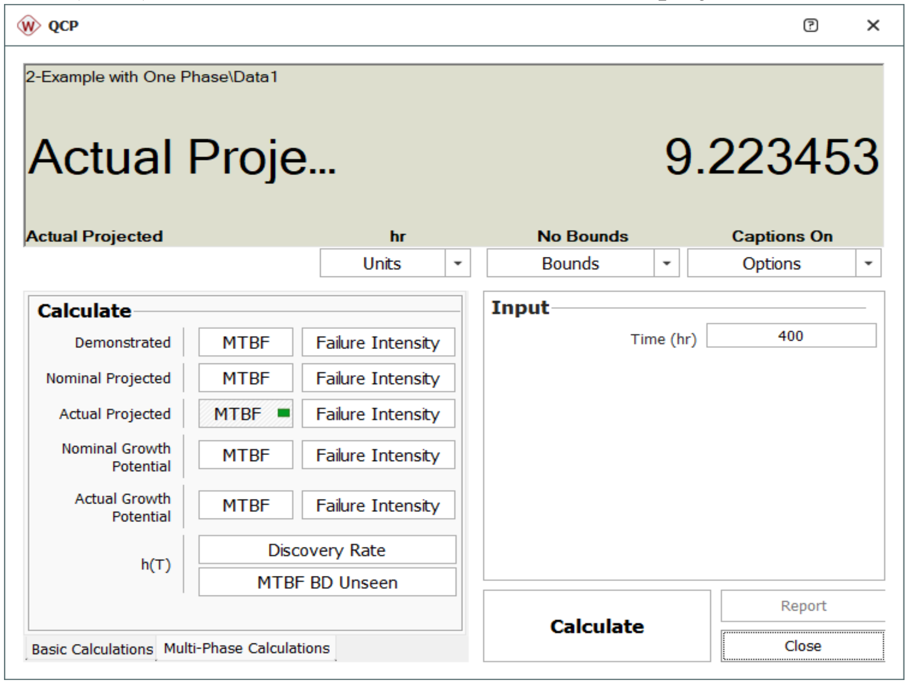
$$\begin{aligned}\lambda_{AP} &= \lambda_{AGP} + d_A \cdot h(400) \\ &= 0.1029 + 0.3075 \cdot 0.0257 \\ &= 0.1108\end{aligned}$$

This is the projected failure intensity based on the current management strategy. The actual projected MTBF at time 400 is:

$$\begin{aligned}MTBF_{AP} &= \frac{1}{\lambda_{AP}} \\ &= \frac{1}{0.1108} \\ &= 9.01\end{aligned}$$

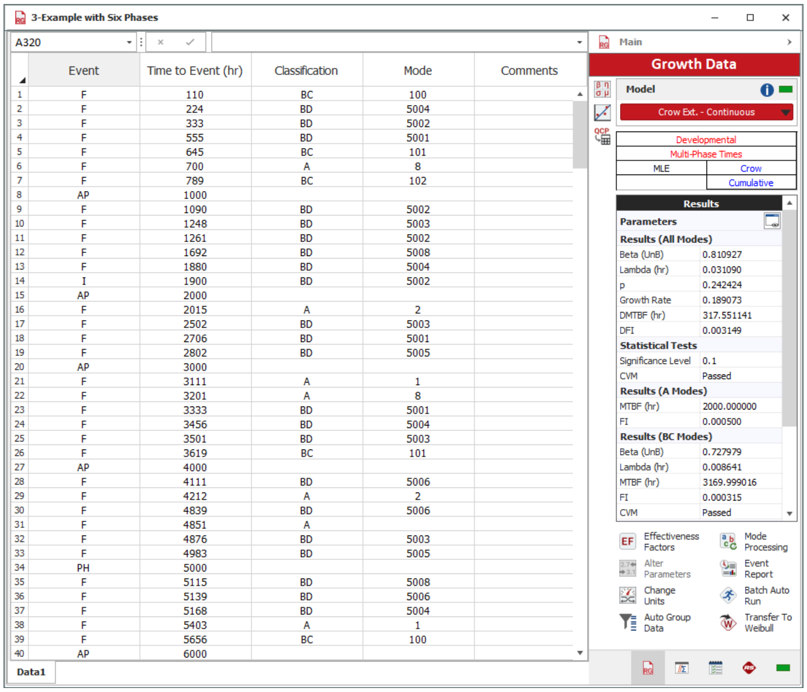
This is the projected MTBF based on the current management strategy. The next figure demonstrates how we can derive the results of this example by using the software's Quick

Calculation Pad (QCP). Here we chose to calculate the actual projected MTBF.



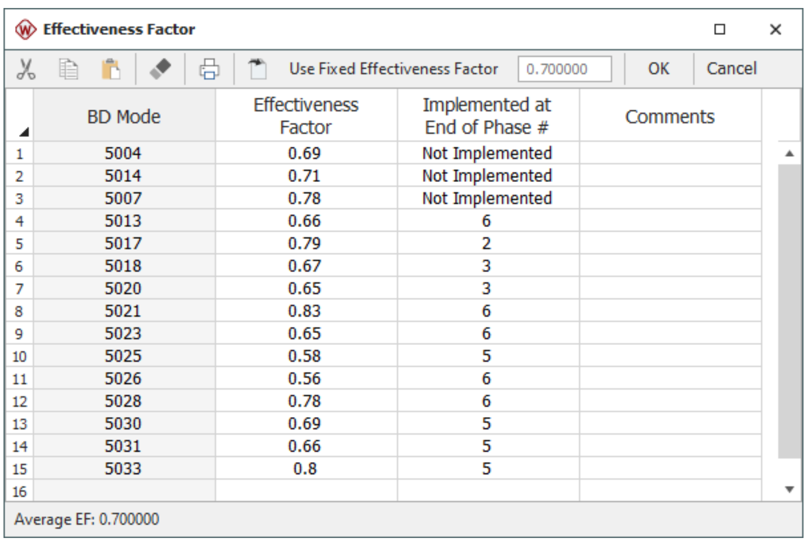
Example - Six Phases

The Crow Extended - Continuous Evaluation model allows data analysis across multiple phases, up to seven individual phases. The next figure shows a portion of failure time test results obtained across six phases. Analysis points are specified for continuous evaluation every 1,000 hours. The cumulative test times at the end of each test phase are 5,000; 15,000; 25,000; 35,000; 45,000 and 60,000 hours.



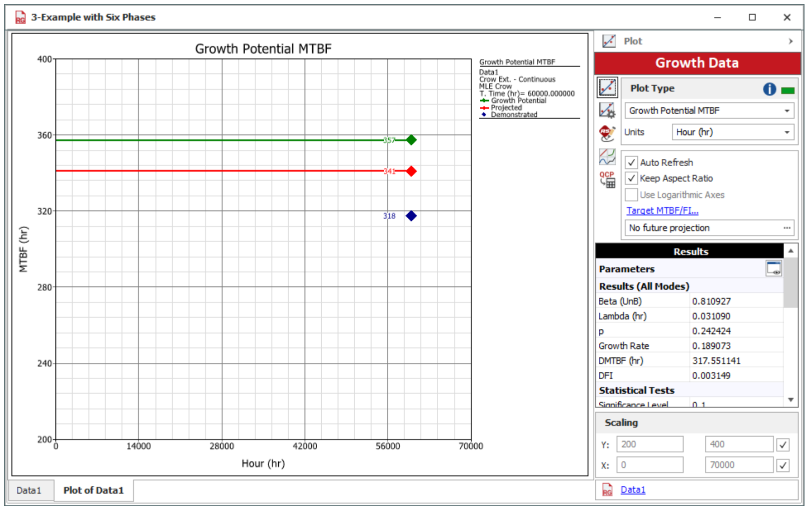
	Event	Time to Event (hr)	Classification	Mode	Comments
1	F	110	BC	100	
2	F	224	BD	5004	
3	F	333	BD	5002	
4	F	555	BD	5001	
5	F	645	BC	101	
6	F	700	A	8	
7	F	789	BC	102	
8	AP	1000			
9	F	1090	BD	5002	
10	F	1248	BD	5003	
11	F	1261	BD	5002	
12	F	1692	BD	5008	
13	F	1880	BD	5004	
14	I	1900	BD	5002	
15	AP	2000			
16	F	2015	A	2	
17	F	2502	BD	5003	
18	F	2706	BD	5001	
19	F	2802	BD	5005	
20	AP	3000			
21	F	3111	A	1	
22	F	3201	A	8	
23	F	3333	BD	5001	
24	F	3456	BD	5004	
25	F	3501	BD	5003	
26	F	3619	BC	101	
27	AP	4000			
28	F	4111	BD	5006	
29	F	4212	A	2	
30	F	4839	BD	5006	
31	F	4851	A		
32	F	4876	BD	5003	
33	F	4983	BD	5005	
34	PH	5000			
35	F	5115	BD	5008	
36	F	5139	BD	5006	
37	F	5168	BD	5004	
38	F	5403	A	1	
39	F	5656	BC	100	
40	AP	6000			

The next figure shows the effectiveness factors for the BD modes that do not have an associated fix implementation event. In other words, these are unfixed BD modes. Note that this specifies the phase after which the BD mode will be fixed, if any.

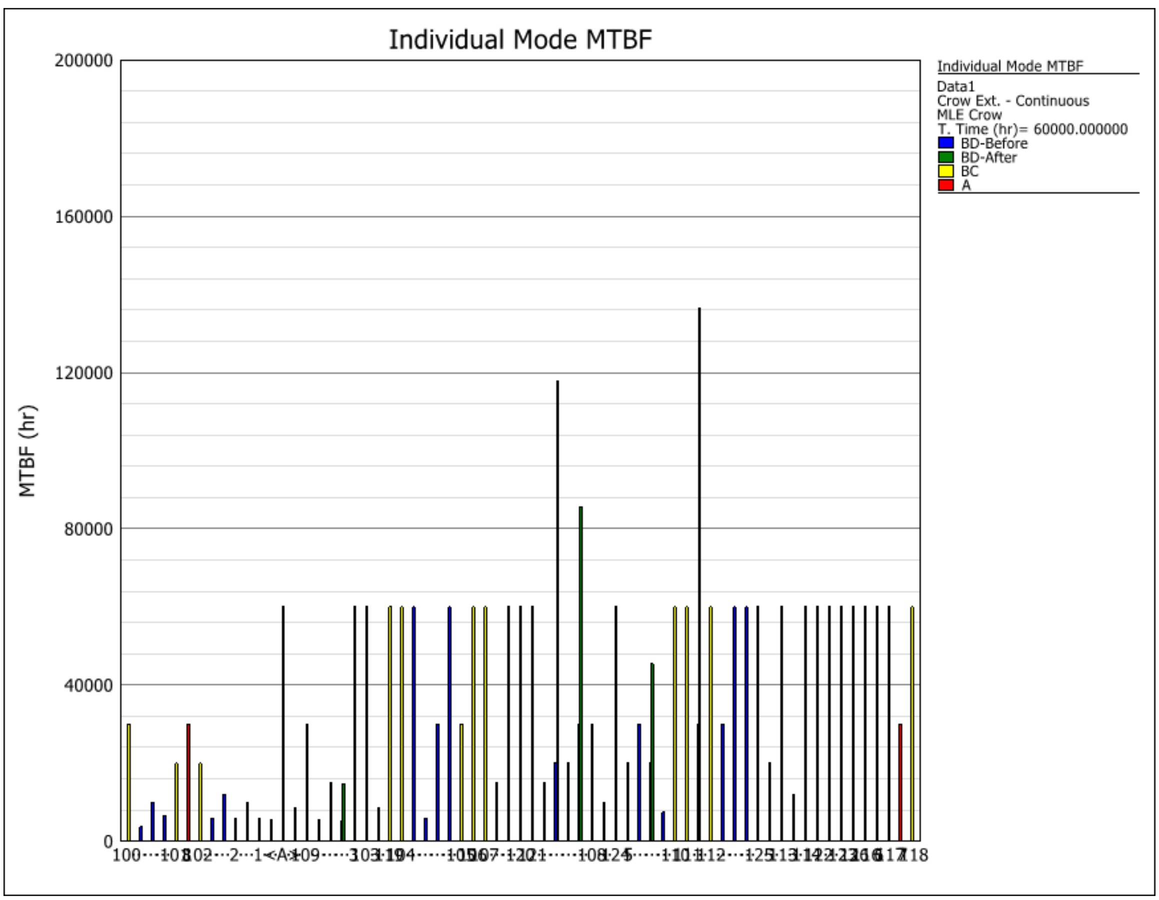


	BD Mode	Effectiveness Factor	Implemented at End of Phase #	Comments
1	5004	0.69	Not Implemented	
2	5014	0.71	Not Implemented	
3	5007	0.78	Not Implemented	
4	5013	0.66	6	
5	5017	0.79	2	
6	5018	0.67	3	
7	5020	0.65	3	
8	5021	0.83	6	
9	5023	0.65	6	
10	5025	0.58	5	
11	5026	0.56	6	
12	5028	0.78	6	
13	5030	0.69	5	
14	5031	0.66	5	
15	5033	0.8	5	
16				

The next plot shows the overall test results in terms of demonstrated, projected and growth potential MTBF.

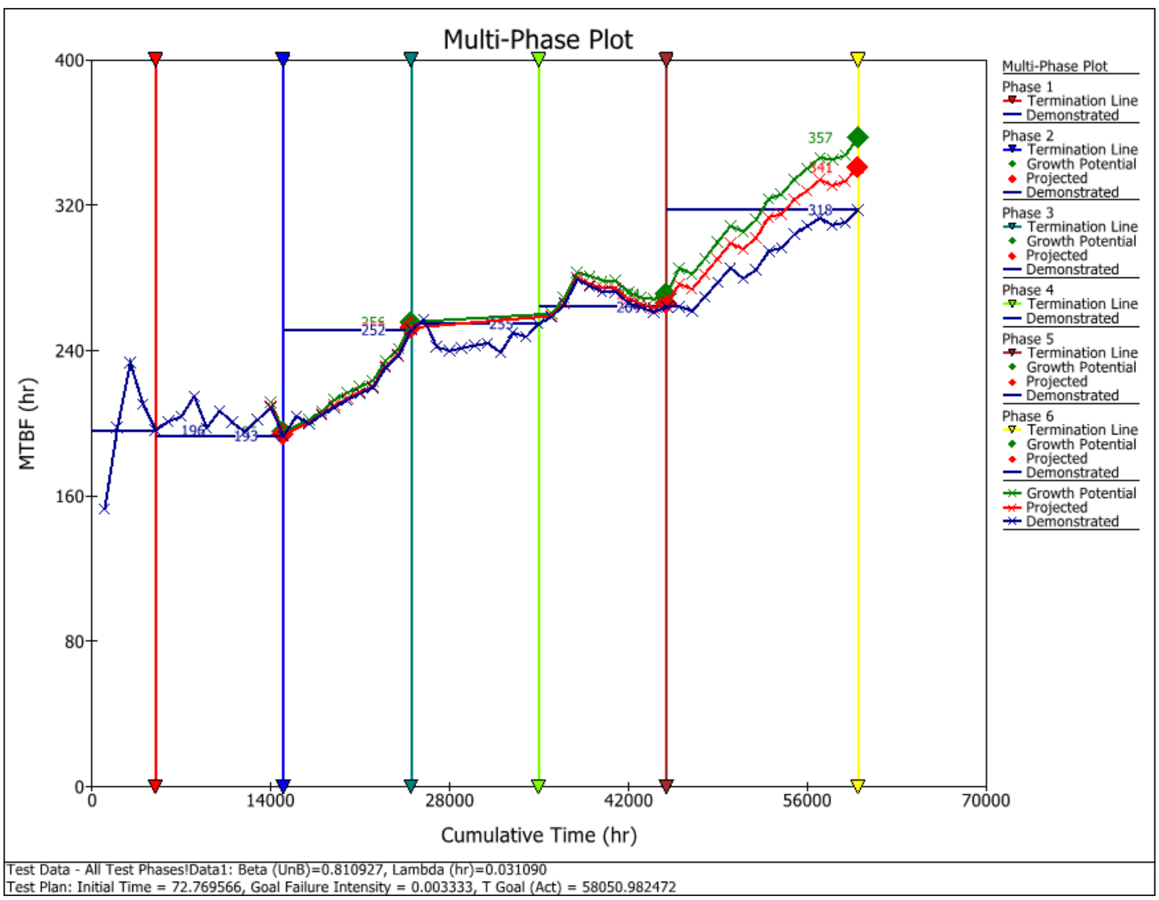


The next chart shows the "before" and "after" MTBFs for all the individual modes. Note that the "after" MTBFs are calculated by taking into account the respective effectiveness factors for each of the unfixed BD modes.



The analysis points are used to track overall growth inside and across phases, at desired intervals. The Weibull++ software allows you to create a multi-phase plot that is associated with a multi-phase data sheet. In addition, if you have an existing growth planning folio in your Weibull++ project, the multi-phase plot can plot the results from the multi-phase data sheet together with the reliability growth model from the growth planning folio. This allows you to track the overall reliability program against the goals, and set plans at each stage of the test. Additional information on combining a growth planning folio and a multi-phase plot is presented in the Reliability Growth Planning chapter.

The next figure shows the multi-phase plot for the six phases of the reliability growth test program. This plot can be a powerful tool for overall tracking of the reliability growth program. It displays the termination time for each phase of testing, along with the demonstrated, projected and growth potential MTBFs at those times. The plot also displays the calculated MTBFs at specified analysis points, which are determined based on the "AP" events in the data sheet.



Mixed Data

The Crow Extended - Continuous Evaluation model also can be applied for discrete data from one-shot (success/failure) testing. In the Weibull++ software, you can create a multi-phase data sheet that is configured for mixed data to accommodate data from tests where a single unit is tested for each successive configuration (individual trial-by-trial), where multiple units are tested for each successive configuration (configurations in groups) or a combination of both.

Corrective actions cannot take place at the time of failure for discrete data. With that in mind, the mixed data type does not allow for BC modes. The delayed corrective actions over time can be either fixed or unfixed, based on a subsequent implementation (I) event. So, for discrete data there are only unfixed BD modes or fixed BD modes. As a practical application, think of a multi-phase program for missile systems. Because the systems are one-shot items, the fixes for the failure modes are delayed until at least the next trial.

Note that for calculation purposes, it is required to have at least three failures in the first interval. If that is not the case, then the data set needs to be grouped until this requirement is met before calculating. The Weibull++ software performs this operation in the background.

Example - Multi-Phase Discrete Data

A one-shot system underwent reliability growth development for a total of 20 trials. The test was performed as a combination of configuration in groups and individual trial-by-trial. The following table shows the obtained data set. The **Failures in Interval** column specifies the number of failures that occurred in each interval, and the **Cumulative Trials** column specifies the cumulative number of trials at the end of that interval.

Mixed Data				
Event	**Failures in Interval**	**Cumulative Trials**	**Classification**	**Mode**
F	1	8	BD	1
F	1	8	BD	2
F	1	8	BD	3
F	0	10	A	
F	0	11	A	
F	0	12	A	
F	1	13	BD	2
F	0	14	A	
F	0	15	A	

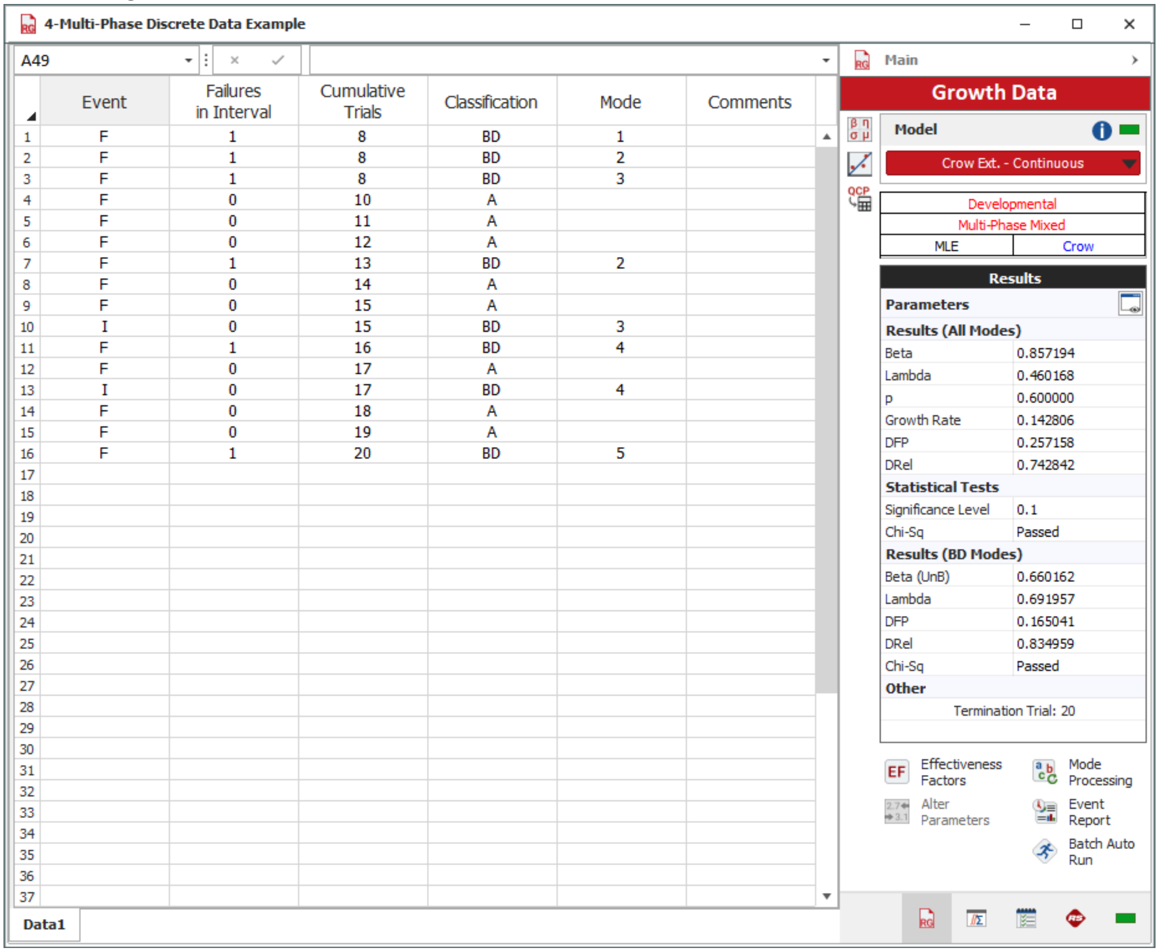
I	0	15	BD	3
F	1	16	BD	4
F	0	17	A	
I	0	17	BD	4
F	0	18	A	
F	0	18	A	
F	1	20	BD	5

The table also gives the classifications of the failure modes. There are 5 BD modes. Of these 5 modes, 2 are corrected during the test (BD3 and BD4) and 3 have not been corrected by time $T = 20$ (BD1, BD2 and BD5). Do the following:

1. Calculate the parameters of the Crow Extended - Continuous Evaluation model.
2. Calculated the demonstrated reliability at the end of the test.
3. Calculate parameter p.
4. Calculate the unfixed BD mode failure probability.
5. Calculate the nominal growth potential factor.
6. Calculate the nominal average effectiveness factor.
7. Calculate the discovery failure intensity function at the end of the test.
8. Calculate the nominal projected reliability at the end of the test.
9. Calculate the nominal growth potential reliability at the end of the test.

Solution

1. The next figure shows the data entered in the Weibull++ software.



The parameters β and λ are calculated as follows (the calculations for these parameters are presented in detail in the Crow-AMSAA (NHPP) chapter):

$$\widehat{\beta} = 0.8572$$

and:

$$\widehat{\lambda} = 0.4602$$

2. The corresponding demonstrated unreliability is calculated as:

$$f_D = \lambda \beta T^{\beta-1}, \text{with } T > 0,\ \lambda > 0 \text{ and } \beta > 0$$

Using the parameter estimates given above, the instantaneous unreliability at the end of the test (or $T = 20$) is equal to:

$$f_D(20) = 0.8572 \cdot 0.4602 \cdot 20^{0.8572-1} = 0.2572$$

The demonstrated reliability is:

$$R_D = 1 - f_D$$
$$= 1 - 0.2572 = 0.7428$$

Note that in discrete data, we calculate the reliability and not MTBF because we are dealing with the number of trials and not test time.

3. Assume that the following effectiveness factors are assigned to the unfixed BD modes:

Classification	Mode	Effectiveness Factor	Implemented at End of Phase
BD	1	0.65	Phase 1
BD	2	0.70	Phase 1
BD	5	0.75	Phase 1

The parameter p is the total number of distinct unfixed BD modes at time T divided by the total number of distinct BD (fixed and unfixed) modes.

In this example:

$$p = \frac{3}{5} = 0.6$$

4. The unfixed BD mode failure probability at time T is the total number of unfixed BD failures (classified at time T) divided by the total trials. Based on the table at the beginning of the example, we have:

$$f_{BD,unfixed} = \frac{4}{20} = 0.2$$

5. The nominal growth potential factor is:

$$\lambda_{NGPFactor} = \sum_{i=1}^{M} (1 - d_i) \frac{N_i}{T}$$
$$= (1 - d_1) \frac{N_1}{T} + (1 - d_2) \frac{N_2}{T} + (1 - d_5) \frac{N_5}{T}$$
$$= (1 - 0.65) \frac{1}{20} + (1 - 0.70) \frac{2}{20} + (1 - 0.75) \frac{1}{20}$$
$$= 0.06$$

6. The nominal average effectiveness factor is:

$$\begin{aligned} d_N =& \frac{\sum^M{}_{i=1} d_{Ni}}{M} \\ =& \frac{0.65+0.70+0.75}{3} \\ =& 0.70 \end{aligned}$$

7. The discovery function at time T is calculated using all the first occurrences of all the BD modes, both fixed and unfixed. In our example, we calculate $\widehat{\beta}_{BD}$ and $\widehat{\lambda}_{BD}$ using only the unfixed BD modes and excluding the second occurrence of BD2, which results in the following:

$$\widehat{\beta}_{BD}=0.6602$$

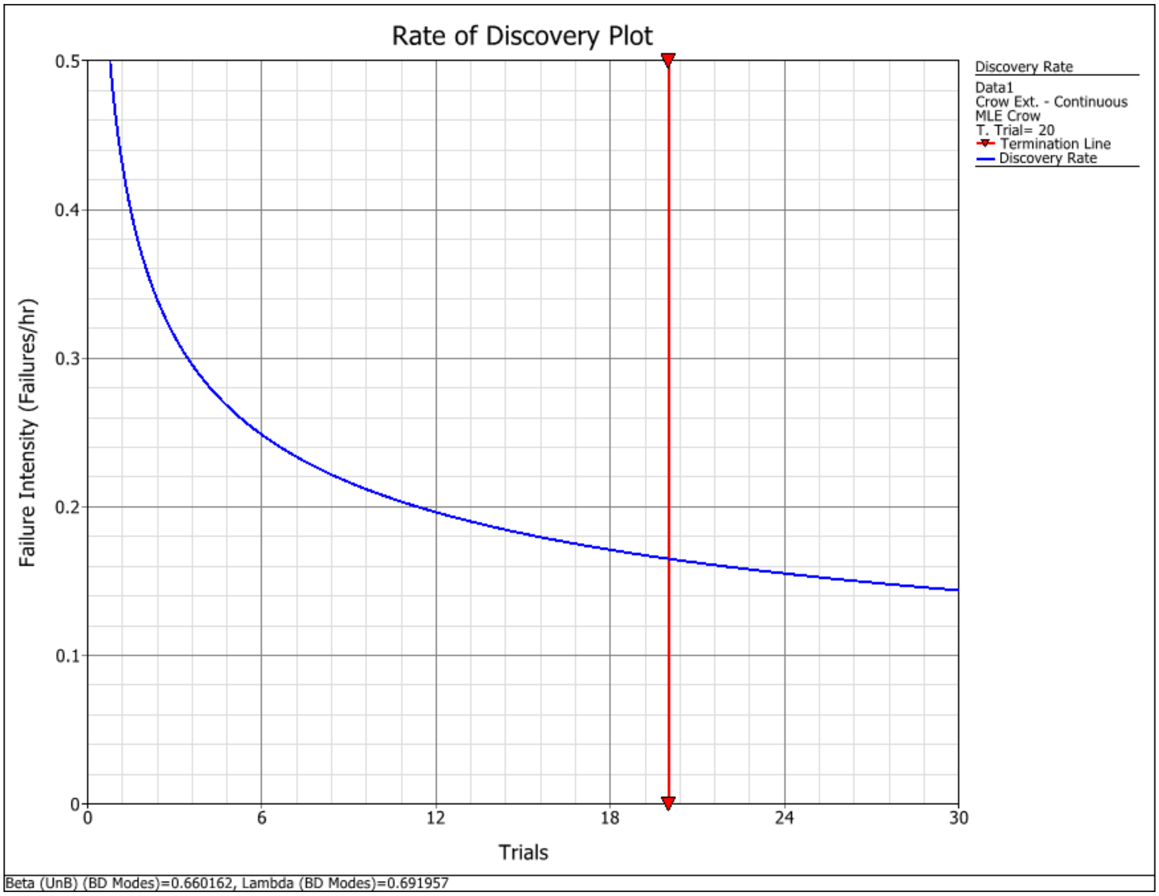
and:

$$\widehat{\lambda}_{BD}=0.6920$$

So the discovery failure intensity function at time $T=20$ is:

$$\begin{aligned} \widehat{h}(T|BD) =& \widehat{\lambda}_{BD}\widehat{\beta}_{BD}T^{\widehat{\beta}_{BD}-1} \\ =& 0.6920\cdot 0.6602\cdot 20^{0.6602-1} \\ =& 0.16507 \end{aligned}$$

The next figure shows the plot for the discovery failure intensity function.



8. The nominal projected failure probability at time $T = 20$ is:

$$\begin{aligned} f_{NP} &= f_{NGP} + d_N h(T) \\ &= 0.0701 + 0.7 \cdot 0.16507 \\ &= 0.1865 \end{aligned}$$

Therefore, the nominal projected reliability is:

$$\begin{aligned} R_P &= 1 - 0.1856 = \\ &= 0.8135 \end{aligned}$$

9. The nominal growth potential unreliability is:

$$f_{NGP} = f_D - f_{BDunfixed} + \lambda_{NGPFactor} + d_N \cdot p \cdot h(T) - d_N h(T)$$

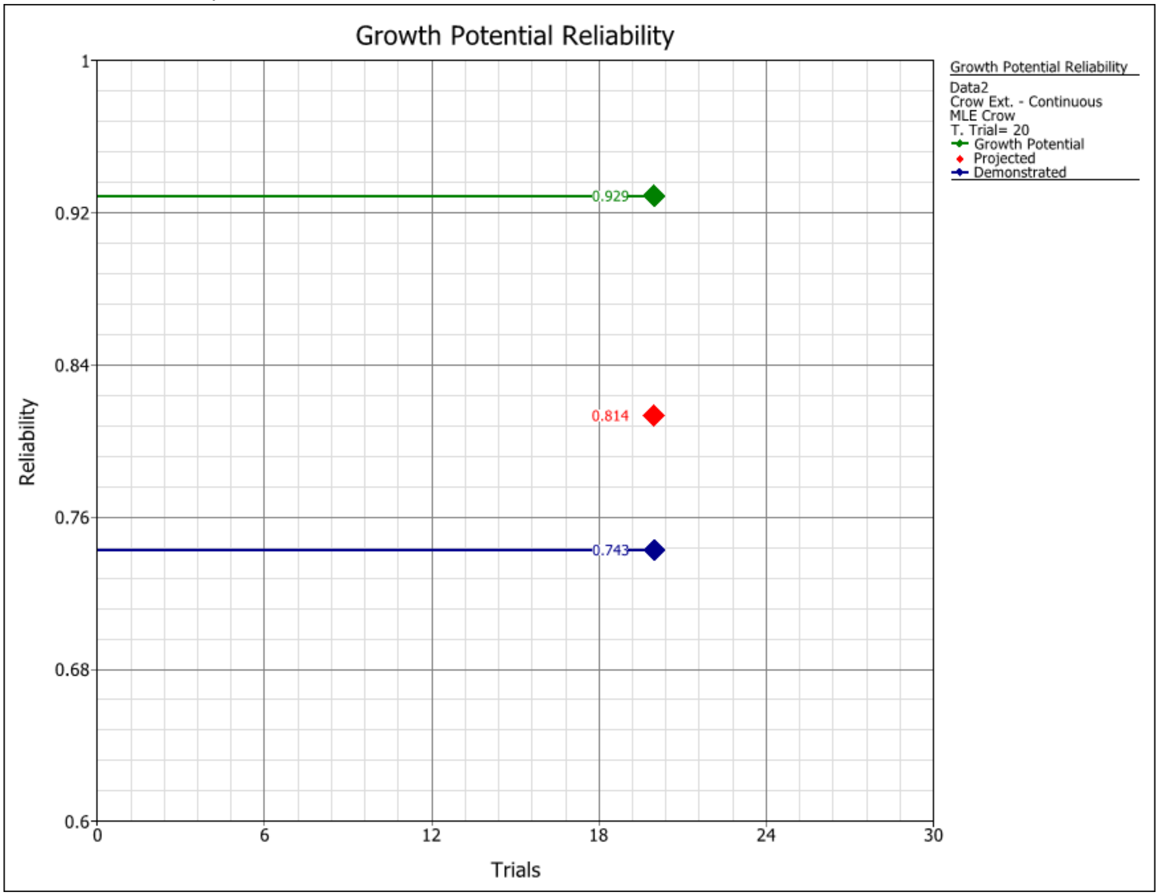
Based on the previous calculation for this example, we have:

$$\begin{aligned} f_{NGP} &= 0.2572 - 0.2 + 0.06 + 0.7 \cdot 0.6 \cdot 0.16507 - 0.7 \cdot 0.16507 \\ &= 0.0709 \end{aligned}$$

So the nominal growth potential reliability is:

$$\begin{aligned} R_{NGP} &= 1 - 0.0709 \\ &= 0.9291 \end{aligned}$$

The next figure shows the plot for the instantaneous demonstrated, projected and growth potential reliability.



Statistical Tests for Effectiveness of Corrective Actions

The purpose of the statistical tests is to explore the effectiveness of corrective actions during or at the end of a phase. Say we have two phases, phase 1 and phase 2. Suppose that corrective actions are incorporated during or at the end of phase 1. The system is then operated during phase 2. The general question is whether or not the corrective actions have been effective.

There are two questions that can be addressed regarding the effectiveness of the corrective actions:

Question 1. Is the average failure intensity for phase 2 statistically less than the average failure intensity for phase1?

Question 2. Is the average failure intensity for phase 2 statistically less than the Crow-AMSAA (NHPP) instantaneous failure intensity at the end of phase 1?

Average Failure Intensities Test

The purpose of this test is to compare the average failure intensity during phase 2 with the average failure intensity during phase 1.

The average failure intensity for phase 1 is:

$$\bar{r}_1 = \frac{N_1}{T_1}$$

where T_1 is the phase 1 test time and N_1 is the number of failures during phase 1.

Similarly, the average failure intensity for phase 2 is:

$$\bar{r}_2 = \frac{N_2}{T_2}$$

where T_2 is the phase 2 test time and N_2 is the number of failures during phase 2.

The overall test time, T, is:

$$T = T_1 + T_2$$

The overall number of failures, N, is:

$$N = N_1 + N_2$$

Define P as:

$$P = \frac{T_2}{T}$$

If the cumulative binomial probability $B(k; P, N)$ of up to N_2 failures is less than or equal to the statistical significance α, then the average failure intensity for phase 2 is statistically less than the average failure intensity for phase 1 at the specific significance level.

The cumulative binomial distribution probability is given by:

$$B(k; P, N) = \sum_{f=0}^{k} \binom{N}{f} P^f (1-P)^{N-f}$$

which gives the probability that the test failures, f, are less than or equal to the number of allowable failures, or acceptance number k in N trials, when each trial has a probability of succeeding of P.

EXAMPLE

Suppose a test is being conducted and is divided into two phases. The test time for the first phase is $T_1 = 27$ days, and the test time for the second phase is $T_2 = 18$ days. The number of failures during phase 1 is $N_1 = 11$, and the number of failures during phase 2 is $N_2 = 2$.

The average failure intensity for phase 1 is:

$$\bar{r}_1 = \frac{N_1}{T_1} = \frac{11}{27} = 0.4074.$$

Similarly, the average failure intensity for phase 2 is:

$$\bar{r}_2 = \frac{N_2}{T2} = \frac{2}{18} = 0.1111$$

Although the average failure intensities are different, we want to see, if at the 10% statistical significance level, the average failure intensity for phase 2 is statistically less than the average failure intensity for phase 1.

Solution

Concerning the total test time, we have:

$$T = T_1 + T_2 = 27 + 18 = 45$$

The total number of failures is equal to:

$$N = N_1 + N_2 = 11 + 2 = 13$$

Then, we calculate P as:

$$P = \frac{T_2}{T} = \frac{18}{45} = 0.4$$

Using the cumulative binomial distribution probability equation, we have:

$$\begin{aligned} B(k;P,N) =& B(N_2;P,N) \\ =& \sum_{f=0}^{N_2} \binom{N}{N_2} P^f (1-P)^{N-f} \\ =& \sum_{f=0}^{2} \binom{13}{f} 0.4^f (1-0.4)^{13-f} \\ =& 0.058 \end{aligned}$$

Because 0.058 is lower than 0.10, the conclusion is that at the 10% significance level of the the average failure intensity for phase 2 is statistically less than the average failure intensity for phase 1. The conclusion would be different for a different significance level. For example, at the 5% significance level, since 0.058 is not lower than 0.05, we fail to reject the null hypothesis. In other words, we cannot statistically prove any significant difference between the average failure intensities at the 5% level.

Average vs. Demonstrated Failure Intensities Test

The purpose of this test is to compare the average failure intensity during phase 2 with the Crow-AMSAA (NHPP) instantaneous (demonstrated) failure intensity at the end of phase 1.

Once again, the average failure intensity for phase 2 is given by:

$$\overline{r_2} = \frac{N_2}{T_2}$$

where T_2 is the phase 2 test time and N_2 is the number of failures during phase 2.

The Crow-AMSAA (NHPP) model estimate of failure intensity at time is $\hat{r}(T_1)$. Dr. Larry H. Crow [16] showed that the Crow-AMSAA (NHPP) estimate is approximately distributed as a random variable with standard deviation $\sqrt{\frac{N_1}{2}}$.

We therefore treat $\hat{r}(T_1)$ as an approximate Poisson random variable with number of failures:

$$N_1^* = \frac{N_1}{2}$$

We also set:

$$T_1^* = \frac{N_1^*}{\hat{r}(T_1)}$$

Then we define:

$$T = T_1^* + T_2$$

and:

$$N = N_1^* + N_2$$

Let:

$$P = \frac{T_2}{T}$$

If the cumulative binomial probability $B(k;P,N)$ of up to N_2 failures is less than or equal to the statistical significance α_1, then the average failure intensity for phase 2 is statistically less than the Crow-AMSAA (NHPP) instantaneous failure intensity at the end of phase 1, at the specific significance level. The cumulative binomial distribution probability is again given by:

$$B(k;P,N)=\sum_{f=0}^{k}\binom{N}{f}P^f(1-P)^{N-f}$$

EXAMPLE

Suppose a test is being conducted and is divided into two phases. The test time for the first phase is $T_1=21$ days, and the test time for the second phase is $T_2=18$ days. The number of failures during phase 1 is $N_1=11$ and the number of failures during phase 2 is $N_2=2$. The Crow-AMSAA (NHPP) parameters for the first phase are $\beta=0.7189$ and $\lambda=1.0288$.

The demonstrated failure intensity at the end of phase 1 is calculated as follows:

$$\begin{aligned}\hat{r}(T_1)&=\lambda\beta T^{\beta-1}\\&=1.0288\cdot 0.7189\cdot 27^{0.7189-1}\\&=0.2929\end{aligned}$$

The average failure intensity for phase 2 is:

$$\bar{r}_2=\frac{N_2}{T_2}=\frac{2}{8}=0.1111$$

Determine if the average failure intensity for phase 2 is statistically less than the demonstrated failure intensity at the end of phase 1 at the 10% significance level.

Solution

We calculate the number of failures N_1^* as:

$$N_1^*=\frac{N_1}{2}=\frac{11}{2}=5.5$$

Then:

$$T_1^*=\frac{N_1^*}{\hat{r}(T_1)}=\frac{5.5}{0.2929}=18.78$$

Concerning the total test time, we have:

$$T=T_1^*+T_2=18.78+18=36.78$$

Concerning the total number of failures, we have:

$$N = N_1^* + N_2 = 5.5 + 2 = 7.5$$

Then we calculate P as:

$$P = \frac{T_2}{T} = \frac{18}{36.78} = 0.4894$$

Since the number of failures $N = 7.5$ is not an integer, we are going to calculate the cumulative binomial probabilities for $N = 7$ and $N = 8$, and then interpolate to $N = 7.5$.

For $N = 7$, we have:

$$\begin{aligned} B(k; P, N) &= B(N_2; P, N) \\ &= B(2; 0.4894, 7) \\ &= \sum_{f=0}^{N_2} \binom{N}{N_2} p^f (1-P)^{N-f} \\ &= \sum_{f=0}^{2} \binom{7}{f} 0.4894^f (1-0.4849)^{7-f} \\ &= 0.244 \end{aligned}$$

And for $N = 8$, we have:

$$\begin{aligned} B(k; P, N) &= B(N_2; P, N) \\ &= B(2; 0.4894, 8) \\ &= \sum_{f=0}^{N_2} \binom{N}{N_2} p^f (1-P)^{N-f} \\ &= \sum_{f=0}^{2} \binom{8}{f} 0.4894^f (1-0.4849)^{8-f} \\ &= 0.158 \end{aligned}$$

A linear interpolation between two data points (x_a, y_a) and (x_b, y_b) at the (x, y) interpolant is given by:

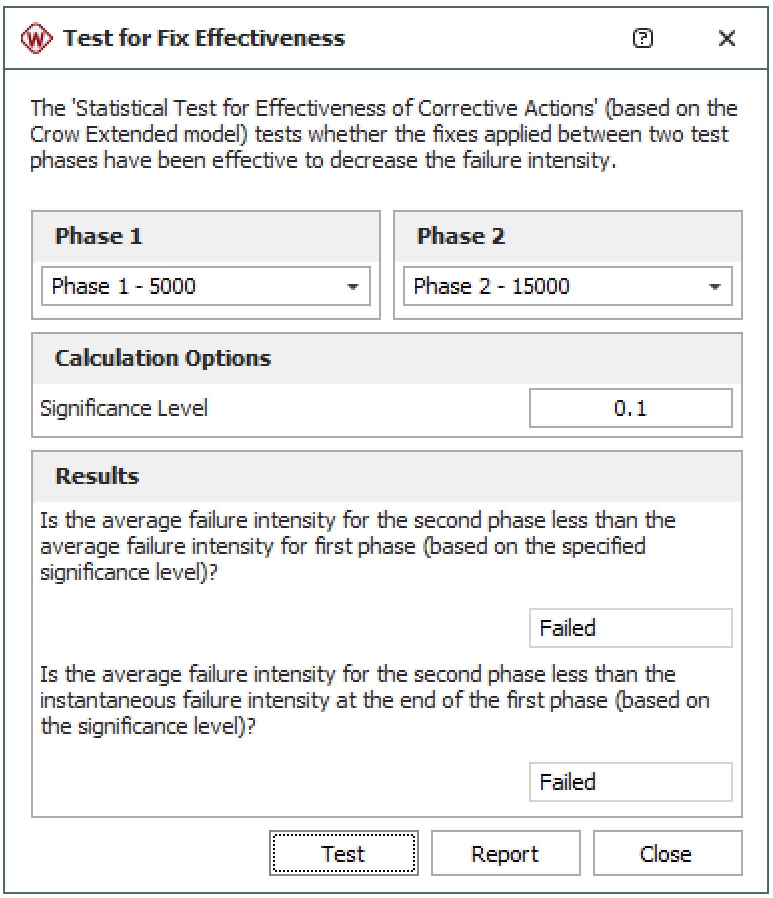
$$y = y_a + (x - x_a)\frac{(y_b - y_a)}{(x_b - x_a)}$$

So for $N = 7.5$ we would have that:

$$\begin{aligned} B(N_2;P,7.5) &= B(N_2;P,7)+(7.5-7)\frac{B(N_2;P,8)-B(N_2;P,7)}{8-7} \\ &= 0.244+(7.5-7)\frac{0.158-0.244}{8-7} \\ &= 0.201 \end{aligned}$$

Since 0.201 is greater than 0.10, the conclusion is that at the 10% significance level the average failure intensity for phase 2 is not statistically different compared to the demonstrated failure intensity at the end of phase 1.

Using the example with the six phases that was given earlier in this chapter, the following figure shows the application of the two statistical tests in Weibull++.



Lloyd-Lipow

Lloyd and Lipow (1962) considered a situation in which a test program is conducted in N stages. Each stage consists of a certain number of trials of an item undergoing testing, and the data set is recorded as successes or failures. All tests in a given stage of testing involve similar items. The results of each stage of testing are used to improve the item for further testing in the next stage. For the k^{th} group of data, taken in chronological order, there are n_k tests with S_k observed successes. The reliability growth function is then given in Lloyd and Lipow [6]:

$$R_k = R_\infty - \frac{\alpha}{k}$$

where:

- $R_k =$ the actual reliability during the k^{th} stage of testing
- $R_\infty =$ the ultimate reliability attained if $k \to \infty$
- $\alpha > 0 =$ modifies the rate of growth

Note that essentially, $R_k = \frac{S_k}{n_k}$. If the data set consists of reliability data, then S_k is assumed to be the observed reliability given and n_k is considered 1.

Parameter Estimation

When analyzing reliability data in the Weibull++ software, you have the option to enter the reliability values in percent or in decimal format. However, $\hat{R}_\infty$ will always be returned in decimal format and not in percent. The estimated parameters in the Weibull++ software are unitless.

Maximum Likelihood Estimators

For the k^{th} stage:

$$L_k = const.\ R_k^{S_k}(1 - R_k)^{n_k - S_k}$$

And assuming that the results are independent between stages:

$$L = \prod_{k=1}^{N} R_k^{S_k}(1 - R_k)^{n_k - S_k}$$

Then taking the natural log gives:

$$\Lambda = \sum_{k=1}^{N} S_k \ln\left(R_\infty - \frac{\alpha}{k}\right) + \sum_{k=1}^{N} (n_k - S_k) \ln\left(1 - R_\infty + \frac{\alpha}{k}\right)$$

Differentiating with respect to R_∞ and α, yields:

$$\frac{\partial \Lambda}{\partial R_\infty} = \sum_{k=1}^{N} \frac{S_k}{R_\infty - \frac{\alpha}{k}} - \sum_{k=1}^{N} \frac{n_k - S_k}{1 - R_\infty + \frac{\alpha}{k}}$$

$$\frac{\partial\Lambda}{\partial\alpha}=-\sum_{k=1}^{N}\frac{\frac{S_k}{k}}{R_\infty-\frac{\alpha}{k}}+\sum_{k=1}^{N}\frac{\frac{n_k-S_k}{k}}{1-R_\infty+\frac{\alpha}{k}}$$

Rearranging the equations and setting them equal to zero gives:

$$\frac{\partial\Lambda}{\partial R_\infty}=\sum_{k=1}^{N}\frac{\frac{S_k}{n_k}-\left(R_\infty-\frac{\alpha}{k}\right)}{\frac{1}{n_k}\left(R_\infty-\frac{\alpha}{k}\right)\left(1-R_\infty+\frac{\alpha}{k}\right)}=0$$

$$\frac{\partial\Lambda}{\partial\alpha}=-\sum_{k=1}^{N}\frac{\frac{1}{k}\frac{S_k}{n_k}-\left(R_\infty-\frac{\alpha}{k}\right)\frac{1}{k}}{\frac{1}{n_k}\left(R_\infty-\frac{\alpha}{k}\right)\left(1-R_\infty+\frac{\alpha}{k}\right)}=0$$

The resulting equations can be solved simultaneously for $\hat{\alpha}$ and $\hat{R}_\infty$. It should be noted that a closed form solution does not exist for either of the parameters; thus, they must be estimated numerically.

Least Squares Estimators

To obtain least squares estimators for R_∞ and α, the sum of squares, Q , of the deviations of the observed success-ratio, S_k/n_k , is minimized from its expected value, $R_\infty-\frac{\alpha}{k}$, with respect to the parameters R_∞ and α. Therefore, Q is expressed as:

$$Q=\sum_{k=1}^{N}\left(\frac{S_k}{n_k}-R_\infty+\frac{\alpha}{k}\right)^2$$

Taking the derivatives with respect to R_∞ and α and setting equal to zero yields:

$$\frac{\partial Q}{\partial R_\infty}=-2\sum_{k=1}^{N}\left(\frac{S_k}{n_k}-R_\infty+\frac{\alpha}{k}\right)=0$$

$$\frac{\partial Q}{\partial\alpha}=2\sum_{k=1}^{N}\left(\frac{S_k}{n_k}-R_\infty+\frac{\alpha}{k}\right)\frac{1}{k}=0$$

Solving the equations simultaneously, the least squares estimates of R_∞ and α are:

$$\hat{R}_\infty=\frac{\sum^{N}{}_{k=1}\frac{1}{k^2}\sum^{N}{}_{k=1}\frac{S_k}{n_k}-\sum^{N}{}_{k=1}\frac{1}{k}\sum^{N}{}_{k=1}\frac{S_k}{kn_k}}{N\sum^{N}{}_{k=1}\frac{1}{k^2}-\left(\sum^{N}{}_{k=1}\frac{1}{k}\right)^2}$$

or:

$$\hat{R}_\infty=\frac{\sum^{N}{}_{k=1}\frac{1}{k^2}\sum^{N}{}_{k=1}R_k-\sum^{N}{}_{k=1}\frac{1}{k}\sum^{N}{}_{k=1}\frac{R_k}{k}}{N\sum^{N}{}_{k=1}\frac{1}{k^2}-\left(\sum^{N}{}_{k=1}\frac{1}{k}\right)^2}$$

and:

$$\hat{\alpha}=\frac{\sum_{k=1}^{N}\frac{1}{k}\sum_{k=1}^{N}\frac{S_k}{n_k}-N\sum_{k=1}^{N}\frac{S_k}{kn_k}}{N\sum_{k=1}^{N}\frac{1}{k^2}-\left(\sum_{k=1}^{N}\frac{1}{k}\right)^2}$$

or:

$$\hat{\alpha}=\frac{\sum_{k=1}^{N}\frac{1}{k}\sum_{k=1}^{N}R_k-N\sum_{k=1}^{N}\frac{R_k}{k}}{N\sum_{k=1}^{N}\frac{1}{k^2}-\left(\sum_{k=1}^{N}\frac{1}{k}\right)^2}$$

EXAMPLE - LEAST SQUARES

After a 20-stage reliability development test program, 20 groups of success/failure data were obtained and are given in the table below. Do the following:

1. Fit the Lloyd-Lipow model to the data using least squares.
2. Plot the reliabilities predicted by the Lloyd-Lipow model along with the observed reliabilities, and compare the results.

The Test Results and Reliabilities of Each Stage Calculated from Raw Data and the Predicted Reliability

Test Stage Number (k)	Number of Tests in Stage (n_k)	Number of Successful Tests (S_k)	Raw Data Reliability	Lloyd-Lipow Reliability
1	9	6	0.667	0.7002
2	9	5	0.556	0.7369
3	8	7	0.875	0.7552
4	10	6	0.600	0.7662
5	9	7	0.778	0.7736
6	10	8	0.800	0.7788
7	10	7	0.700	0.7827
8	10	6	0.600	0.7858
9	11	7	0.636	0.7882
10	11	9	0.818	0.7902

11	9	9	1.000	0.7919
12	12	10	0.833	0.7933
13	12	9	0.750	0.7945
14	11	8	0.727	0.7956
15	10	7	0.700	0.7965
16	10	8	0.800	0.7973
17	11	10	0.909	0.7980
18	10	9	0.900	0.7987
19	9	8	0.889	0.7992
20	8	7	0.875	0.7998

Solution

1. The least squares estimates are:

$$\sum_{k=1}^{N}\frac{1}{k}=\sum_{k=1}^{20}\frac{1}{k}=3.5977$$
$$\sum_{k=1}^{N}\frac{1}{k^2}=\sum_{k=1}^{20}\frac{1}{k^2}=1.5962$$
$$\sum_{k=1}^{N}\frac{S_k}{n_k}=\sum_{k=1}^{20}\frac{S_k}{n_k}=15.4131$$

and:

$$\sum_{k=1}^{N}\frac{S_k}{k\cdot n_k}=\sum_{k=1}^{20}\frac{S_k}{k\cdot n_k}=2.5632$$

Using these estimates to obtain $\hat{R_\infty}$ and $\hat{\alpha}$ yields:

$$\begin{aligned}\hat{R}_\infty &=\frac{\sum^{N}{}_{k=1}\frac{1}{k^2}\sum^{N}{}_{k=1}R_k-\sum^{N}{}_{k=1}\frac{1}{k}\sum^{N}{}_{k=1}\frac{R_k}{k}}{N\sum^{N}{}_{k=1}\frac{1}{k^2}-\left(\sum^{N}{}_{k=1}\frac{1}{k}\right)^2}\\ &=\frac{(1.5962)(15.413)-(3.5977)(2.5637)}{(20)(1.5962)-(3.5977)^2}\\ &=0.8104\end{aligned}$$

and:

$$\hat{\alpha} = \frac{\sum^{N}{}_{k=1} \frac{1}{k} \sum^{N}{}_{k=1} R_k - N \sum^{N}{}_{k=1} \frac{R_k}{k}}{N \sum^{N}{}_{k=1} \frac{1}{k^2} - \left(\sum^{N}{}_{k=1} \frac{1}{k}\right)^2}$$

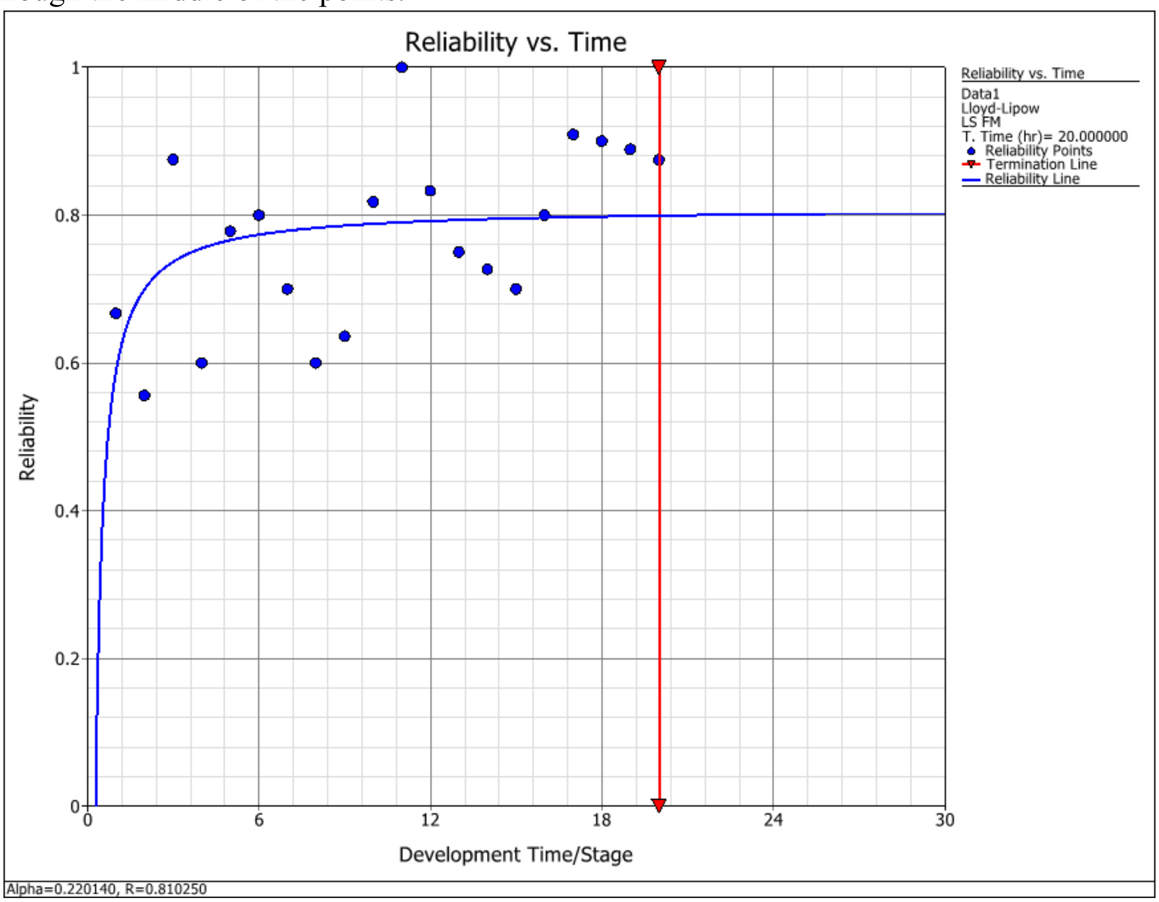
$$= \frac{(3.5977)(15.413) - (20)(2.5637)}{(20)(1.5962) - (3.5977)^2}$$

$$= 0.2207$$

Therefore, the Lloyd-Lipow reliability growth model is as follows, where k is the test stage.

$$R_k = 0.8104 - \frac{0.2201}{k}$$

2. The reliabilities from the raw data and the reliabilities predicted from the Lloyd-Lipow reliability growth model are given in the last two columns of the table. The figure below shows the plot. Based on the given data, the model cannot do much more than to basically fit a line through the middle of the points.



Confidence Bounds

This section presents the methods used in the Weibull++ software to estimate the confidence bounds under the Lloyd-Lipow model. One of the properties of maximum likelihood estimators is that they are asymptotically normal. This indicates that they are normally distributed for large samples [6, 7]. Additionally, since the parameter α must be positive, $\ln\alpha$ is treated as being normally distributed as well. The parameter R_∞ represents the ultimate reliability that would be attained if $k\rightarrow\infty$. R_k is the actual reliability during the k^{th} stage of testing. Therefore, R_∞ and R_k will be between 0 and 1. Consequently, the endpoints of the confidence intervals of the parameters R_∞ and R_k also will be between 0 and 1. To obtain the confidence interval, it is common practice to use the logit transformation.

The confidence bounds on the parameters α and R_∞ are given by:

$$CB_\alpha=\hat{\alpha}e^{\pm z_{\alpha/2}\sqrt{Var(\hat{\alpha})}/\hat{\alpha}}$$

$$CB_{R_\infty}=\frac{\hat{R}_\infty}{\hat{R}_\infty+(1-\hat{R}_\infty)e^{\pm z_{\alpha/2}\sqrt{Var(\hat{R}_\infty)}/[\hat{R}_\infty(1-\hat{R}_\infty)]}}$$

where $z_{\alpha/2}$ represents the percentage points of the $N(0,1)$ distribution such that $P\{z\geq z_{\alpha/2}\}=\alpha/2$.

The confidence bounds on reliability are given by:

$$CB=\frac{\hat{R}_k}{\hat{R}_k+(1-\hat{R}_k)e^{\pm z_{\alpha/2}\sqrt{Var(\hat{R}_k)}/[\hat{R}_k(1-\hat{R}_k)]}}$$

where:

$$Var(\widehat{R}_k)=Var(\widehat{R}_\infty)+\frac{1}{k^2}\cdot Var(\widehat{\alpha})-\frac{2}{k}\cdot Cov(\widehat{R}_\infty,\widehat{\alpha})$$

All the variances can be calculated using the Fisher Matrix:

$$\begin{bmatrix}-\frac{\partial^2\Lambda}{\partial R_\infty^2} & -\frac{\partial^2\Lambda}{\partial\alpha\partial R_\infty}\\ -\frac{\partial^2\Lambda}{\partial\alpha\partial R_\infty} & -\frac{\partial^2\Lambda}{\partial\alpha^2}\end{bmatrix}^{-1}=\begin{bmatrix}Var(\widehat{R}_\infty) & Cov(\widehat{R}_\infty,\widehat{\alpha})\\ Cov(\widehat{R}_\infty,\widehat{\alpha}) & Var(\widehat{\alpha})\end{bmatrix}$$

From the ML estimators of the Lloyd-Lipow model, taking the second partial derivatives yields:

$$\frac{\partial^2\Lambda}{\partial R_\infty^2}=-\sum_{k=1}^{N}\frac{S_k}{\left(R_\infty-\frac{\alpha}{k}\right)^2}-\sum_{k=1}^{N}\frac{n_k-S_k}{\left(1-R_\infty+\frac{\alpha}{k}\right)^2}$$

$$\frac{\partial^2\Lambda}{\partial\alpha^2}=-\sum_{k=1}^{N}\frac{\frac{S_k}{k^2}}{\left(R_\infty-\frac{\alpha}{k}\right)^2}-\sum_{k=1}^{N}\frac{\frac{n_k-S_k}{k^2}}{\left(1-R_\infty+\frac{\alpha}{k}\right)^2}$$

and:

$$\frac{\partial^2 \Lambda}{\partial R_\infty \partial \alpha} = \sum_{k=1}^{N} \frac{\frac{S_k}{k}}{\left(R_\infty - \frac{\alpha}{k}\right)^2} - \sum_{k=1}^{N} \frac{\frac{n_k - S_k}{k}}{\left(1 - R_\infty + \frac{\alpha}{k}\right)^2}$$

The confidence bounds can be obtained by solving for the three equations shown above and the equation for $Var(\hat{R}_k)$, and then substituting the values into the Fisher Matrix.

As an example, you can calculate and plot the confidence bounds for the data set given above in the Least Squares example as:

$$\frac{\partial^2 \Lambda}{\partial R_\infty^2} = -255.3835 - 937.2902 = -1192.6737$$
$$\frac{\partial^2 \Lambda}{\partial \alpha^2} = -24.4575 - 43.3930 = -67.8505$$
$$\frac{\partial^2 \Lambda}{\partial R_\infty \partial \alpha} = 48.6606 - 140.7518 = -92.0912$$

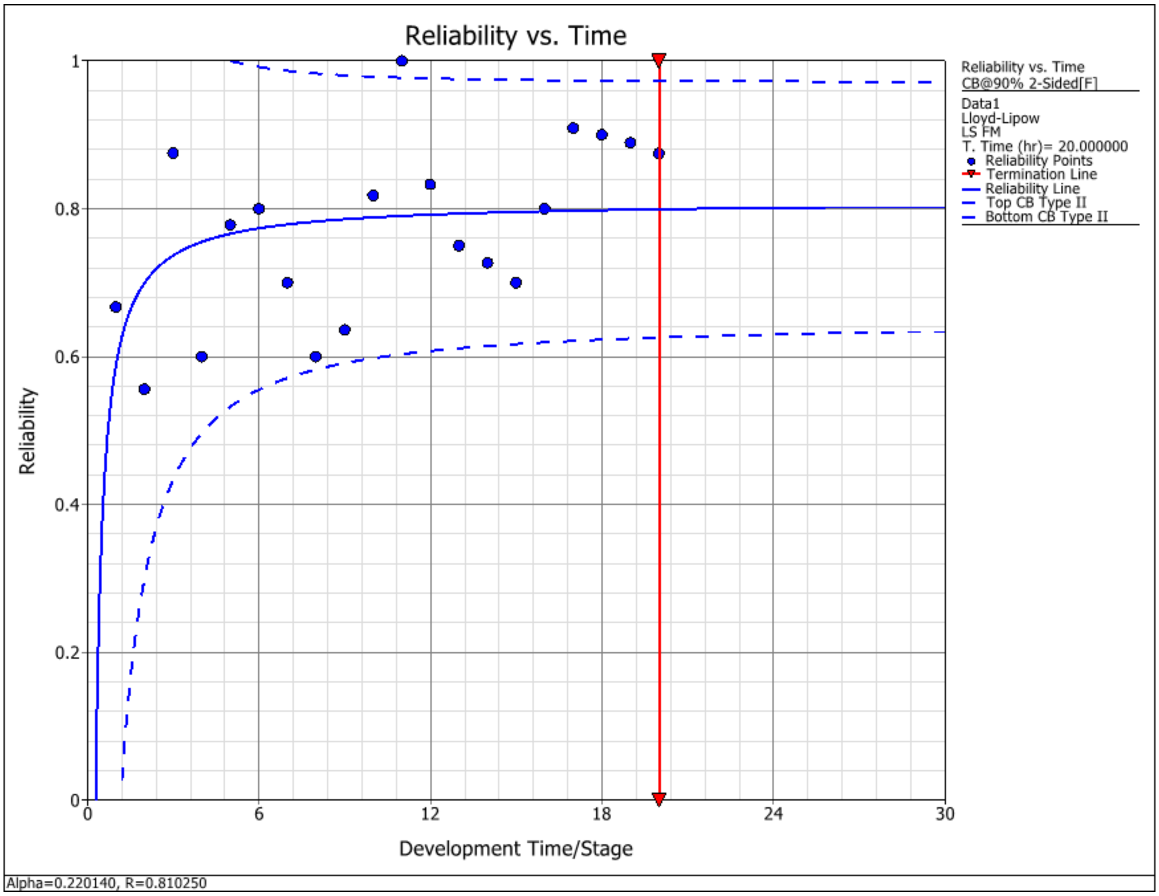
The variances can be calculated using the Fisher Matrix:

$$\begin{bmatrix} 1192.6737 & 92.0912 \\ 92.0912 & 67.8505 \end{bmatrix}^{-1} = \begin{bmatrix} Var(\widehat{R}_\infty) & Cov(\widehat{R}_\infty, \widehat{\alpha}) \\ Cov(\widehat{R}_\infty, \widehat{\alpha}) & Var(\widehat{\alpha}) \end{bmatrix} = \begin{bmatrix} 0.00093661 & -0.00127123 \\ -0.00127123 & 0.01646371 \end{bmatrix}$$

The variance of R_k is obtained such that:

$$Var(\widehat{R}_k) = Var(\widehat{R}_\infty) + \frac{1}{k^2} \cdot Var(\widehat{\alpha}) - \frac{2}{k} \cdot Cov(\widehat{R}_\infty, \widehat{\alpha})$$
$$Var(\widehat{R}_k) = 0.00093661 + \frac{1}{k^2} \cdot 0.01646371 + \frac{2}{k} \cdot 0.00127123$$

The confidence bounds on reliability can now be calculated. The associated confidence bounds at the 90% confidence level are plotted in the figure below with the predicted reliability, R_k .



Example - Lloyd-Lipow Confidence Bounds

Consider the success/failure data given in the following table. Solve for the Lloyd-Lipow parameters using least squares analysis, and plot the Lloyd-Lipow reliability with 2-sided confidence bounds at the 90% confidence level.

Success/Failure Data for a Variable Number of Tests Performed in Each Test Stage

Test Stage Number (k)	Result	Number of Tests (n_k >	Successful Tests ($S_k = R_i$)
1	F	1	0
2	F	1	0
3	F	1	0
4	S	1	0.2500
5	F	1	0.2000

6	F	1	0.1667
7	S	1	0.2857
8	S	1	0.3750
9	S	1	0.4444
10	S	1	0.5000
11	S	1	0.5455
12	S	1	0.5833
13	S	1	0.6154
14	S	1	0.6429
15	S	1	0.6667
16	S	1	0.6875
17	F	1	0.6471
18	S	1	0.6667
19	F	1	0.6316
20	S	1	0.6500
21	S	1	0.6667
22	S	1	0.6818

Solution

Note that the data set contains three consecutive failures at the beginning of the test. These failures will be ignored throughout the analysis because it is considered that the test starts when the reliability is not equal to zero or one. The number of data points is now reduced to 19. Also, note that the only time that the first three first failures are considered is to calculate the observed reliability in the test. For example, given this data set, the observed reliability at stage 4 is $1/4=0.25$. This is considered to be the reliability at stage 1.

From the table, the least squares estimates can be calculated as follows:

$$\sum_{k=1}^{N}\frac{1}{k}=\sum_{k=1}^{19}\frac{1}{k}=3.54774$$
$$\sum_{k=1}^{N}\frac{1}{k^2}=\sum_{k=1}^{19}\frac{1}{k^2}=1.5936$$
$$\sum_{k=1}^{N}\frac{S_k}{n_k}=\sum_{k=1}^{19}\frac{S_k}{n_k}=9.907$$

and:

$$\sum_{k=1}^{N}\frac{S_k}{k\cdot n_k}=\sum_{k=1}^{19}\frac{S_k}{k\cdot n_k}=1.3002$$

Using these estimates to obtain $\hat{R}_\infty$ and $\hat{\alpha}$ yields:

$$\begin{aligned}\hat{R}_\infty &=\frac{(1.5936)(9.907)-(3.5477)(1.3002)}{(19)(1.5936)-(3.5477)^2}\\ &=0.6316\end{aligned}$$

and:

$$\begin{aligned}\hat{\alpha} &=\frac{(3.5477)(9.907)-(19)(1.3002)}{(19)(1.5936)-(3.5477)^2}\\ &=0.5902\end{aligned}$$

Therefore, the Lloyd-Lipow reliability growth model is as follows, where k is the number of the test stage.

$$R_k=0.6316-\frac{0.5902}{k}$$

Using the data from the table:

$$\begin{aligned}\frac{\partial^2\Lambda}{\partial R_\infty^2} &=-176.847-40.500=-217.347\\ \frac{\partial^2\Lambda}{\partial\alpha^2} &=-146.763-2.1274=-148.891\\ \frac{\partial^2\Lambda}{\partial R_\infty\partial\alpha} &=149.909-6.5660=143.343\end{aligned}$$

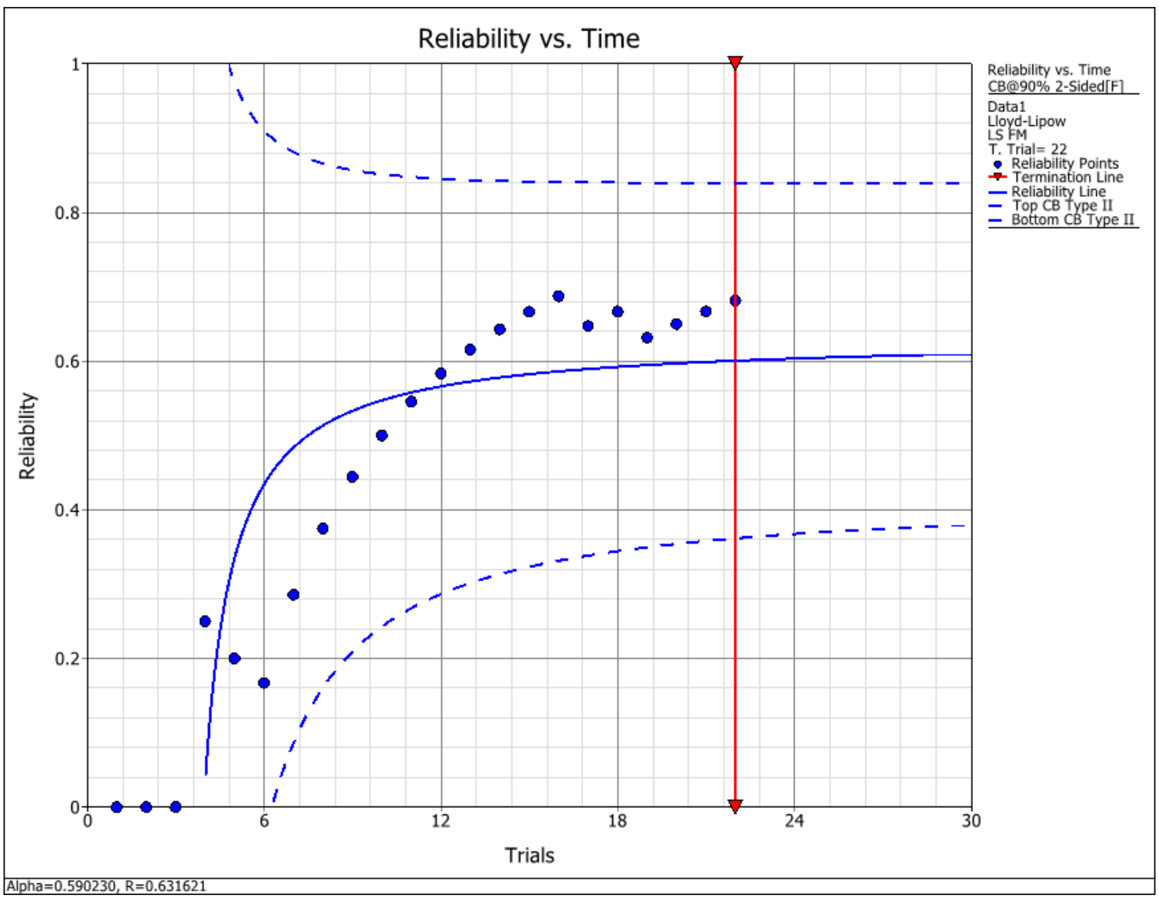
The variances can be calculated using the Fisher Matrix:

$$\begin{bmatrix} 217.347 & -143.343 \\ -143.343 & 148.891 \end{bmatrix}^{-1} = \begin{bmatrix} Var(\widehat{R}_{\infty}) & Cov(\widehat{R}_{\infty}, \widehat{\alpha}) \\ Cov(\widehat{R}_{\infty}, \widehat{\alpha}) & Var(\widehat{\alpha}) \end{bmatrix}$$
$$= \begin{bmatrix} 0.0126033 & 0.0121335 \\ 0.0121335 & 0.0183977 \end{bmatrix}$$

The variance of R_k is therefore:

$$Var(\widehat{R}_k) = 0.0126031 + \frac{1}{k^2} \cdot 0.0183977 - \frac{2}{k} \cdot 0.0121335$$

The confidence bounds on reliability can now can be calculated. The associated confidence bounds on reliability at the 90% confidence level are plotted in the following figure, with the predicted reliability, R_k .



More Examples

Grouped per Configuration Example

A 15-stage reliability development test program was performed. The grouped per configuration data set is shown in the following table. Do the following:

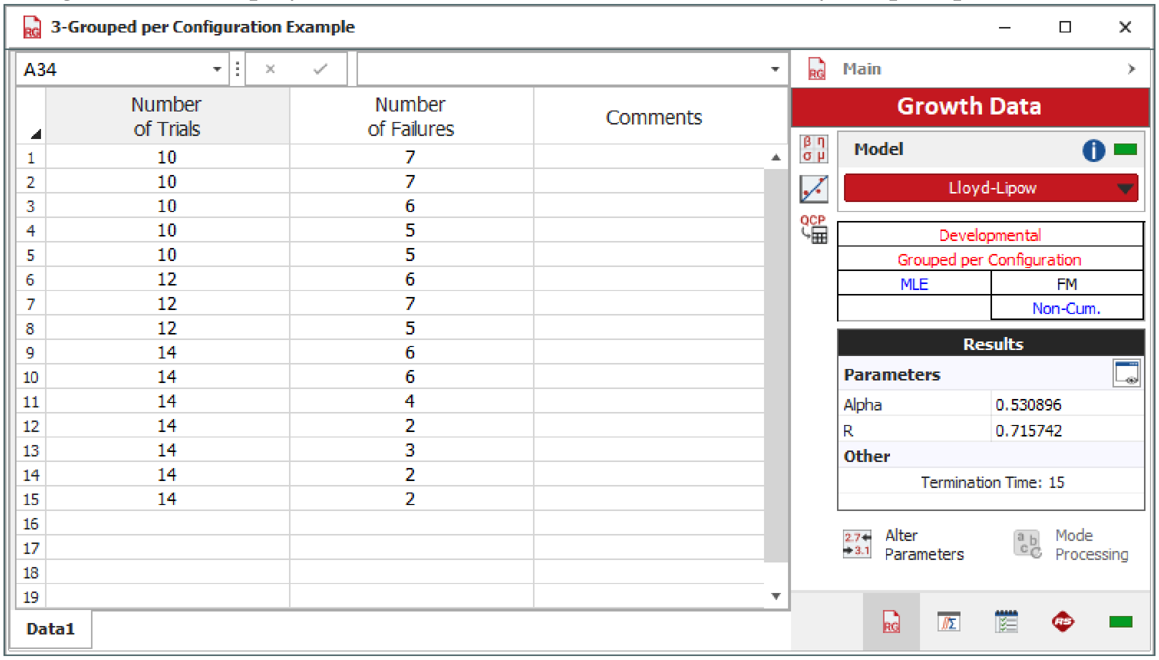
1. Fit the Lloyd-Lipow model to the data using MLE.
2. What is the maximum reliability attained as the number of test stages approaches infinity?
3. What is the maximum achievable reliability with a 90% confidence level?

Grouped per Configuration Data

Stage, k	Number of Tests (n_k)	Number of Successes (S_k)
1	10	3
2	10	3
3	10	4
4	10	5
5	10	5
6	12	6
7	12	5
8	12	7
9	14	8
10	14	8
11	14	10
12	14	12
13	14	11
14	14	12
15	14	12

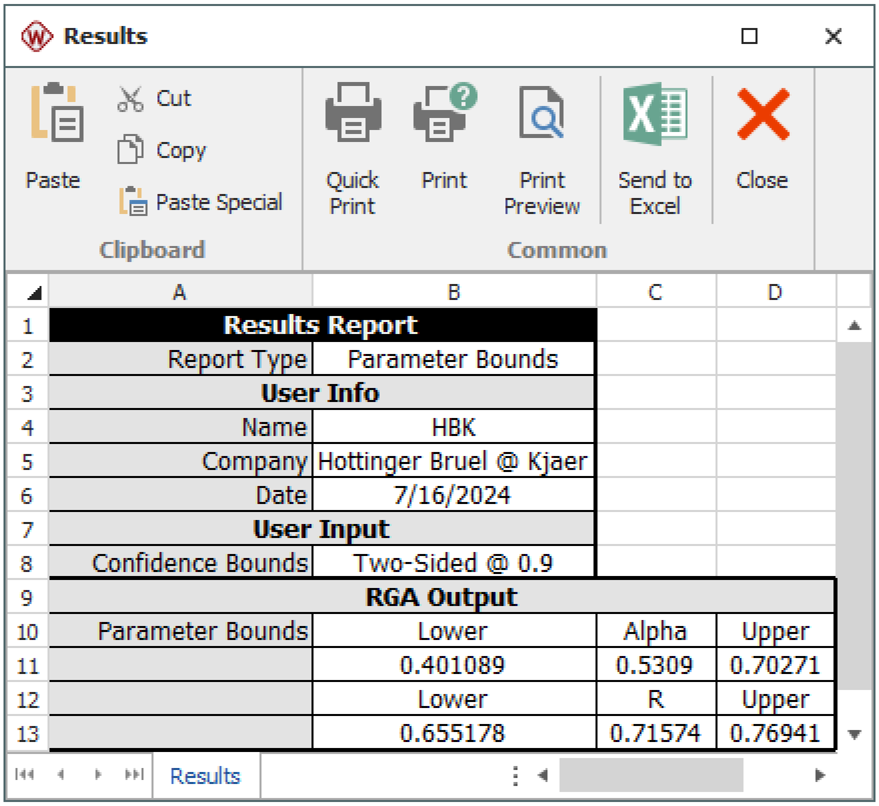
Solution

1. The figure below displays the entered data and the estimated Lloyd-Lipow parameters.



2. The maximum achievable reliability as the number of test stages approaches infinity is equal to the value of R. Therefore, $R = 0.7157$.

3. The maximum achievable reliability with a 90% confidence level can be estimated by viewing the confidence bounds on the parameters in the QCP, as shown in the figure below. The lower bound on the value of R is equal to 0.6551 .



	A	B	C	D
1	Results Report			
2	Report Type	Parameter Bounds		
3	User Info			
4	Name	HBK		
5	Company	Hottinger Bruel @ Kjaer		
6	Date	7/16/2024		
7	User Input			
8	Confidence Bounds	Two-Sided @ 0.9		
9	RGA Output			
10	Parameter Bounds	Lower	Alpha	Upper
11		0.401089	0.5309	0.70271
12		Lower	R	Upper
13		0.655178	0.71574	0.76941

Reliability Data Example

Given the reliability data in the table below, do the following:

1. Fit the Lloyd-Lipow model to the data using least squares analysis.
2. Plot the Lloyd-Lipow reliability with 90% 2-sided confidence bounds.
3. How many months of testing are required to achieve a reliability goal of 90%?
4. What is the attainable reliability if the maximum duration of testing is 30 months?

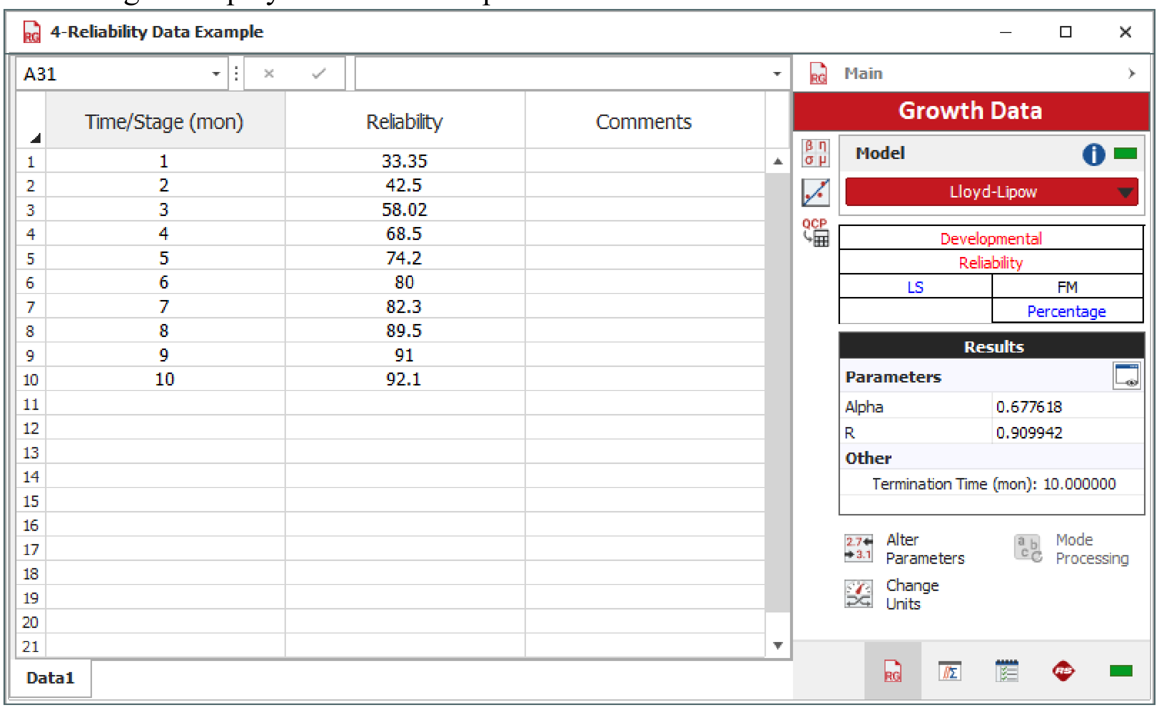
Reliability Data

Time (months)	Reliability (%)
1	33.35
2	42.50
3	58.02
4	68.50

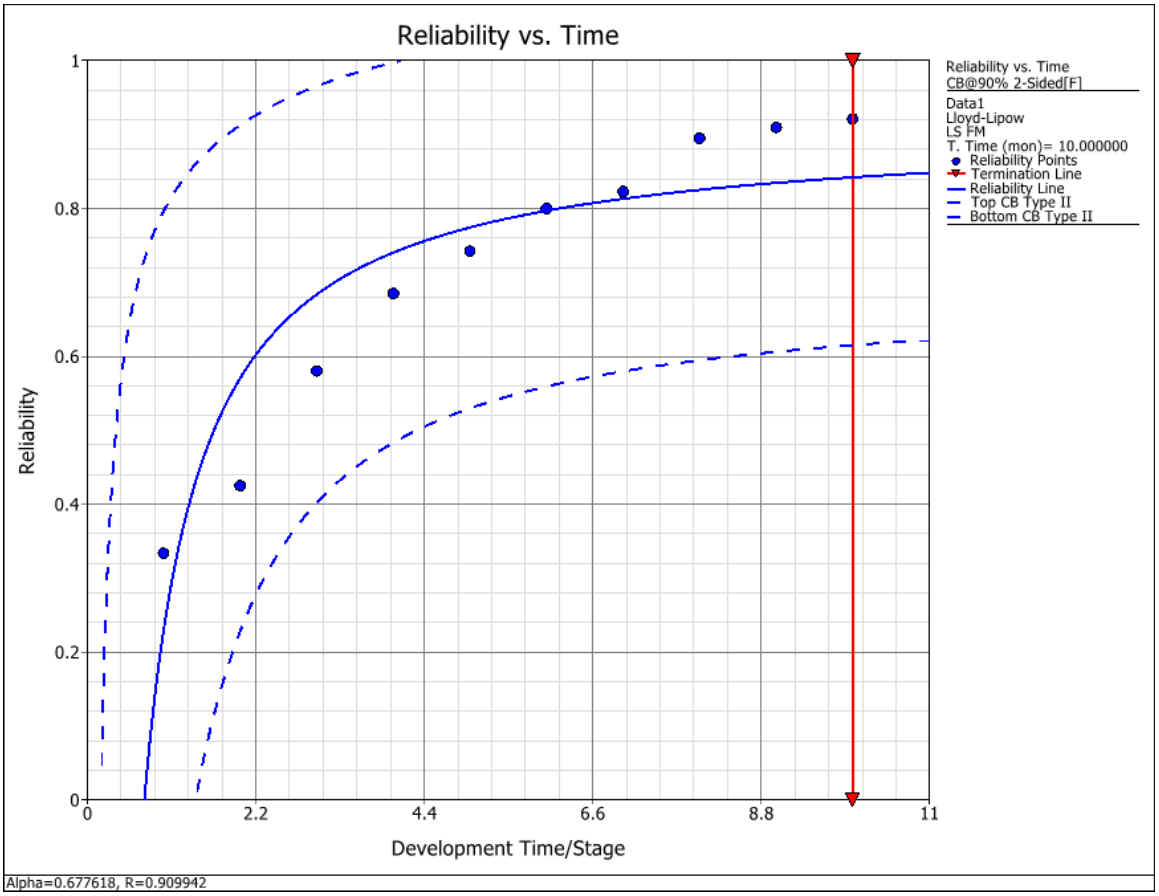
5	74.20
6	80.00
7	82.30
8	89.50
9	91.00
10	92.10

Solution

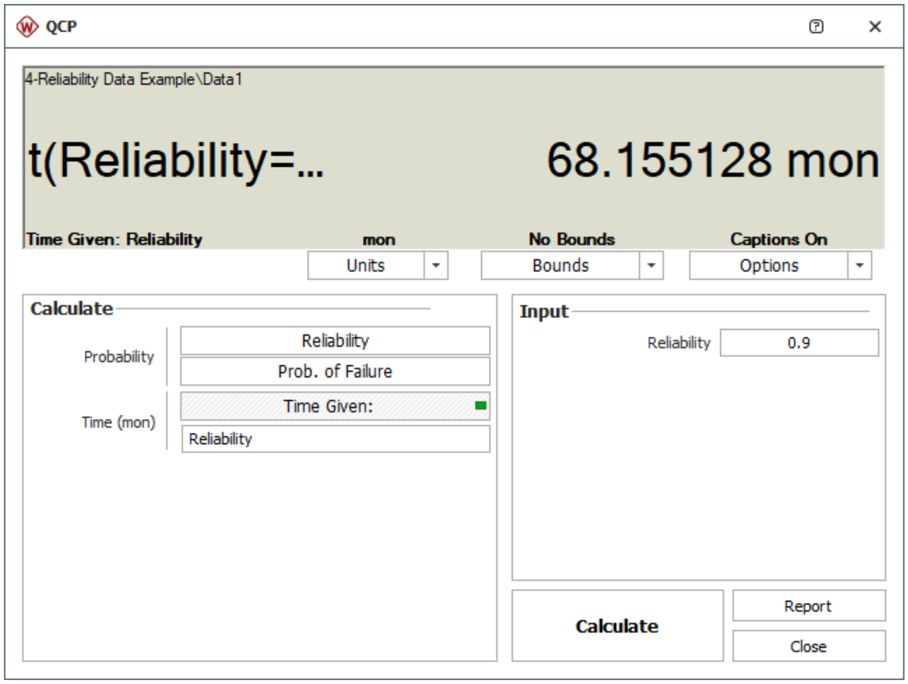
1. The next figure displays the estimated parameters.



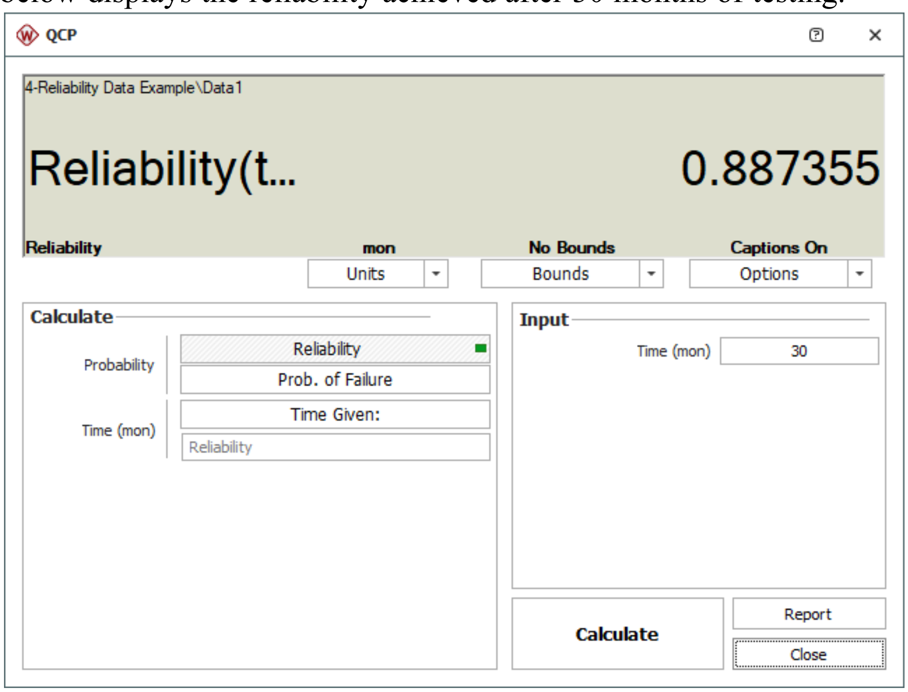
2. The figure below displays Reliability vs. Time plot with 90% 2-sided confidence bounds.



3. The next figure shows the number of months of testing required to achieve a reliability goal of 90%.



4. The figure below displays the reliability achieved after 30 months of testing.



Sequential Data Example

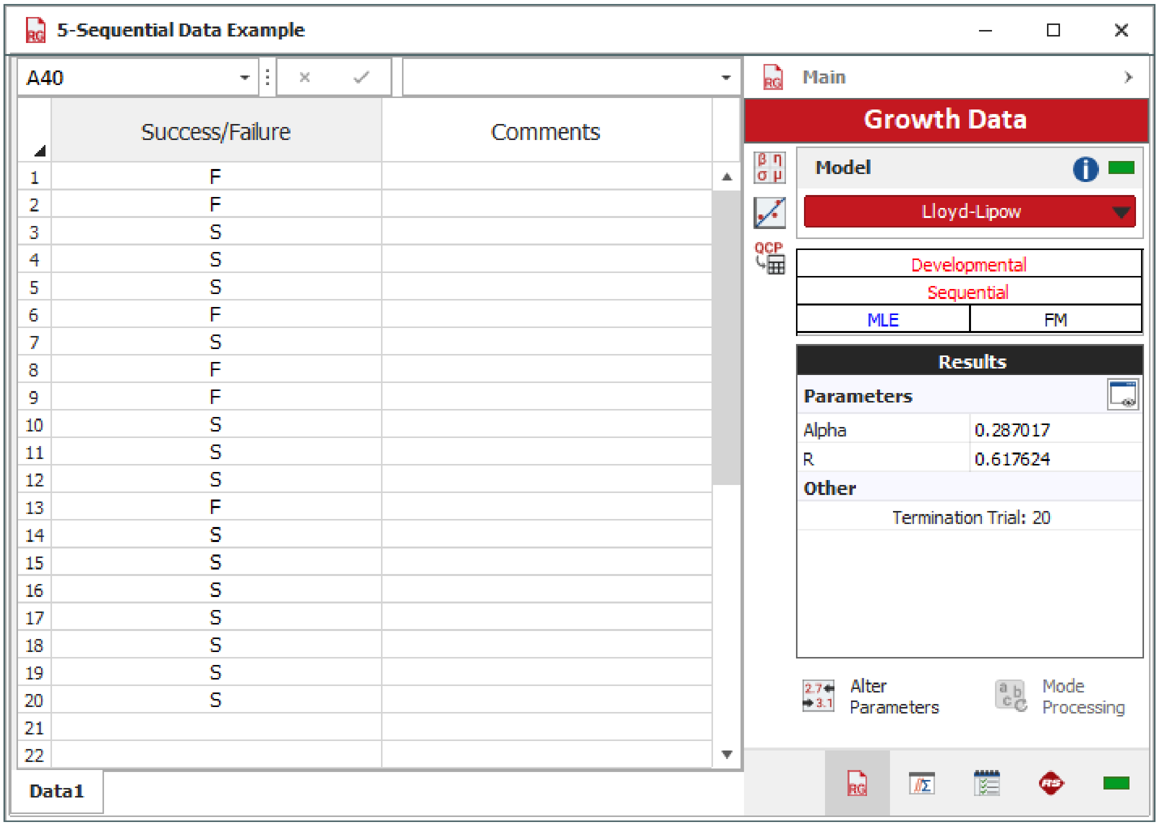
Use MLE to find the Lloyd-Lipow model that represents the data in the following table, and plot it along with the 95% 2-sided confidence bounds. Does the model follow the data?

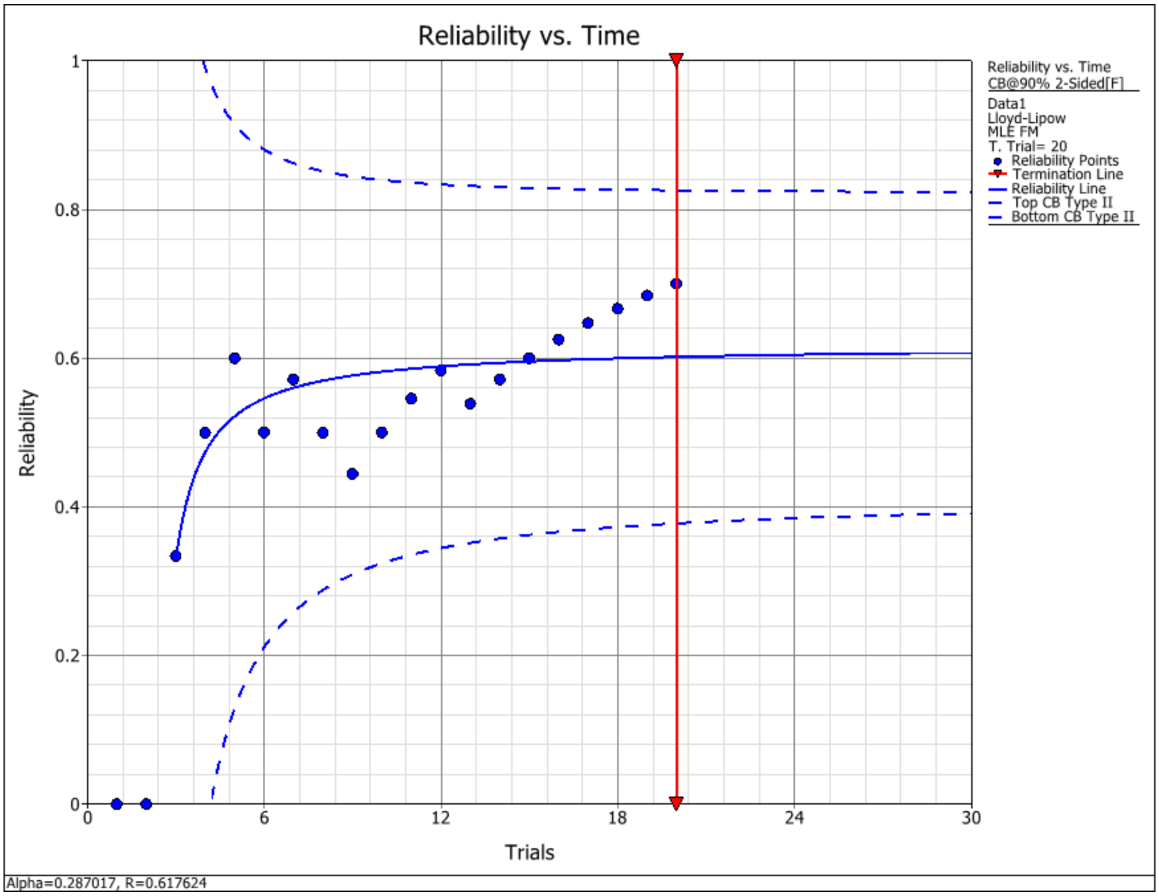
Sequential Data

Run Number	Result
1	F
2	F
3	S
4	S
5	S
6	F
7	S
8	F
9	F
10	S
11	S
12	S
13	F
14	S
15	S
16	S
17	S
18	S
19	S
20	S

Solution

The two figures below demonstrate the solution. As can be seen from the Reliability vs. Time plot with 95% 2-sided confidence bounds, the model does not seem to follow the data. You may want to consider another model for this data set.





Sequential Data with Failure Modes Example

Use least squares to find the Lloyd-Lipow model that represents the data in the following table. This data set includes information about the failure mode that was responsible for each failure, so that the probability of each failure mode recurring is taken into account in the analysis.

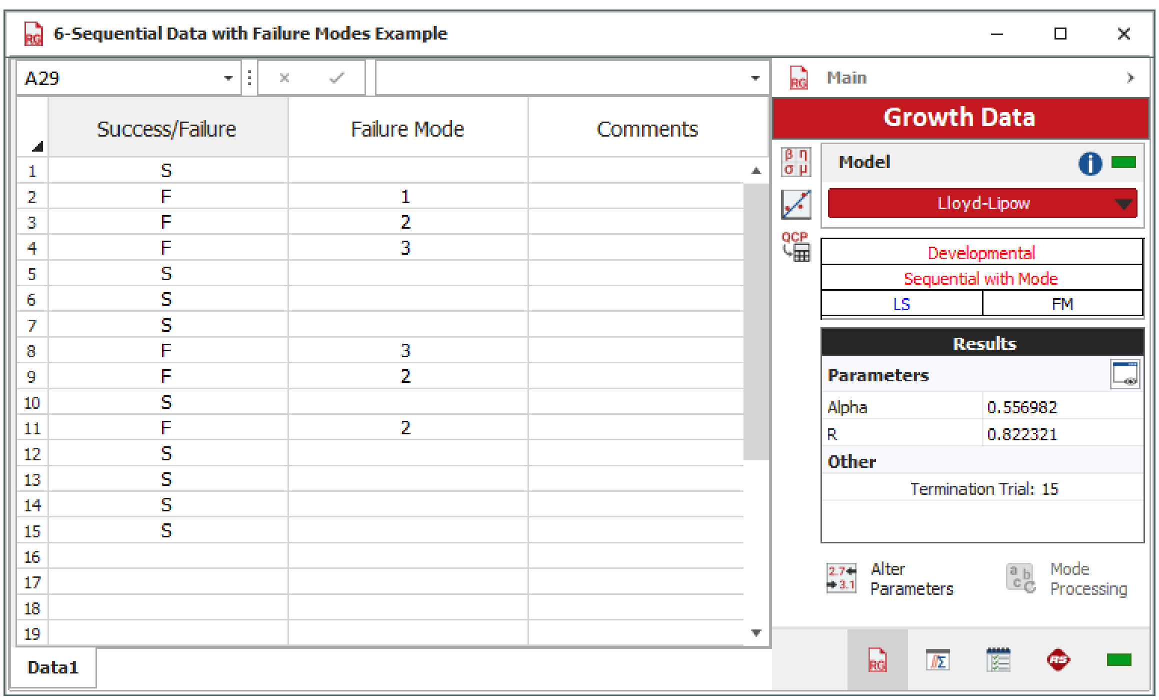
Sequential with Mode Data

Run Number	Result	Mode
1	S	
2	F	1
3	F	2
4	F	3
5	S	

6	S	
7	S	
8	F	3
9	F	2
10	S	
11	F	2
12	S	
13	S	
14	S	
15	S	

Solution

The following figure shows the analysis.

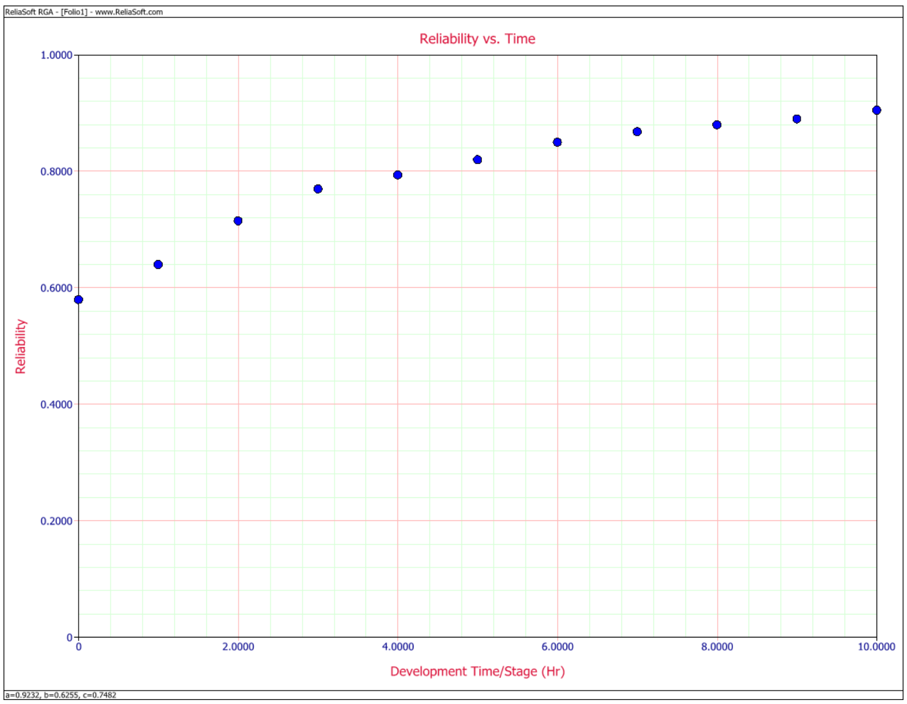


Gompertz Models

This chapter discusses the two Gompertz models that are used in Weibull++: the standard Gompertz and the modified Gompertz.

The Standard Gompertz Model

The Gompertz reliability growth model is often used when analyzing reliability data. It is most applicable when the data set follows a smooth curve, as shown in the plot below.



The Gompertz model is mathematically given by Virene [1]:

$$R = ab^{c^T}$$

where:

- $0 < a \leq 1$
- $0 < b < 1$
- $0 < c < 1$
- $T > 0$
- $R =$ the system's reliability at development time, launch number or stage number, T

- $a =$ the upper limit that the reliability approaches asymptotically as $T \to \infty$, or the maximum reliability that can be attained
- $ab =$ initial reliability at $T = 0$
- $c =$ the growth pattern indicator (small values of c indicate rapid early reliability growth and large values of c indicate slow reliability growth)

As it can be seen from the mathematical definition, the Gompertz model is a 3-parameter model with the parameters a, b and c. The solution for the parameters, given T_i and R_i, is accomplished by fitting the best possible line through the data points. Many methods are available; all of which tend to be numerically intensive. When analyzing reliability data in the Weibull++ software, you have the option to enter the reliability values in percent or in decimal format. However, a will always be returned in decimal format and not in percent. The estimated parameters in the Weibull++ software are unitless. The next section presents an overview and background on some of the most commonly used algorithms/methods for obtaining these parameters.

Parameter Estimation

LINEAR REGRESSION

The method of least squares requires that a straight line be fitted to a set of data points. If the regression is on Y, then the sum of the squares of the vertical deviations from the points to the line is minimized. If the regression is on X, the line is fitted to a set of data points such that the sum of the squares of the horizontal deviations from the points to the line is minimized. To illustrate the method, this section presents a regression on Y. Consider the linear model given by Seber and Wild [2]:

$$Y_i = \widehat{\beta}_0 + \widehat{\beta}_1 X_{i1} + \widehat{\beta}_2 X_{i2} + \ldots + \widehat{\beta}_p X_{ip}$$

or in matrix form where bold letters indicate matrices:

$$Y = X\beta$$

where:

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_N \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & X_{1,1} & \cdots & X_{1,p} \\ 1 & X_{2,1} & \cdots & X_{2,p} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & X_{N,1} & \cdots & X_{N,p} \end{bmatrix}$$

and:

$$\beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_p \end{bmatrix}$$

The vector β holds the values of the parameters. Now let $\widehat{\beta}$ be the estimates of these parameters, or the regression coefficients. The vector of estimated regression coefficients is denoted by:

$$\widehat{\beta} = \begin{bmatrix} \widehat{\beta}_0 \\ \widehat{\beta}_1 \\ \vdots \\ \widehat{\beta}_p \end{bmatrix}$$

Solving for β in the matrix form of the equation requires the analyst to left multiply both sides by the transpose of X, X^T , or :

$$Y = X\beta$$
$$(X^T X)\widehat{\beta} = X^T Y$$

Now the term $(X^T X)$ becomes a square and invertible matrix. Then taking it to the other side of the equation gives:

$$\widehat{\beta} = (X^T X)^{-1} X^T Y$$

NON-LINEAR REGRESSION

Non-linear regression is similar to linear regression, except that a curve is fitted to the data set instead of a straight line. Just as in the linear scenario, the sum of the squares of the horizontal and vertical distances between the line and the points are to be minimized. In the case of the non-linear Gompertz model $R = ab^{c^T}$, let:

$$Y_i = f(T_i, \delta) = ab^{c^{T_i}}$$

where:

$$T_i = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_N \end{bmatrix}, \quad i = 1, 2, \ldots, N$$

and:

$$\delta = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

The Gauss-Newton method can be used to solve for the parameters a, b and c by performing a Taylor series expansion on $f(T_i, \delta)$. Then approximate the non-linear model with linear terms and employ ordinary least squares to estimate the parameters. This procedure is performed in an iterative manner and it generally leads to a solution of the non-linear problem.

This procedure starts by using initial estimates of the parameters a, b and c, denoted as $g_1^{(0)}, g_2^{(0)}$ and $g_3^{(0)}$, where (0) is the iteration number. The Taylor series expansion approximates the mean response, $f(T_i, \delta)$, around the starting values, $g_1^{(0)}, g_2^{(0)}$ and $g_3^{(0)}$. For the i^{th} observation:

$$f(T_i, \delta) \simeq f(T_i, g^{(0)}) + \sum_{k=1}^{p} \left[\frac{\partial f(T_i, \delta)}{\partial \delta_k} \right]_{\delta = g^{(0)}} (\delta_k - g_k^{(0)})$$

where:

$$g^{(0)} = \begin{bmatrix} g_1^{(0)} \\ g_2^{(0)} \\ g_3^{(0)} \end{bmatrix}$$

Let:

$$\begin{aligned} f_i^{(0)} &= f(T_i, g^{(0)}) \\ \nu_k^{(0)} &= (\delta_k - g_k^{(0)}) \\ D_{ik}^{(0)} &= \left[\frac{\partial f(T_i, \delta)}{\partial \delta_k} \right]_{\delta = g^{(0)}} \end{aligned}$$

So the equation $Y_i = f(T_i, \delta) = ab^{c^{T_i}}$ becomes:

$$Y_i \simeq f_i^{(0)} + \sum_{k=1}^{p} D_{ik}^{(0)} \nu_k^{(0)}$$

or by shifting $f_i^{(0)}$ to the left of the equation:

$$Y_i^{(0)} \simeq \sum_{k=1}^{p} D_{ik}^{(0)} \nu_k^{(0)}$$

In matrix form this is given by:

$$Y^{(0)} \simeq D^{(0)} \nu^{(0)}$$

where:

$$Y^{(0)} = \begin{bmatrix} Y_1 - f_1^{(0)} \\ Y_2 - f_2^{(0)} \\ \vdots \\ Y_N - f_N^{(0)} \end{bmatrix} = \begin{bmatrix} Y_1 - g_1^{(0)} g_2^{(0) g_3^{(0)T_1}} \\ Y_1 - g_1^{(0)} g_2^{(0) g_3^{(0)T_2}} \\ \vdots \\ Y_N - g_1^{(0)} g_2^{(0) g_3^{(0)T_N}} \end{bmatrix}$$

$$D^{(0)} = \begin{bmatrix} D_{11}^{(0)} & D_{12}^{(0)} & D_{13}^{(0)} & \\ D_{21}^{(0)} & D_{22}^{(0)} & D_{23}^{(0)} & \\ . & . & . & . \\ . & . & . & . \\ D_{N1}^{(0)} & D_{N2}^{(0)} & D_{N3}^{(0)} & \end{bmatrix}$$
$$= \begin{bmatrix} g_2^{(0) g_3^{(0)T_1}} & \frac{g_1^{(0)}}{g_2^{(0)}} g_3^{(0)T_1} g_2^{(0) g_3^{(0)T_1}} & \frac{g_1^{(0)}}{g_3^{(0)}} g_3^{(0)T_1} \ln(g_2^{(0)}) T_1 g_2^{(0) g_3^{(0)T_1}} & 1 \\ g_2^{(0) g_3^{(0)T_2}} & \frac{g_1^{(0)}}{g_2^{(0)}} g_3^{(0)T_2} g_2^{(0) g_3^{(0)T_2}} & \frac{g_1^{(0)}}{g_3^{(0)}} g_3^{(0)T_2} \ln(g_2^{(0)}) T_2 g_2^{(0) g_3^{(0)T_2}} & 1 \\ . & . & . & . \\ . & . & . & . \\ g_2^{(0) g_3^{(0)T_N}} & \frac{g_1^{(0)}}{g_2^{(0)}} g_3^{(0)T_N} g_2^{(0) g_3^{(0)T_N}} & \frac{g_1^{(0)}}{g_3^{(0)}} g_3^{(0)T_N} \ln(g_2^{(0)}) T_N g_2^{(0) g_3^{(0)T_N}} & 1 \end{bmatrix}$$

and:

$$\nu^{(0)} = \begin{bmatrix} g_1^{(0)} \\ g_2^{(0)} \\ g_3^{(0)} \end{bmatrix}$$

Note that the equation $Y^{(0)} \simeq D^{(0)} \nu^{(0)}$ is in the form of the general linear regression model given in the Linear Regression section. Therefore, the estimate of the parameters $\nu^{(0)}$ is given by:

$$\widehat{\nu}^{(0)} = \left(D^{(0)^T} D^{(0)} \right)^{-1} D^{(0)^T} Y^{(0)}$$

The revised estimated regression coefficients in matrix form are:

$$g^{(1)} = g^{(0)} + \widehat{\nu}^{(0)}$$

The least squares criterion measure, Q, should be checked to examine whether the revised regression coefficients will lead to a reasonable result. According to the Least Squares Principle, the solution to the values of the parameters are those values that minimize Q. With the starting coefficients, $g^{(0)}$, Q is:

$$Q^{(0)} = \sum_{i=1}^{N} \left[Y_i - f\left(T_i, g^{(0)}\right)\right]^2$$

And with the coefficients at the end of the first iteration, $g^{(1)}$, Q is:

$$Q^{(1)} = \sum_{i=1}^{N} \left[Y_i - f\left(T_i, g^{(1)}\right)\right]^2$$

For the Gauss-Newton method to work properly and to satisfy the Least Squares Principle, the relationship $Q^{(k+1)} < Q^{(k)}$ has to hold for all k, meaning that $g^{(k+1)}$ gives a better estimate than $g^{(k)}$. The problem is not yet completely solved. Now $g^{(1)}$ are the starting values, producing a new set of values $g^{(2)}$. The process is continued until the following relationship has been satisfied:

$$Q^{(s-1)} - Q^{(s)} \simeq 0$$

When using the Gauss-Newton method or some other estimation procedure, it is advisable to try several sets of starting values to make sure that the solution gives relatively consistent results.

CHOICE OF INITIAL VALUES

The choice of the starting values for the nonlinear regression is not an easy task. A poor choice may result in a lengthy computation with many iterations. It may also lead to divergence, or to a convergence due to a local minimum. Therefore, good initial values will result in fast computations with few iterations, and if multiple minima exist, it will lead to a solution that is a minimum.

Various methods were developed for obtaining valid initial values for the regression parameters. The following procedure is described by Virene [1] in estimating the Gompertz parameters. This procedure is rather simple. It will be used to get the starting values for the Gauss-Newton method, or for any other method that requires initial values. Some analysts use this method to calculate the parameters when the data set is divisible into three groups of equal size. However, if the data set is not equally divisible, it can still provide good initial estimates.

Consider the case where m observations are available in the form shown next. Each reliability value, R_i , is measured at the specified times, T_i .

$$\begin{array}{cc} T_i & R_i \\ T_0 & R_0 \\ T_1 & R_1 \\ T_2 & R_2 \\ \vdots & \vdots \\ T_{m-1} & R_{m-1} \end{array}$$

where:

- $m = 3n, n$ is equal to the number of items in each equally sized group
- $T_i - T_{i-1} = const$
- $i = 0, 1, \ldots, m-1$

The Gompertz reliability equation is given by:

$$R = ab^{c^T}$$

and:

$$\ln(R) = \ln(a) + c^T \ln(b)$$

Define:

$$S_1 = \sum_{i=0}^{n-1} ln(R_i) = nln(a) + ln(b) \sum_{i=0}^{n-1} c^{T_i}$$
$$S_2 = \sum_{i=n}^{2n-1} ln(R_i) = nln(a) + ln(b) \sum_{i=n}^{2n-1} c^{T_i}$$
$$S_3 = \sum_{i=2n}^{m-1} ln(R_i) = nln(a) + ln(b) \sum_{i=2n}^{m-1} c^{T_i}$$

Then:

$$\frac{S_3 - S_2}{S_2 - S_1} = \frac{\sum_{i=2n} m - 1 c^{T_i} - \sum_{i=n}^{2n-1} c_i^T}{\sum_{i=0}^{n-1} c^{T_i}}$$
$$\frac{S_3 - S_2}{S_2 - S_1} = c_{2n}^T \frac{\sum_{i=0} n - 1 c^{T_i} - c^{T_n} \sum_{i=0}^{n-1} c_i^T}{c^{T_n} \sum_{i=0}^{n-1} c^{T_i}}$$
$$\frac{S_3 - S_2}{S_2 - S_1} = \frac{c^{T_2 n} - c^{T_n}}{c^{T_n} - 1} = c^{T_{an}} = c^{n \cdot I + T_0}$$

Without loss of generality, take $T_{a_0} = 0$; then:

$$\frac{S_3 - S_2}{S_2 - S_1} = c^{n \cdot I}$$

Solving for c yields:

$$c = \left(\frac{S_3 - S_2}{S_2 - S_1}\right)^{\frac{1}{n \cdot I}}$$

Considering the definitions for S_1 and S_2 , given above, then:

$$S_1 - n \cdot \ln(a) = \ln(b)\sum_{i=0}^{n-1} c^{T_i}$$

$$S_2 - n \cdot \ln(a) = \ln(b)\sum_{i=n}^{2n-1} c^{T_i}$$

or:

$$\frac{S_1 - n \cdot \ln(a)}{S_2 - n \cdot \ln(a)} = \frac{1}{c^{n \cdot I}}$$

Reordering the equation yields:

$$\ln(a) = \frac{1}{n}\left(S_1 + \frac{S_2 - S_1}{1 - c^{n \cdot I}}\right)$$

$$a = e^{\left[\frac{1}{n}\left(S_1 + \frac{S_2 - S_1}{1 - c^{n \cdot I}}\right)\right]}$$

If the reliability values are in percent then a needs to be divided by 100 to return the estimate in decimal format. Consider the definitions for S_1 and S_2 again, where:

$$S_1 - ln(b)\sum_{i=0}^{n-1} c^{T_i} = nln(a)$$

$$S_2 - ln(b)\sum_{i=n}^{2n-1} c^{T_i} = nln(a)$$

$$\frac{S_1 - ln(b)\sum_{i=0}^{n-1} c^{T_i}}{S_2 - ln(b)\sum_{i=n}^{2n-1} c^{T_i}} = 1$$

$$S_1 - ln(b)\sum_{i=0}^{n-1} c^{T_i} = S_2 - ln(b)\sum_{i=n}^{2n-1} c^{T_i}$$

Reordering the equation above yields:

$$\ln(b) = \frac{(S_2 - S_1)(c^I - 1)}{(1 - c^{n \cdot I})^2}$$

$$b = e^{\left[\frac{(S_2 - S_1)(c^I - 1)}{(1 - c^{n \cdot I})^2}\right]}$$

Therefore, for the special case where $I = 1$, the parameters are:

$$c = \left(\frac{S_3 - S_2}{S_2 - S_1}\right)^{\frac{1}{n}}$$

$$a = e^{\left[\frac{1}{n}\left(S_1 + \frac{S_2 - S_1}{1 - c^n}\right)\right]}$$

$$b = e^{\left[\frac{(S_2 - S_1)(c - 1)}{(1 - c^n)^2}\right]}$$

To estimate the values of the parameters a, b and c, do the following:

1. Arrange the currently available data in terms of T and R as in the table below. The T values should be chosen at equal intervals and increasing in value by 1, such as one month, one hour, etc.

Design and Development Time vs. Demonstrated Reliability Data for a Device

Group Number	Growth Time T (months)	Reliability R (%)	$\ln R$
	0	58	4.060
1	1	66	4.190
			S_1 = 8.250
	2	72.5	4.284
2	3	78	4.357
			S_2 = 8.641
	4	82	4.407
3	5	85	4.443
			S_3 = 8.850

2. Calculate the natural log R.

3. Divide the column of values for log R into three groups of equal size, each containing n items. There should always be three groups. Each group should always have the same number, n , of items, measurements or values.

4. Add the values of the natural log R in each group, obtaining the sums identified as S_1 , S_2 and S_3 , starting with the lowest values of the natural log R .

5. Calculate c using the following equation:

$$c = \left(\frac{S_3 - S_2}{S_2 - S_1} \right)^{\frac{1}{n}}$$

6. Calculate a using the following equation:

$$a = e^{\left[\frac{1}{n} \left(S_1 + \frac{S_2 - S_1}{1 - c^n} \right) \right]}$$

7. Calculate b using the following equation:

$$b = e^{\left[\frac{(S_2 - S_1)(c-1)}{(1-c^n)^2} \right]}$$

8. Write the Gompertz reliability growth equation.

9. Substitute the value of T , the time at which the reliability goal is to be achieved, to see if the reliability is indeed to be attained or exceeded by T .

Confidence Bounds

The approximate reliability confidence bounds under the Gompertz model can be obtained with non-linear regression. Additionally, the reliability is always between 0 and 1. In order to keep the endpoints of the confidence interval, the logit transformation is used to obtain the confidence bounds on reliability.

$$CB = \frac{\hat{R}_i}{\hat{R}_i + (1 - \hat{R}_i) e^{\pm z_\alpha \hat{\sigma}_R / [\hat{R}_i (1 - \hat{R}_i)]}}$$

$$\hat{\sigma}^2 = \frac{SSE}{n - p}$$

where p is the total number of groups (in this case 3) and n is the total number of items in each group.

Example - Standard Gompertz for Reliability Data

A device is required to have a reliability of 92% at the end of a 12-month design and development period. The following table gives the data obtained for the first five moths.

1. What will the reliability be at the end of this 12-month period?
2. What will the maximum achievable reliability be if the reliability program plan pursued during the first 5 months is continued?
3. How do the predicted reliability values compare with the actual values?

Design and Development Time vs. Demonstrated Reliability Data for a Device

Group Number	Growth Time T (months)	Reliability R (%)	$\ln R$
	0	58	4.060
1	1	66	4.190
			S_1 = 8.250
	2	72.5	4.284
2	3	78	4.357
			S_2 = 8.641
	4	82	4.407
3	5	85	4.443
			S_3 = 8.850

Solution

After generating the table above and calculating the last column to find S_1 , S_2 and S_3 , proceed as follows:

a. Solve for the value of c :

$$c = \left(\frac{8.850 - 8.641}{8.641 - 8.250}\right)^{\frac{1}{2}}$$
$$= 0.731$$

b. Solve for the value of a:

$$\begin{aligned} a &= e^{\left[\left(\frac{1}{2} \left(8.250 + \frac{S_2 - S_1}{1 - c^{n \cdot 1}} \right) \right] \right.} \\ &= e^{4.545} \\ &= 94.16\% \end{aligned}$$

This is the upper limit for the reliability as $T \rightarrow \infty$.

c. Solve for the value of b:

$$\begin{aligned} b &= e^{\left[\frac{(8.641 - 8.250)(0.731 - 1)}{(1 - 0.731^2)^2} \right]} \\ &= e^{(-0.485)} \\ &= 0.615 \end{aligned}$$

Now, that the initial values have been determined, the Gauss-Newton method can be used. Therefore, substituting $Y_i = R_i, g_1^{(0)} = 94.16, g_2^{(0)} = 0.615, g_3^{(0)} = 0.731, Y^{(0)}, D^{(0)}, \nu^{(0)}$ become:

$$Y^{(0)} = \begin{bmatrix} 0.0916 \\ 0.0015 \\ -0.1190 \\ 0.1250 \\ 0.0439 \\ -0.0743 \end{bmatrix}$$

$$D^{(0)} = \begin{bmatrix} 0.6150 & 94.1600 & 0.0000 \\ 0.7009 & 78.4470 & -32.0841 \\ 0.7712 & 63.0971 & -51.6122 \\ 0.8270 & 49.4623 & -60.6888 \\ 0.8704 & 38.0519 & -62.2513 \\ 0.9035 & 28.8742 & -59.0463 \end{bmatrix}$$

$$\nu^{(0)} = \begin{bmatrix} g_1^{(0)} \\ g_2^{(0)} \\ g_3^{(0)} \end{bmatrix} = \begin{bmatrix} 94.16 \\ 0.615 \\ 0.731 \end{bmatrix}$$

The estimate of the parameters $\nu^{(0)}$ is given by:

$$\begin{aligned} \hat{v} &= \left(D^{(0)} D^{(0)} \right)^{-1} D^{(0)} Y^{(0)} \\ &= \begin{bmatrix} 0.061575 \\ 0.000222 \\ 0.001123 \end{bmatrix} \end{aligned}$$

The revised estimated regression coefficients in matrix form are:

$$\begin{aligned} g^{(1)} &= g^{(0)} + \hat{v}^{(0)} \\ &= \begin{bmatrix} 94.16 \\ 0.615 \\ 0.731 \end{bmatrix} + \begin{bmatrix} 0.061575 \\ 0.000222 \\ 0.001123 \end{bmatrix} \\ &= \begin{bmatrix} 94.2216 \\ 0.6152 \\ 0.7321 \end{bmatrix} \end{aligned}$$

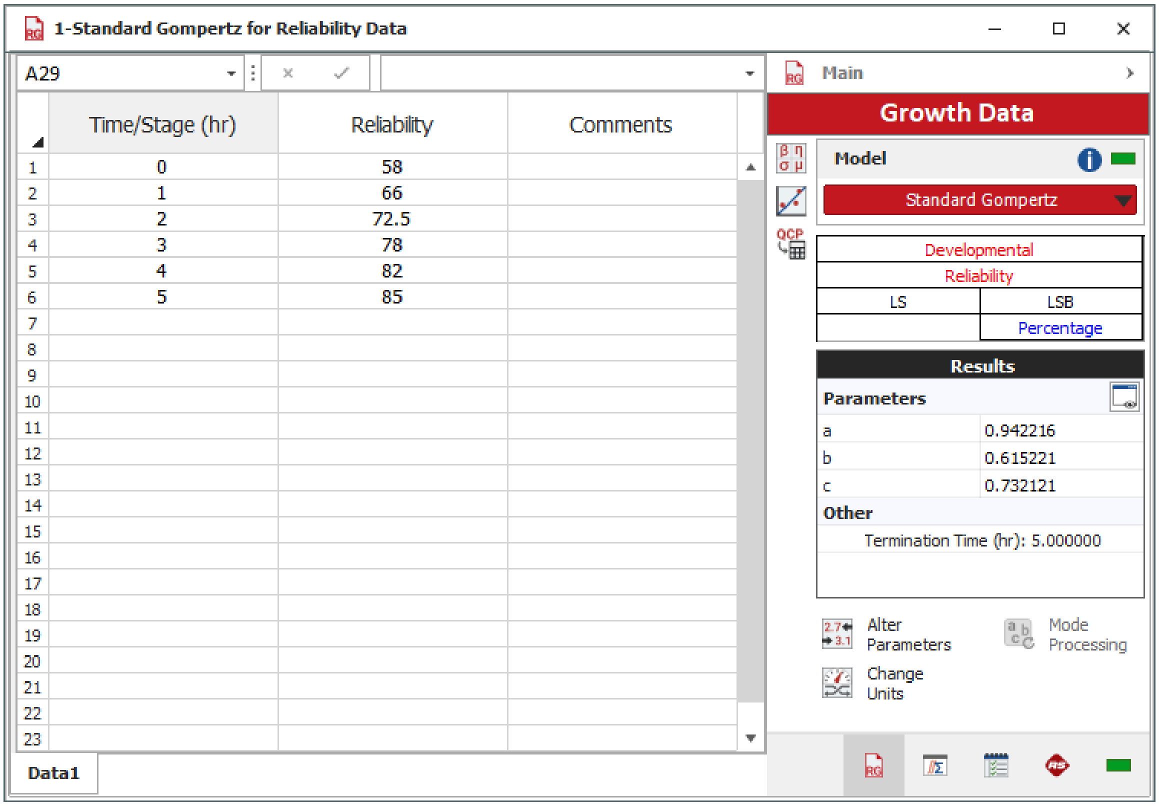
If the Gauss-Newton method works effectively, then the relationship $Q^{(k+1)} < Q^{(k)}$ has to hold, meaning that $g^{(k+1)}$ gives better estimates than $g^{(k)}$, after k. With the starting coefficients, $g^{(0)}$, Q is:

$$\begin{aligned} Q^0 &= \sum_{i=1}^{N} \left[Y_i - f\left(T_i, g^{(0)}\right) \right]^2 \\ &= 0.045622 \end{aligned}$$

And with the coefficients at the end of the first iteration, $g^{(1)}$, Q is:

$$\begin{aligned} Q^1 &= \sum_{i=1}^{N} \left[Y_i - f\left(T_i, g^{(1)}\right) \right]^2 \\ &= 0.041439 \end{aligned}$$

Therefore, it can be justified that the Gauss-Newton method works in the right direction. The iterations are continued until the relationship $Q^{(s-1)} - Q^{(s)} \simeq 0$ is satisfied. Note that the Weibull++ software uses a different analysis method called the *Levenberg-Marquardt*. This method utilizes the best features of the Gauss-Newton method and the method of the steepest descent, and occupies a middle ground between these two methods. The estimated parameters using Weibull++ are shown in the figure below.



They are:

$$\hat{a} = 0.9422$$
$$\hat{b} = 0.6152$$
$$\hat{c} = 0.7321$$

The Gompertz reliability growth curve is:

$$R = 0.9422(0.6152)^{0.7321^{T}}$$

1. The achievable reliability at the end of the 12-month period of design and development is:

   $$R = 0.9422(0.6152)^{0.7321} = 0.9314$$

   The required reliability is 92%. Consequently, from the previous result, this requirement will be barely met. Every effort should therefore be expended to implement the reliability program plan fully, and perhaps augment it slightly to assure that the reliability goal will be met.

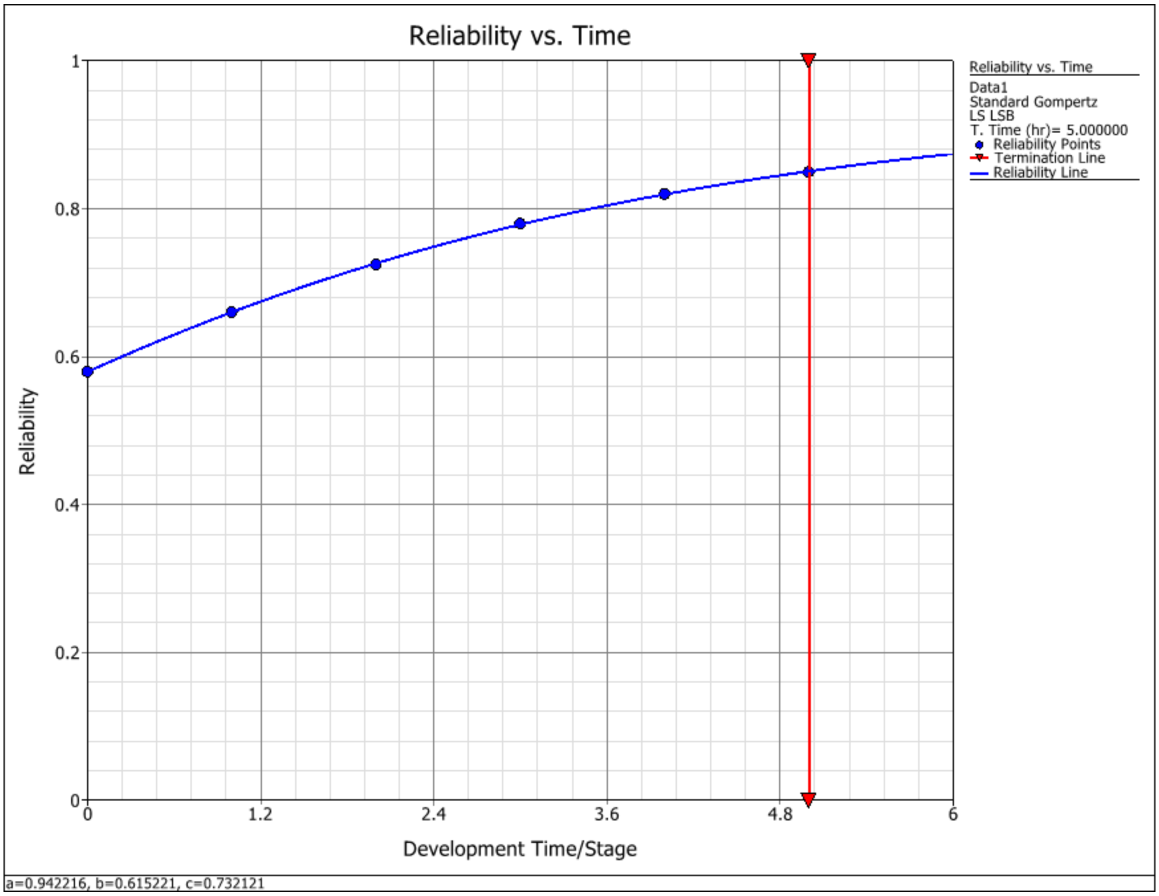
2. The maximum achievable reliability from Step 2, or from the value of a, is 0.9422.

3. The predicted reliability values, as calculated from the standard Gompertz model, are compared with the actual data in the table below. It may be seen in the table that the Gompertz curve appears to provide a very good fit for the data used because the equation reproduces

the available data with less than 1% error. The standard Gompertz model is plotted in the figure below the table. The plot identifies the type of reliability growth curve that the equation represents.

Comparison of the Predicted Reliabilities with the Actual Data

Growth Time T (months)	Gompertz Reliability (%)	Raw Data Reliability (%)
0	57.97	58.00
1	66.02	66.00
2	72.62	72.50
3	77.87	78.00
4	81.95	82.00
5	85.07	85.00
6	87.43	
7	89.20	
8	90.52	
9	91.50	
10	92.22	
11	92.75	
12	93.14	



Example - Standard Gompertz for Sequential Data

Calculate the parameters of the Gompertz model using the sequential data in the following table.

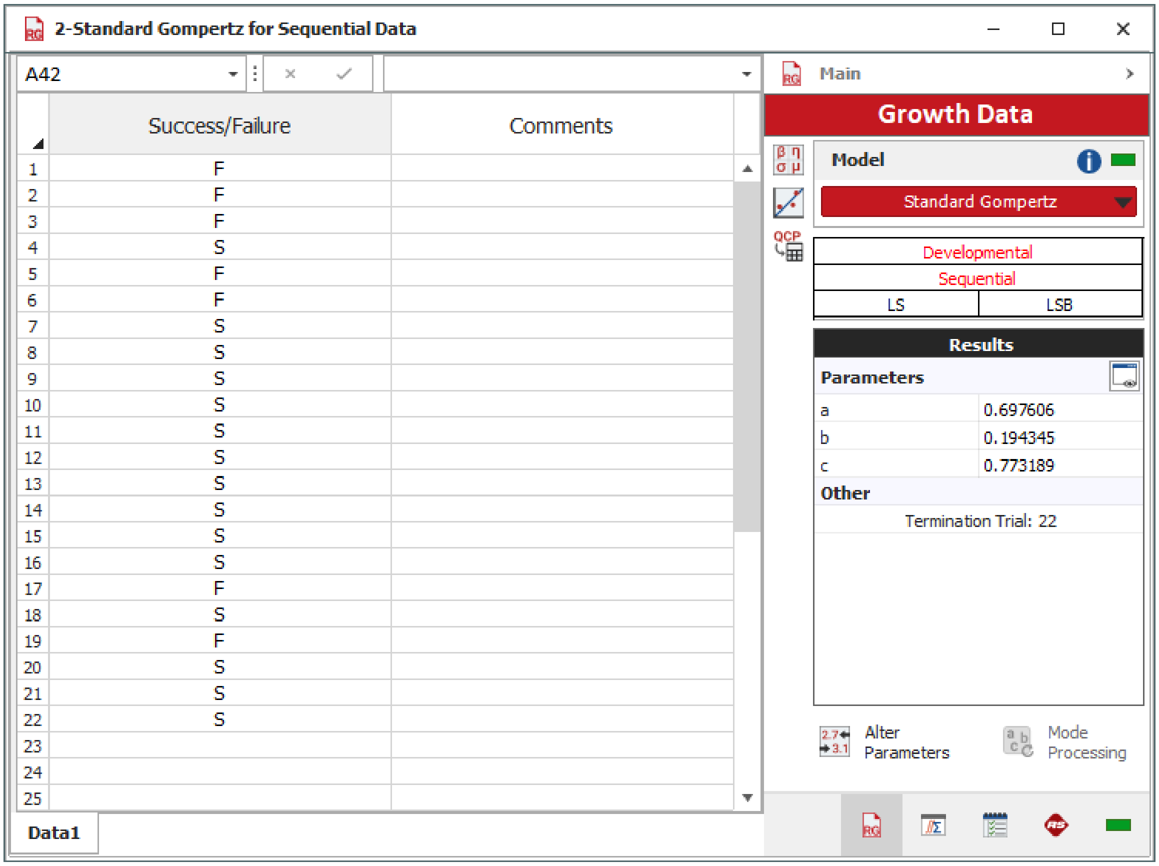
Sequential Data

Run Number	Result	Successes	Observed Reliability (%)
1	F	0	
2	F	0	
3	F	0	
4	S	1	25.00
5	F	1	20.00
6	F	1	16.67

7	S	2	28.57
8	S	3	37.50
9	S	4	44.44
10	S	5	50.00
11	S	6	54.55
12	S	7	58.33
13	S	8	61.54
14	S	9	64.29
15	S	10	66.67
16	S	11	68.75
17	F	11	64.71
18	S	12	66.67
19	F	12	63.16
20	S	13	65.00
21	S	14	66.67
22	S	15	68.18

Solution

Using Weibull++, the parameter estimates are shown in the following figure.



Cumulative Reliability

For many kinds of equipment, especially missiles and space systems, only success/failure data (also called discrete or attribute data) is obtained. Conservatively, the cumulative reliability can be used to estimate the trend of reliability growth. The cumulative reliability is given by Kececioglu [3]:

$$\bar{R}(N) = \frac{N - r}{N}$$

where:

- N is the current number of trials
- r is the number of failures

It must be emphasized that the instantaneous reliability of the developed equipment is increasing as the test-analyze-fix-and-test process continues. In addition, the instantaneous reliability is higher than the cumulative reliability. Therefore, the reliability growth curve based on the cumulative reliability can be thought of as the lower bound of the true reliability growth curve.

The Modified Gompertz Model

Sometimes, reliability growth data with an S-shaped trend cannot be described accurately by the Standard Gompertz or Logistic curves. Because these two models have fixed values of reliability at the inflection points, only a few reliability growth data sets following an S-shaped reliability growth curve can be fitted to them. A modification of the Gompertz curve, which overcomes this shortcoming, is given next [5].

If we apply a shift in the vertical coordinate, then the Gompertz model is defined by:

$$R = d + ab^{c^T}$$

where:

- $0 < a + d \leq 1$
- $0 < b < 1, 0 < c < 1, \text{and} T \geq 0$
- R is the system's reliability at development time T or at launch number T, or stage number T
- d is the shift parameter
- $d + a$ is the upper limit that the reliability approaches asymptotically as $T \to \infty$
- $d + ab$ is the initial reliability at $T = 0$
- c is the growth pattern indicator (small values of c indicate rapid early reliability growth and large values of c indicate slow reliability growth)

The modified Gompertz model is more flexible than the original, especially when fitting growth data with S-shaped trends.

Parameter Estimation

To implement the modified Gompertz growth model, initial values of the parameters a, b, c and d must be determined. When analyzing reliability data in Weibull++, you have the option to enter the reliability values in percent or in decimal format. However, a and d will always be returned in decimal format and not in percent. The estimated parameters in W eibull++are unitless.

Given that $R = d + ab^{c^T}$ and $\ln(R - d) = \ln(a) + c^T \ln(b)$, it follows that S_1, S_2 and S_3, as defined in the derivation of the Standard Gompertz model, can be expressed as functions of d.

$$S_1(d) = \sum_{i=0}^{n-1} \ln(R_i - d) = n\ln(a) + \ln(b)\sum_{i=0}^{n-1} c^{T_i}$$
$$S_2(d) = \sum_{i=n}^{2n-1} \ln(R_i - d) = n\ln(a) + \ln(b)\sum_{i=n}^{2n-1} c^{T_i}$$
$$S_3(d) = \sum_{i=2n}^{m-1} \ln(R_i - d) = n\ln(a) + \ln(b)\sum_{i=2n}^{m-1} c^{T_i}$$

Modifying the equations for estimating parameters c, a, b, as functions of d, yields:

$$c(d) = \left[\frac{S_3(d) - S_2(d)}{S_2(d) - S_1(d)}\right]^{\frac{1}{n \cdot I}}$$
$$a(d) = e^{\left[\frac{1}{n}\left(S_1(d) + \frac{S_2(d) - S_1(d)}{1 - [c(d)]^{n \cdot I}}\right)\right]}$$
$$b(d) = e^{\left[\frac{[S_2(d) - S_1(d)]\left[[c(d)]^I - 1\right]}{\left[1 - [c(d)]^{n \cdot I}\right]^2}\right]}$$

where I is the time interval increment. At this point, you can use the initial constraint of:

$$d + ab = \text{original level of reliability at } T = 0$$

Now there are four unknowns, a, b, c and d, and four corresponding equations. The simultaneous solution of these equations yields the four initial values for the parameters of the modified Gompertz model. This procedure is similar to the one discussed before. It starts by using initial estimates of the parameters, a, b, c and d, denoted as $g_1^{(0)}, g_2^{(0)}, g_3^{(0)}$, and $g_4^{(0)}$, where (0) is the iteration number.

The Taylor series expansion approximates the mean response, $f(T_i, \delta)$, around the starting values, $g_1^{(0)}, g_2^{(0)}, g_3^{(0)}$ and $g_4^{(0)}$. For the i^{th} observation:

$$f(T_i, \delta) \simeq f(T_i, g^{(0)}) + \sum_{k=1}^{p} \left[\frac{\partial f(T_i, \delta)}{\partial \delta_k}\right]_{\delta = g^{(0)}} \cdot (\delta_k - g_k^{(0)})$$

where:

$$g^{(0)} = \begin{bmatrix} g_1^{(0)} \\ g_2^{(0)} \\ g_3^{(0)} \\ g_4^{(0)} \end{bmatrix}$$

Let:

$$\begin{aligned} f_i^{(0)} &= f(T_i, g^{(0)}) \\ \nu_k^{(0)} &= (\delta_k - g_k^{(0)}) \\ D_{ik}^{(0)} &= \left[\frac{\partial f(T_i,\delta)}{\partial \delta_k}\right]_{\delta=g^{(0)}} \end{aligned}$$

Therefore:

$$Y_i = f_i^{(0)} + \sum_{k=1}^{p} D_{ik}^{(0)} \nu_k^{(0)}$$

or by shifting $f_i^{(0)}$ to the left of the equation:

$$Y_i^{(0)} - f_i^{(0)} = \sum_{k=1}^{p} D_{ik}^{(0)} \nu_k^{(0)}$$

In matrix form, this is given by:

$$Y^{(0)} \simeq D^{(0)} \nu^{(0)}$$

where:

$$Y^{(0)} = \begin{bmatrix} Y_1 - f_1^{(0)} \\ . \\ . \\ Y_N - f_N^{(0)} \end{bmatrix} = \begin{bmatrix} Y_1 - g_4^{(0)} + g_1^{(0)} g_2^{(0) g_3^{(0)T_1}} \\ . \\ . \\ Y_N - g_4^{(0)} + g_1^{(0)} g_2^{(0) g_3^{(0)T_N}} \end{bmatrix}$$

$$D^{(0)} = \begin{bmatrix} D_{11}^{(0)} & D_{12}^{(0)} & D_{13}^{(0)} & D_{14}^{(0)} \\ . & . & . & . \\ . & . & . & . \\ D_{N1}^{(0)} & D_{N2}^{(0)} & D_{N3}^{(0)} & D_{N4}^{(0)} \end{bmatrix}$$
$$= \begin{bmatrix} g_2^{(0) g_3^{(0)T_1}} & \frac{g_1^{(0)}}{g_2^{(0)}} g_3^{(0)T_1} g_2^{(0) g_3^{(0)T_1}} & \frac{g_1^{(0)}}{g_3^{(0)}} g_3^{(0)T_1} \ln(g_2^{(0)}) T_1 g_2^{(0) g_3^{(0)T_1}} & 1 \\ . & . & . & . \\ . & . & . & . \\ g_2^{(0) g_3^{(0)T_N}} & \frac{g_1^{(0)}}{g_2^{(0)}} g_3^{(0)T_N} g_2^{(0) g_3^{(0)T_N}} & \frac{g_1^{(0)}}{g_3^{(0)}} g_3^{(0)T_N} \ln(g_2^{(0)}) T_N g_2^{(0) g_3^{(0)T_N}} & 1 \end{bmatrix}$$

$$\nu^{(0)} = \begin{bmatrix} g_1^{(0)} \\ g_2^{(0)} \\ g_3^{(0)} \\ g_4^{(0)} \end{bmatrix}$$

The same reasoning as before is followed here, and the estimate of the parameters $\nu^{(0)}$ is given by:

$$\widehat{\nu}^{(0)} = \left(D^{(0)^T} D^{(0)}\right)^{-1} D^{(0)^T} Y^{(0)}$$

The revised estimated regression coefficients in matrix form are:

$$g^{(1)} = g^{(0)} + \widehat{\nu}^{(0)}$$

To see if the revised regression coefficients will lead to a reasonable result, the least squares criterion measure, Q, should be checked. According to the Least Squares Principle, the solution to the values of the parameters are those values that minimize Q. With the starting coefficients, $g^{(0)}$, Q is:

$$Q^{(0)} = \sum_{i=1}^{N} \left(Y_i - f(T_i, g^{(0)})\right)^2$$

With the coefficients at the end of the first iteration, $g^{(1)}$, Q is:

$$Q^{(1)} = \sum_{i=1}^{N} \left(Y_i - f(T_i, g^{(1)})\right)^2$$

For the Gauss-Newton method to work properly, and to satisfy the Least Squares Principle, the relationship $Q^{(k+1)} < Q^{(k)}$ has to hold for all k, meaning that $g^{(k+1)}$ gives a better estimate than $g^{(k)}$. The problem is not yet completely solved. Now $g^{(1)}$ are the starting values, producing a new set of values $g^{(2)}$. The process is continued until the following relationship has been satisfied.

$$Q^{(s-1)} - Q^{(s)} \simeq 0$$

As mentioned previously, when using the Gauss-Newton method or some other estimation procedure, it is advisable to try several sets of starting values to make sure that the solution gives relatively consistent results. Note that Weibull++ uses a different analysis method called the Levenberg-Marquardt. This method utilizes the best features of the Gauss-Newton method and the method of the steepest descent, and occupies a middle ground between these two methods.

Confidence Bounds

The approximate reliability confidence bounds under the modified Gompertz model can be obtained using non-linear regression. Additionally, the reliability is always between 0 and 1. In order to keep the endpoints of the confidence interval, the logit transformation can be used to obtain the confidence bounds on reliability.

$$CB = \frac{\hat{R}_i}{\hat{R}_i + (1 - \hat{R}_i)e^{\pm z_\alpha \hat{\sigma}_R / [\hat{R}_i(1-\hat{R}_i)]}}$$

$$\hat{\sigma}^2 = \frac{SSE}{n - p}$$

where p is the total number of groups (in this case 4) and n is the total number of items in each group.

Example - Modified Gompertz for Reliability Data

A reliability growth data set is given in columns 1 and 2 of the following table. Find the modified Gompertz curve that represents the data and plot it comparatively with the raw data.

Development Time vs. Observed Reliability Data and Predicted Reliabilities

Time (months)	Raw Data Reliability (%)	Gompertz Reliability (%)	Logistic Reliability (%)	Modified Gompertz Reliability (%)
0	31.00	25.17	22.70	31.18
1	35.50	38.33	38.10	35.08
2	49.30	51.35	56.40	49.92
3	70.10	62.92	73.00	69.23
4	83.00	72.47	85.00	83.72
5	92.20	79.94	93.20	92.06
6	96.40	85.59	96.10	96.29
7	98.60	89.75	98.10	98.32
8	99.00	92.76	99.10	99.27

Solution

To determine the parameters of the modified Gompertz curve, use:

$$S_1(d) = \sum_{i=0}^{2} \ln(R_{oi} - d)$$
$$S_2(d) = \sum_{i=3}^{5} \ln(R_{oi} - d)$$
$$S_3(d) = \sum_{i=6}^{8} \ln(R_{oi} - d)$$

$$c(d) = \left[\frac{S_3(d) - S_2(d)}{S_2(d) - S_1(d)}\right]^{\frac{1}{3}}$$

$$a(d) = e^{\left[\frac{1}{3}\left(S_1(d) + \frac{S_2(d) - S_1(d)}{1 - [c(d)]^3}\right)\right]}$$

$$b(d) = e^{\left[\frac{(S_2(d) - S_1(d))(c(d) - 1)}{\left[1 - [c(d)]^3\right]^2}\right]}$$

and:

$$R_0 = d + a(d) \cdot b(d)$$

for $R_0 = 31\%$, the equation above may be rewritten as:

$$d - 31 + a(d) \cdot b(d) = 0$$

The equations for parameters c, a and b can now be solved simultaneously. One method for solving these equations numerically is to substitute different values of d, which must be less than R_0, into the last equation shown above, and plot the results along the y-axis with the value of d along the x-axis. The value of d can then be read from the x-intercept. This can be repeated for greater accuracy using smaller and smaller increments of d. Once the desired accuracy on d has been achieved, the value of d can then be used to solve for a, b and c. For this case, the initial estimates of the parameters are:

$$\begin{aligned}
\widehat{a} &= 69.324 \\
\hat{b} &= 0.002524 \\
\hat{c} &= 0.46012 \\
\hat{d} &= 30.825
\end{aligned}$$

Now, since the initial values have been determined, the Gauss-Newton method can be used. Therefore, substituting $Y_i = R_i, g_1^{(0)} = 69.324, g_2^{(0)} = 0.002524, g_3^{(0)} = 0.46012,$ and $g_4^{(0)} = 30.825$, $Y^{(0)}, D^{(0)}, \nu^{(0)}$ become:

$$Y^{(0)} = \begin{bmatrix} 0.000026 \\ 0.253873 \\ -1.062940 \\ 0.565690 \\ -0.845260 \\ 0.096737 \\ 0.076450 \\ 0.238155 \\ -0.320890 \end{bmatrix}$$

$$D^{(0)} = \begin{bmatrix} 0.002524 & 69.3240 & 0.0000 & 1 \\ 0.063775 & 805.962 & -26.4468 & 1 \\ 0.281835 & 1638.82 & -107.552 & 1 \\ 0.558383 & 1493.96 & -147.068 & 1 \\ 0.764818 & 941.536 & -123.582 & 1 \\ 0.883940 & 500.694 & -82.1487 & 1 \\ 0.944818 & 246.246 & -48.4818 & 1 \\ 0.974220 & 116.829 & -26.8352 & 1 \\ 0.988055 & 54.5185 & -14.3117 & 1 \end{bmatrix}$$

$$\nu^{(0)} = \begin{bmatrix} g_1^{(0)} \\ g_2^{(0)} \\ g_3^{(0)} \\ g_4^{(0)} \end{bmatrix} = \begin{bmatrix} 69.324 \\ 0.002524 \\ 0.46012 \\ 30.825 \end{bmatrix}$$

The estimate of the parameters $\nu^{(0)}$ is given by:

$$\begin{aligned} \hat{\nu}^{(0)} &= \left(D^{(0)^T} D^{(0)}\right)^{-1} D^{(0)^T} Y^{(0)} \\ &= \begin{bmatrix} -0.275569 \\ -0.000549 \\ -0.003202 \\ 0.209458 \end{bmatrix} \end{aligned}$$

The revised estimated regression coefficients in matrix form are given by:

$$\begin{aligned} g^{(1)} =& g^{(0)} + \widehat{\nu}^{(0)}. \\ =& \begin{bmatrix} 69.324 \\ 0.002524 \\ 0.46012 \\ 30.825 \end{bmatrix} + \begin{bmatrix} -0.275569 \\ -0.000549 \\ -0.003202 \\ 0.209458 \end{bmatrix} \\ =& \begin{bmatrix} 69.0484 \\ 0.00198 \\ 0.45692 \\ 31.0345 \end{bmatrix} \end{aligned}$$

With the starting coefficients $g^{(0)}$, Q is:

$$\begin{aligned} Q^{(0)} =& \sum_{i=1}^{N} \left(Y_i - f(T_i, g^{(0)})\right)^2 \\ =& 2.403672 \end{aligned}$$

With the coefficients at the end of the first iteration, $g^{(1)}$, Q is:

$$\begin{aligned} Q^{(1)} =& \sum_{i=1}^{N} \left[Y_i - f\left(T_i, g^{(1)}\right)\right]^2 \\ =& 2.073964 \end{aligned}$$

Therefore:

$$Q^{(1)} < Q^{(0)}$$

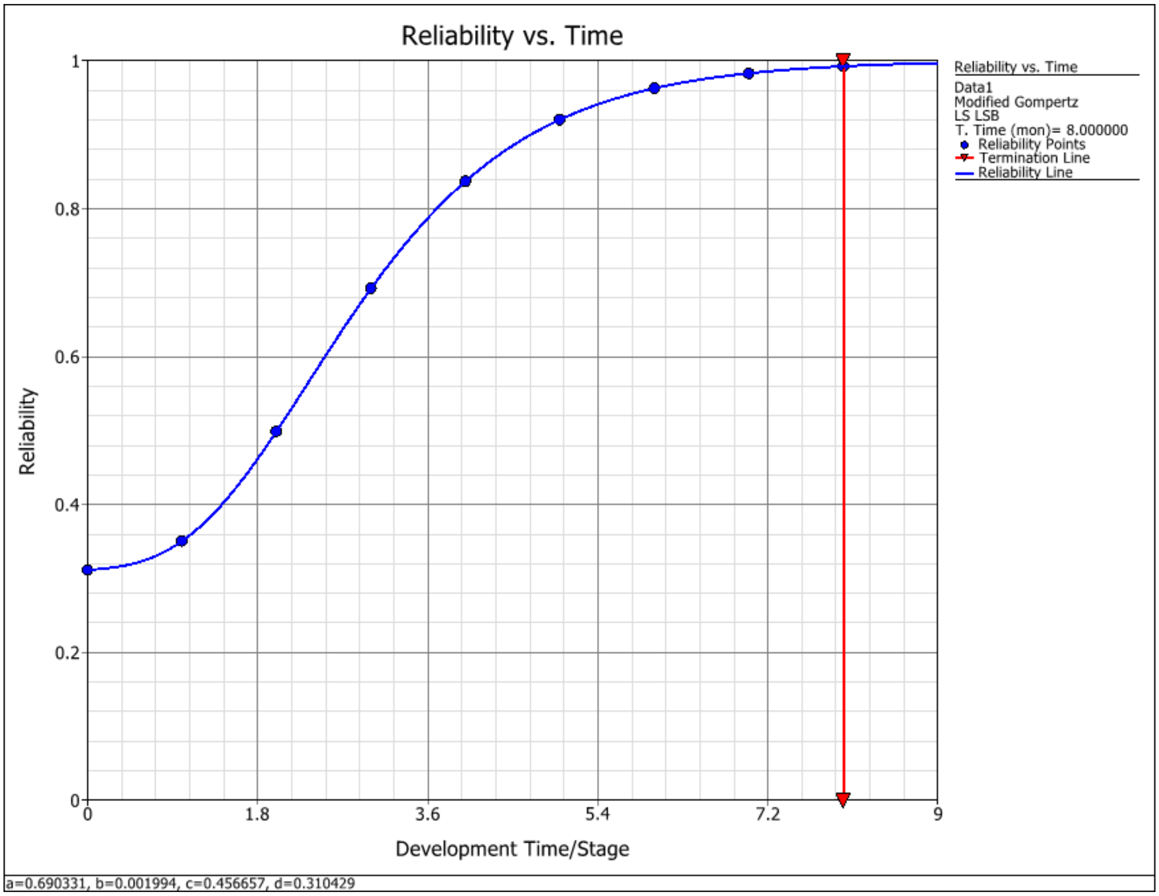
Hence, the Gauss-Newton method works in the right direction. The iterations are continued until the relationship of $Q^{(s-1)} - Q^{(s)} \simeq 0$ has been satisfied. Using Weibull++RGA, the estimators of the parameters are:

$$\begin{aligned} \widehat{a} =& 0.6904 \\ \hat{b} =& 0.0020 \\ \hat{c} =& 0.4567 \\ \hat{d} =& 0.3104 \end{aligned}$$

Therefore, the modified Gompertz model is:

$$R = 0.3104 + (0.6904)(0.0020)^{0.4567^T}$$

Using this equation, the predicted reliability is plotted in the following figure along with the raw data. As you can see, the modified Gompertz curve represents the data very well.



More Examples

Standard Gompertz for Grouped per Configuration Data

A new design is put through a reliability growth test. The requirement is that after the ninth stage the design will exhibit an 85% reliability with a 90% confidence level. Given the data in the following table, do the following:

1. Estimate the parameters of the standard Gompertz model.
2. What is the initial reliability at $T = 0$?
3. Determine the reliability at the end of the ninth stage and check to see whether the goal has been met.

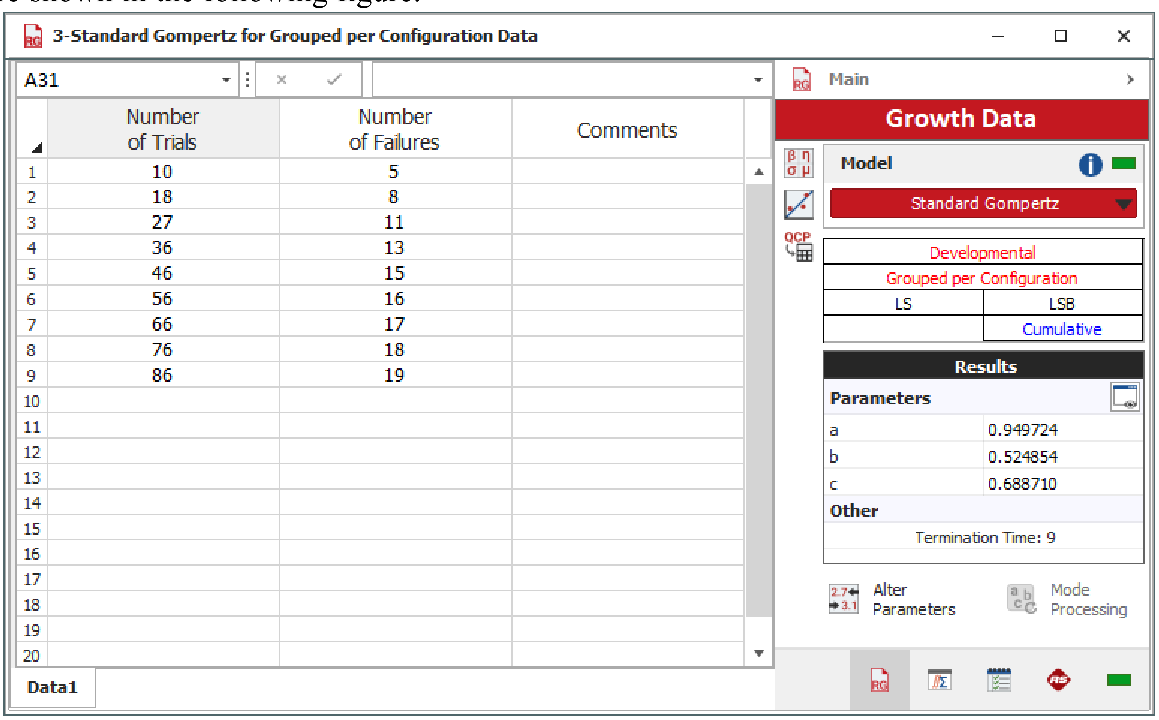
Grouped per Configuration Data

Stage	Number of Units	Number of Failures
1	10	5

2	8	3
3	9	3
4	9	2
5	10	2
6	10	1
7	10	1
8	10	1
9	10	1

Solution

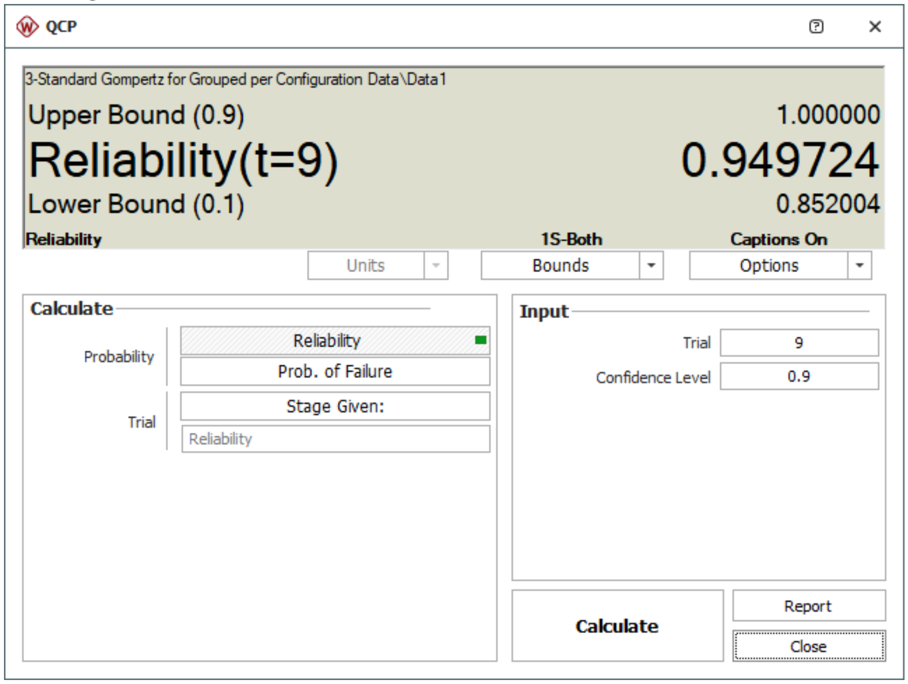
1. The data is entered in cumulative format and the estimated standard Gompertz parameters are shown in the following figure.



2. The initial reliability at $T = 0$ is equal to:

$$\begin{aligned} R_{T=0} &= a \cdot b \\ &= 0.9497 \cdot 0.5249 \\ &= 0.4985 \end{aligned}$$

3. The reliability at the ninth stage can be calculated using the Quick Calculation Pad (QCP) as shown in the figure below.



The estimated reliability at the end of the ninth stage is equal to 94.97%. However, the lower limit at the 90% 1-sided confidence bound is equal to 85.2%. Therefore, the required goal of 85% reliability at a 90% confidence level has been met.

Comparing Standard and Modified Gompertz

Using the data in the following table, determine whether the standard Gompertz or modified Gompertz would be better suited for analyzing the given data.

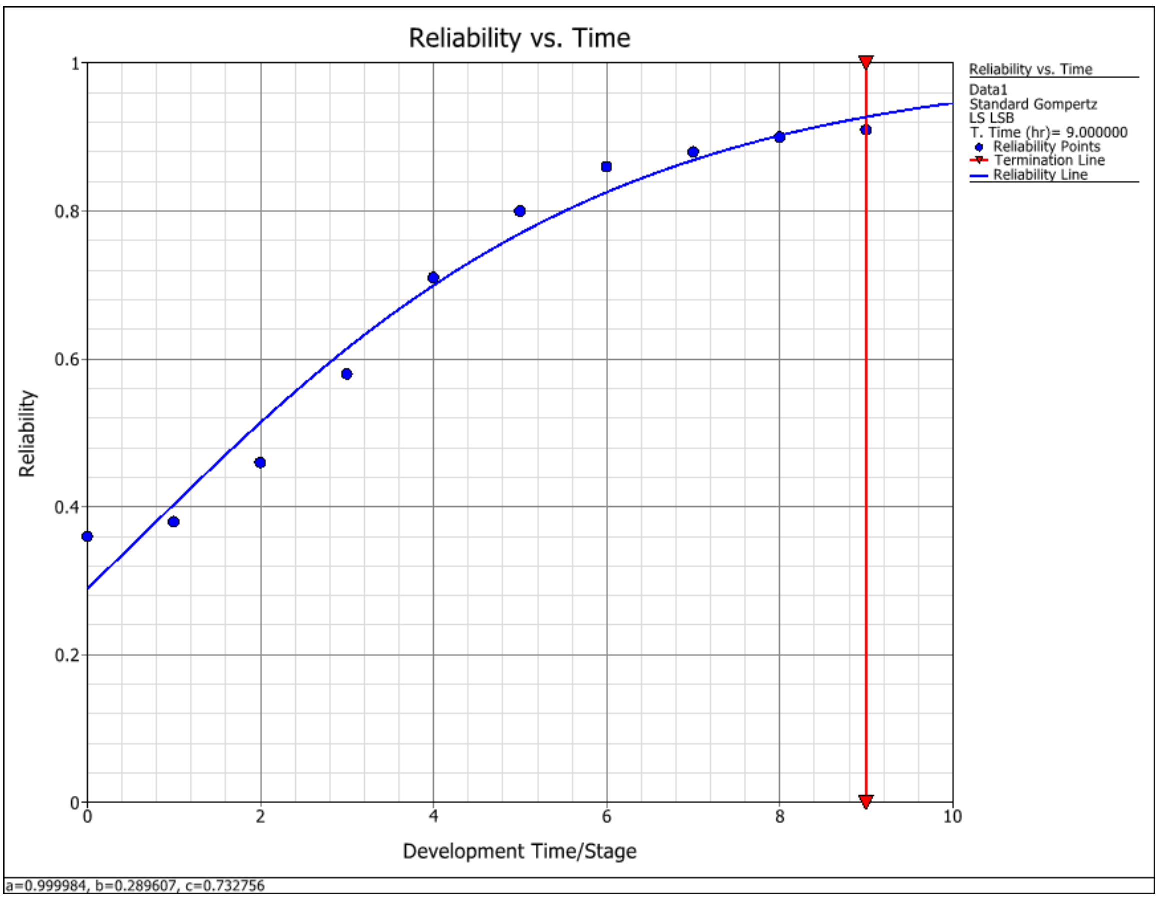
Reliability Data

Stage	Reliability (%)
0	36
1	38
2	46
3	58
4	71
5	80

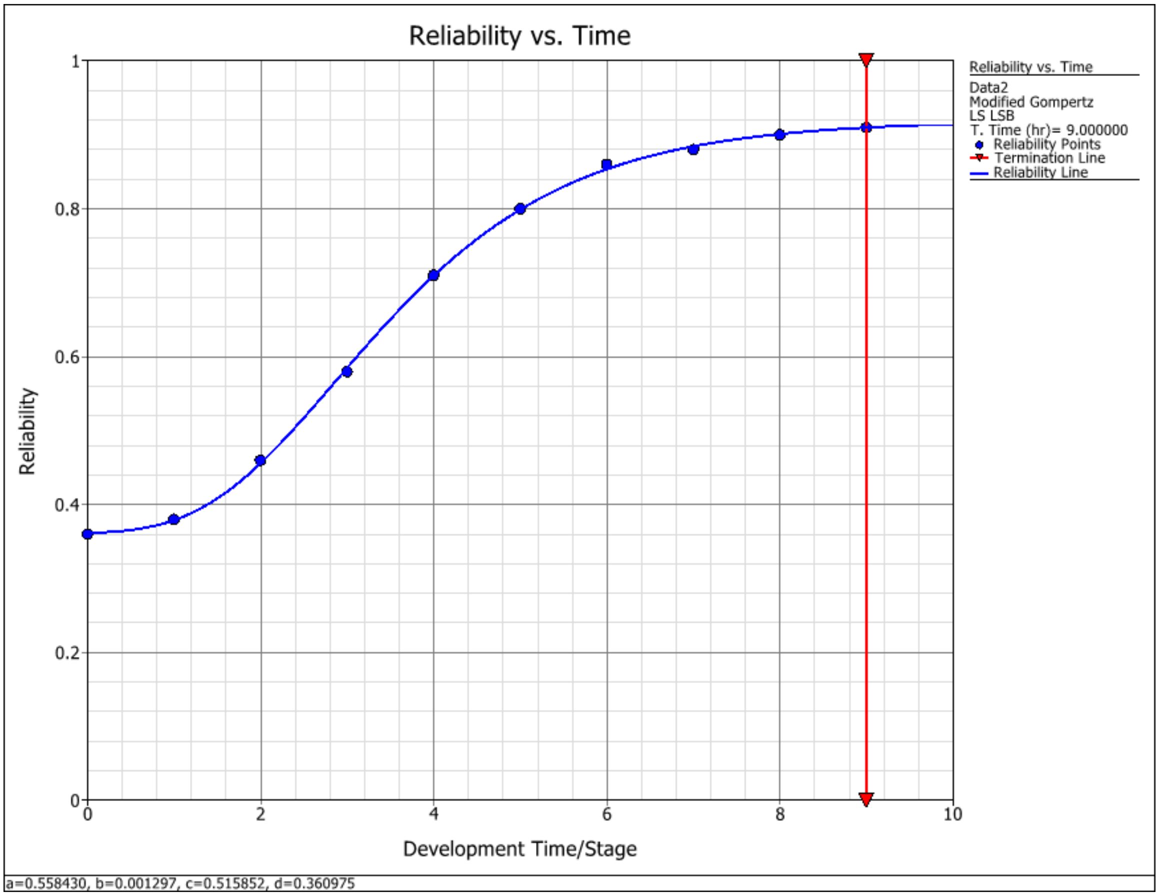
6	86
7	88
8	90
9	91

Solution

The standard Gompertz Reliability vs. Time plot is shown next.



The standard Gompertz seems to do a fairly good job of modeling the data. However, it appears that it is having difficulty modeling the S-shape of the data. The modified Gompertz Reliability vs. Time plot is shown next. As expected, the modified Gompertz does a much better job of handling the S-shape presented by the data and provides a better fit for this data.



Logistic

The Logistic reliability growth model has an S-shaped curve and is given by Kececioglu [3]:

$$R = \frac{1}{1+be^{-kt}}, b > 0, k > 0, T \simeq 0$$

where b and k are parameters. Similar to the analysis given for the Gompertz curve, the following may be concluded:

1. The point of inflection is given by:

$$T_i = \frac{\ln(b)}{k}$$

2. When $b > 1$, then $T_i > 0$ and an S-shaped curve will be generated. However, when $0 < b \leq 1$, then $T_i \leq 0$ and the Logistic reliability growth model will not be described by an S-shaped curve.

3. The value of R is equal to 0.5 at the inflection point.

Parameter Estimation

In this section, we will demonstrate the parameter estimation method for the Logistic model using three examples for different types of data.

Example: Logistic for Reliability Data

Using the reliability growth data given in the table below, do the following:

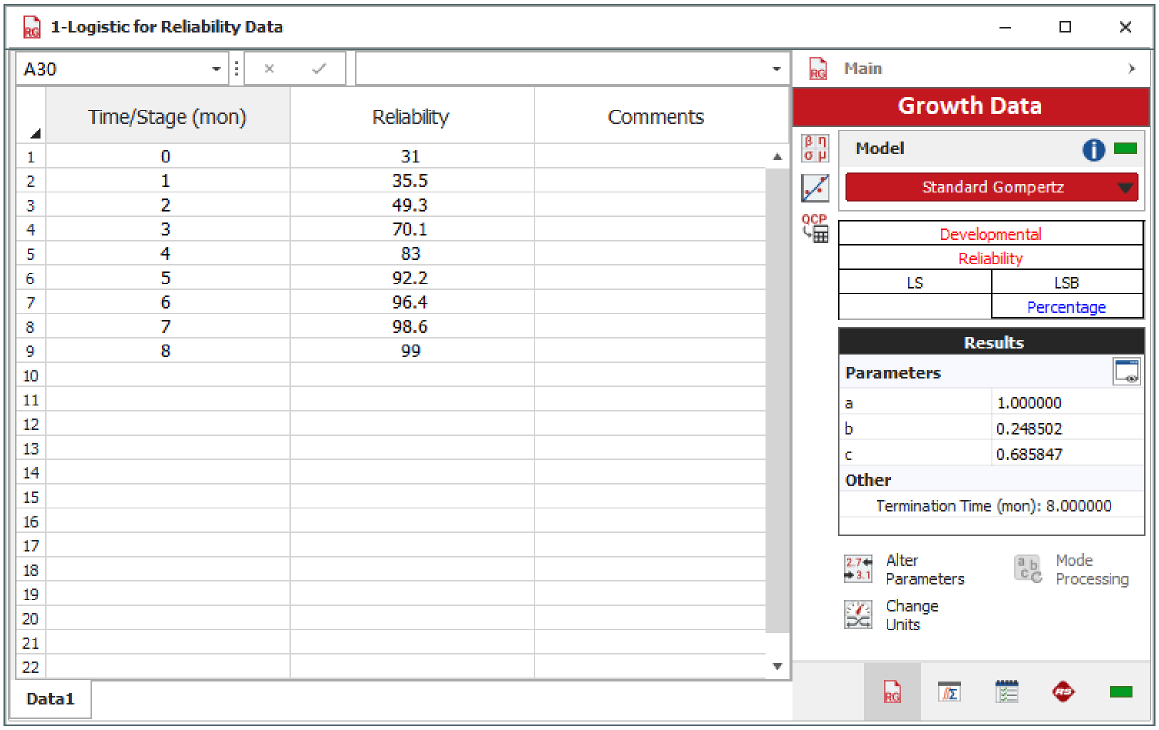
1. Find a Gompertz curve that represents the data and plot it with the raw data.
2. Find a Logistic reliability growth curve that represents the data and plot it with the raw data.

Development Time vs. Observed Reliability data and Predicted Reliabilities

Time, months	Raw Data Reliability (%)	Gompertz Reliability (%)	Logistic Reliability (%)
0	31.00	24.85	22.73
1	35.50	38.48	38.14
2	49.30	51.95	56.37
3	70.10	63.82	73.02
4	83.00	73.49	85.01
5	92.20	80.95	92.24
6	96.40	86.51	96.14
7	98.60	90.54	98.12
8	99.00	93.41	99.09

Solution

1. The figure below shows the entered data and the estimated parameters using the standard Gompertz model.

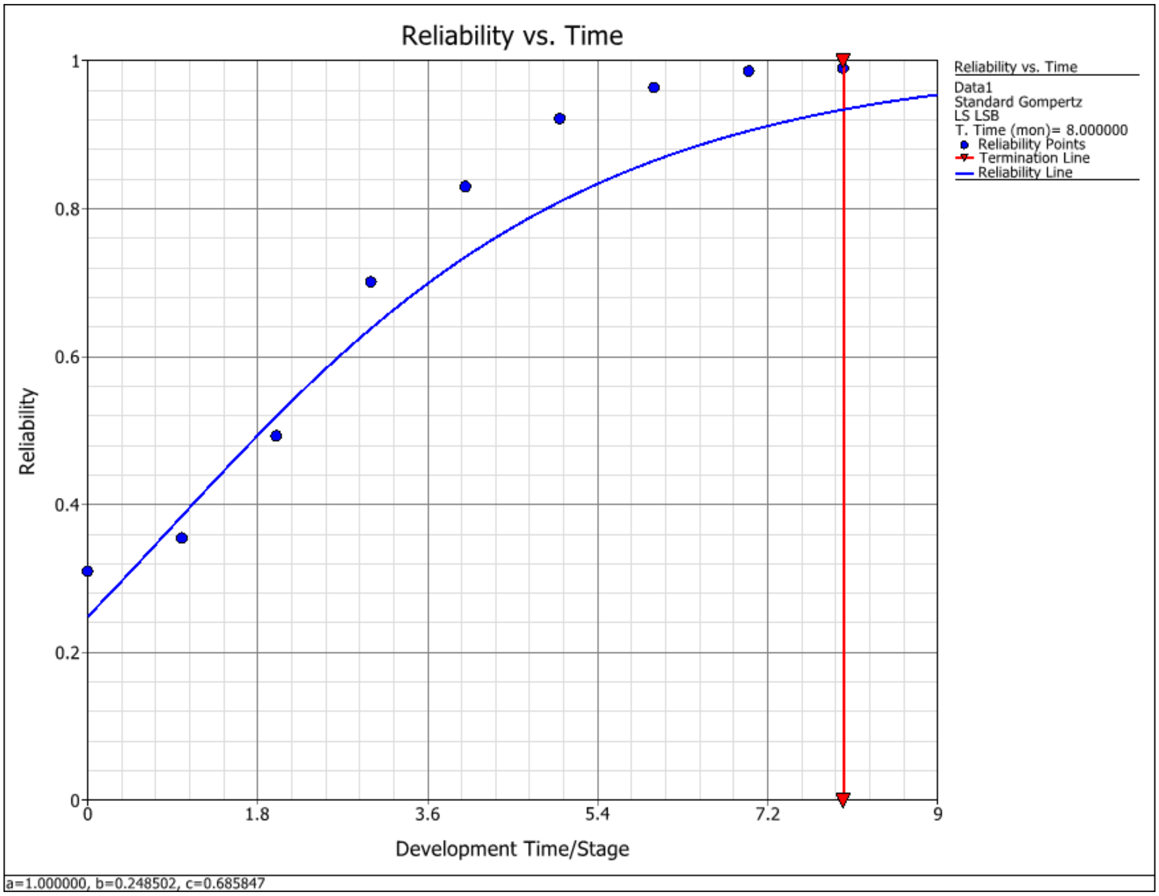


Therefore:

$$\hat{a} = 0.9999$$
$$\hat{b} = 0.2485$$
$$\hat{c} = 0.6858$$

$$R = (0.9999)(0.2485)^{0.6858^{T}}$$

The values of the predicted reliabilities are plotted in the figure below.



Notice how the standard Gompertz model is not really capable of handling the S-shaped characteristics of this data.

2. The least squares estimators of the Logistic growth curve parameters are given by Crow [9]: where:

$$\hat{b}_1 = \frac{\sum^{N-1}{}_{i=0} T_i Y_i - N \cdot \bar{T} \cdot \bar{Y}}{\sum^{N-1}{}_{i=0} T_i^2 - N \cdot \bar{T}^2}$$

$$\hat{b}_0 = \bar{Y} - \hat{b}_1 \bar{T}$$

$$Y_i = \ln\left(\frac{1}{R_i} - 1\right)$$

$$\bar{Y} = \frac{1}{N} \sum_{i=0}^{N-1} Y_i$$

In this example $N = 9$, which gives:

$$\overline{Y} = \frac{1}{9}\sum_{i=0}^{8} ln\left(\frac{1}{R_i} - 1\right)$$
$$= -1.7355$$

$$\overline{T} = \frac{1}{9}\sum_{i=0}^{8} T_i = 4$$

$$\sum_{i=0}^{8} T_i^2 = 204$$

$$\sum_{i=0}^{8} T_i Y_i = -106.8630$$

From the equations for b_i and $\hat{b_0}$:

$$\hat{b_1} = \frac{-106.8630 - 9(4)(-1.7355)}{204 - 9(4)_2}$$
$$= 0.7398$$
$$\hat{b_0} = -1.7355 - (-0.7398)(4)$$
$$= 1.2235$$

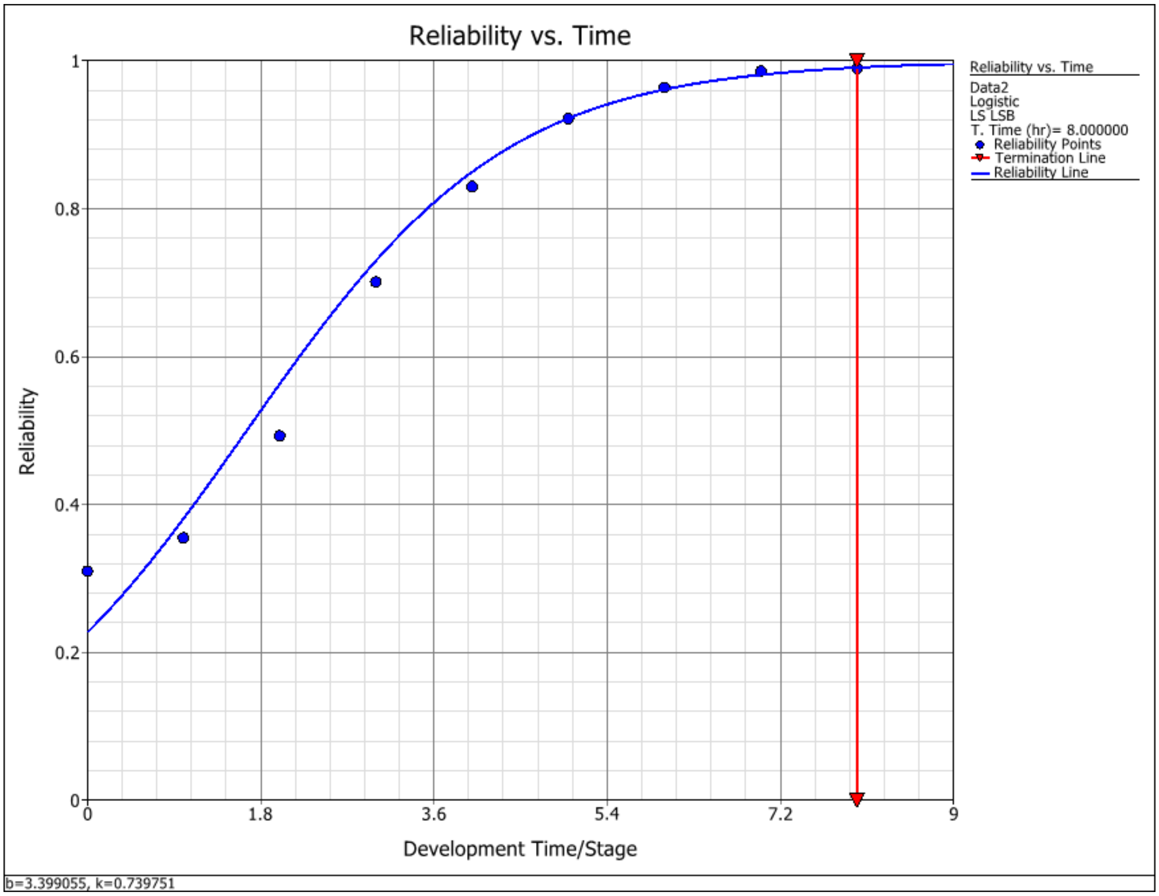
And from the least squares estimators for $\hat{b}$ and $\hat{k}$:

$$\hat{b} = e^{1.2235}$$
$$= 3.3991$$
$$\hat{k} = -(-0.7398)$$
$$= 0.7398$$

Therefore, the Logistic reliability growth curve that represents this data set is given by:

$$R = \frac{1}{1 + 3.3991\, e^{-0.7398\, T}}$$

The following figure shows the Reliability vs. Time plot. The plot shows that the observed data set is estimated well by the Logistic reliability growth curve, except in the region closely surrounding the inflection point of the observed reliability. This problem can be overcome by using the modified Gompertz model.



Example: Logistic for Sequential Success/Failure Data

A prototype was tested under a success/failure pattern. The test consisted of 15 runs. The following table presents the data from the test. Find the Logistic model that best fits the data set, and plot it along with the reliability observed from the raw data.

Sequential Success/Failure Data with Observed Reliability Values

Time	Result	Observed Reliability
0	F	0.5000
1	F	0.3333
2	S	0.5000
3	S	0.6000
4	F	0.5000

5	S	0.5714
6	S	0.6250
7	S	0.6667
8	S	0.7000
9	F	0.6364
10	S	0.6667
11	S	0.6923
12	S	0.7143
13	S	0.7333

Solution

The first run is ignored because it was a success, and the reliability at that point was 100%. This failure will be ignored throughout the analysis because it is considered that the test starts when the reliability is not equal to zero or one. The test essentially begins at time 1, and is now considered as time 0 with $N = 14$. The observed reliability is shown in the last column of the table. Keep in mind that the observed reliability values still account for the initial suspension.

Therefore:

$$\begin{aligned}\bar{Y} &= \frac{1}{N}\sum_{i=0}^{N-1} Y_i \\ &= \frac{1}{14}\sum_{i=0}^{13} \ln\left(\frac{1}{R_i} - 1\right) \\ &= -0.43163\end{aligned}$$

and:

$$\begin{aligned}\bar{T} &= \frac{1}{14}\sum_{i=0}^{13} T_i \\ &= 6.5\end{aligned}$$

$$\sum_{i=0}^{13} T_i^2 = 819.0$$

$$\sum_{i=0}^{13} T_i Y_i = -61.69$$

Now, from the least squares estimators, the values are:

$$\hat{b}_1 = \frac{\sum^{N-1}{}_{i=0} T_i Y_i - N \cdot \bar{T} \cdot \bar{Y}}{\sum^{N-1}{}_{i=0} T_i^2 - N \cdot \bar{T}^2}$$

$$= \frac{-61.69 - 14 \cdot 6.5 \cdot (-.43163)}{819.0 - 14 \cdot 6.5^2}$$
$$= -0.0985$$

$$\hat{b_0} = \overline{Y} - \hat{b_1}\overline{T}$$
$$= (-.043163) - (-0.0985) \cdot 6.5$$
$$= 0.2087$$

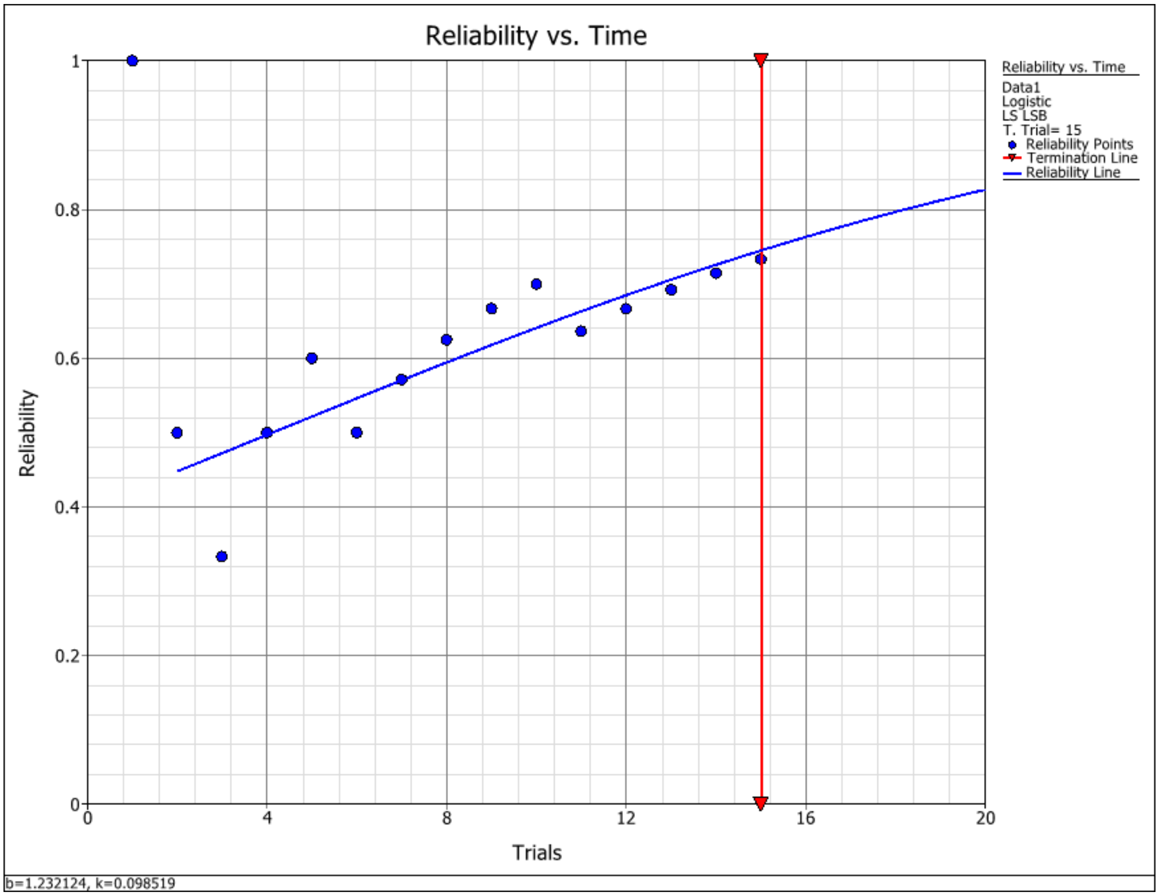
Therefore:

$$\hat{b} = e^{0.2087}$$
$$=1.2321$$
$$\hat{k} = -(-0.0985)$$
$$=0.0985$$

The Logistic reliability model that best fits the data is given by:

$$R = \frac{1}{1 + 1.2321 \cdot \ e^{-0.0985T}}$$

The following figure shows the Reliability vs. Time plot.



Example: Logistic for Grouped per Configuration Data

Some equipment underwent testing in different stages. The testing may have been performed in subsequent days, weeks or months with an unequal number of units tested every day. Each group was tested and several failures occurred. The data set is given in columns 1 and 2 of the following table. Find the Logistic model that best fits the data, and plot it along with the reliability observed from the raw data.

Grouped per Configuration Data

Number of Units	Number of Failures	T_i	Observed Reliability
10	5	0	0.5000
8	3	1	0.6250
9	3	2	0.6667
9	2	3	0.7778

10	2	4	0.8000
10	1	5	0.9000
10	1	6	0.9000
10	1	7	0.9000
10	1	8	0.9000

Solution

The observed reliability is $1-\frac{\text{\# of failures}}{\text{\# of units}}$ and the last column of the table above shows the values for each group. With $N=9$, the least square estimator $\overline{Y}$ becomes:

$$\begin{aligned}\bar{Y}=&\frac{1}{9}\sum_{i=0}^{8}\ln\left(\frac{1}{R_i}-1\right)\\ =&-1.4036\end{aligned}$$

and:

$$\begin{aligned}\bar{T}=&\frac{1}{9}\sum_{i=0}^{8}T_i\\ =&4\\ \sum_{i=0}^{8}T_i^2=&204\\ \sum_{i=0}^{8}T_iY_i=&-68.33\end{aligned}$$

Now from the least squares estimators, $\hat{b_i}$ and $\hat{b_0}$, we have:

$$\begin{aligned}\hat{b}_1=&\frac{\sum_{i=0}^{8}T_iY_i-N\cdot\bar{T}\cdot\bar{Y}}{\sum_{i=0}^{8}T_i^2-N\cdot\bar{T}^2}\\ =&\frac{-68.33-9\cdot4\cdot(-1.4036)}{204-9\cdot4^2}\\ =&-0.2967\end{aligned}$$

$$\begin{aligned}\hat{b}_0=&\bar{Y}-\hat{b}_1\bar{T}\\ =&(-1.4036)-(-0.2967)\cdot4.0\\ =&-0.2168\end{aligned}$$

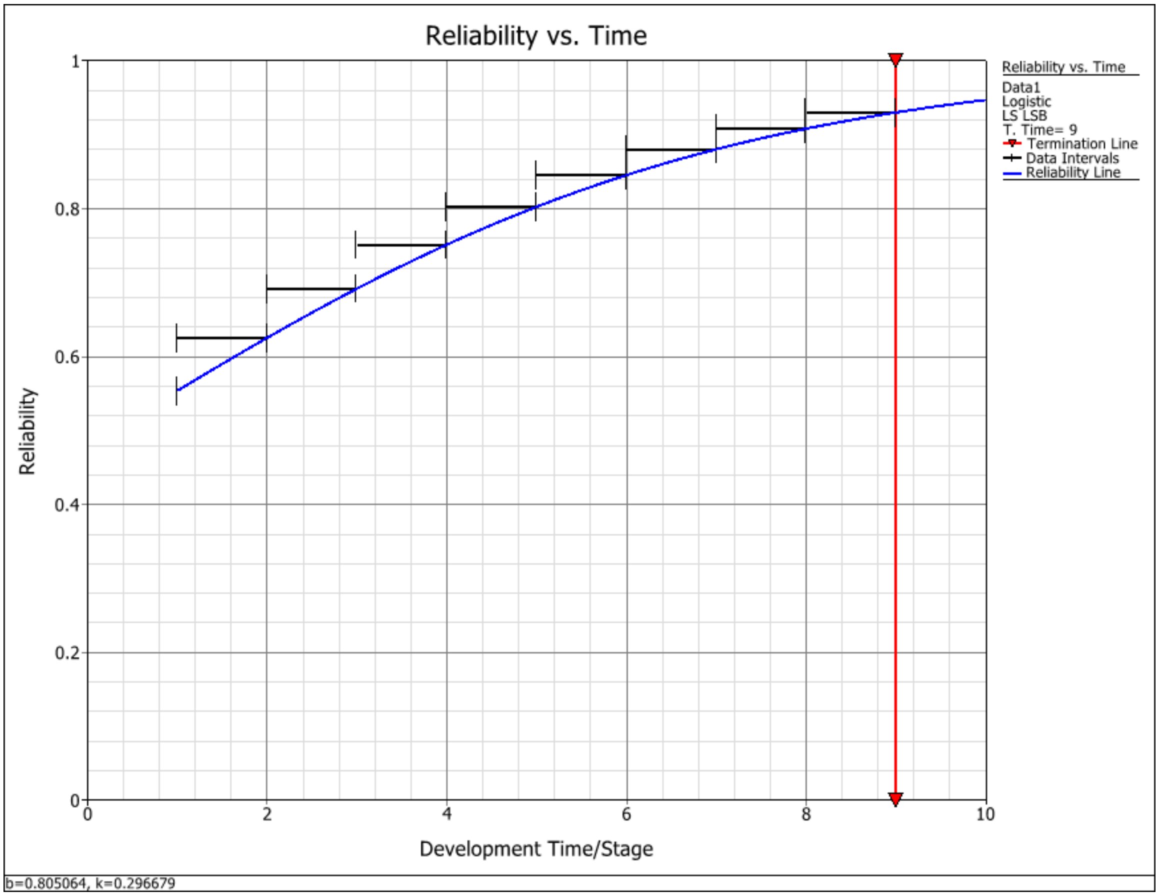
Therefore:

$$\hat{b} = e^{-0.2168}$$
$$= 0.8051$$
$$\hat{k} = -(-0.2967)$$
$$= 0.2967$$

The Logistic reliability model that best fits the data is given by:

$$R = \frac{1}{1 + 0.8051 \cdot \ e^{-0.2967T}}$$

The figure below shows the Reliability vs. Time plot.



Confidence Bounds

Least squares is used to estimate the parameters of the following Logistic model.

$$\ln\left(\frac{1}{\hat{R}_i} - 1\right) = \ln(b) - kT_i$$

Thus, the confidence bounds on the parameter b are given by:

$$b = \hat{b}e^{t_{n-2,\alpha/2}SE(\ln \hat{b})}$$

where:

$$SE(\ln \hat{b}) = \sigma \cdot \sqrt{\frac{\sum^{n}{}_{i=1}(T_i)^2}{n \cdot S_{xx}}}, \quad S_{xx} = \left[\sum_{i=1}^{n}(T_i)^2\right] - \frac{1}{n}\left(\sum_{i=1}^{n}T_i\right)^2$$

$$\sigma = \sqrt{SSE/(n-2)}$$

and the confidence bounds on the parameter k are:

$$k = \hat{k} \pm t_{n-2,\alpha/2}SE(\hat{k})$$

where:

$$SE(\hat{k}) = \frac{\sigma}{\sqrt{S_{xx}}}, \quad S_{xx} = \left[\sum_{i=1}^{n}(T_i)^2\right] - \frac{1}{n}\left(\sum_{i=1}^{n}T_i\right)^2$$

Since the reliability is always between 0 and 1, the logit transformation is used to obtain the confidence bounds on reliability, which is:

$$CB = \frac{\hat{R}_i}{\hat{R}_i + (1-\hat{R}_i)e^{\pm z_\alpha \hat{\sigma}_R/[\hat{R}_i(1-\hat{R}_i)]}}$$

Example: Logistic Confidence Bounds

For the data given above for the reliability data example, calculate the 2-sided 90% confidence bounds under the Logistic model for the following:

1. The parameters b and k.
2. Reliability at month 5.

Solution

1. The values of $\hat{b}$ and $\hat{k}$ that were estimated from the least squares analysis in the reliability data example are:

   $\hat{b}$ =3.3991
   $\widehat{\alpha}$ =0.7398

   Thus, the 2-sided 90% confidence bounds on parameter b are:

$$b_{lower} = 2.5547$$
$$b_{upper} = 4.5225$$

The 2-sided 90% confidence bounds on parameter k are:

$$k_{lower} = 0.6798$$
$$k_{upper} = 0.7997$$

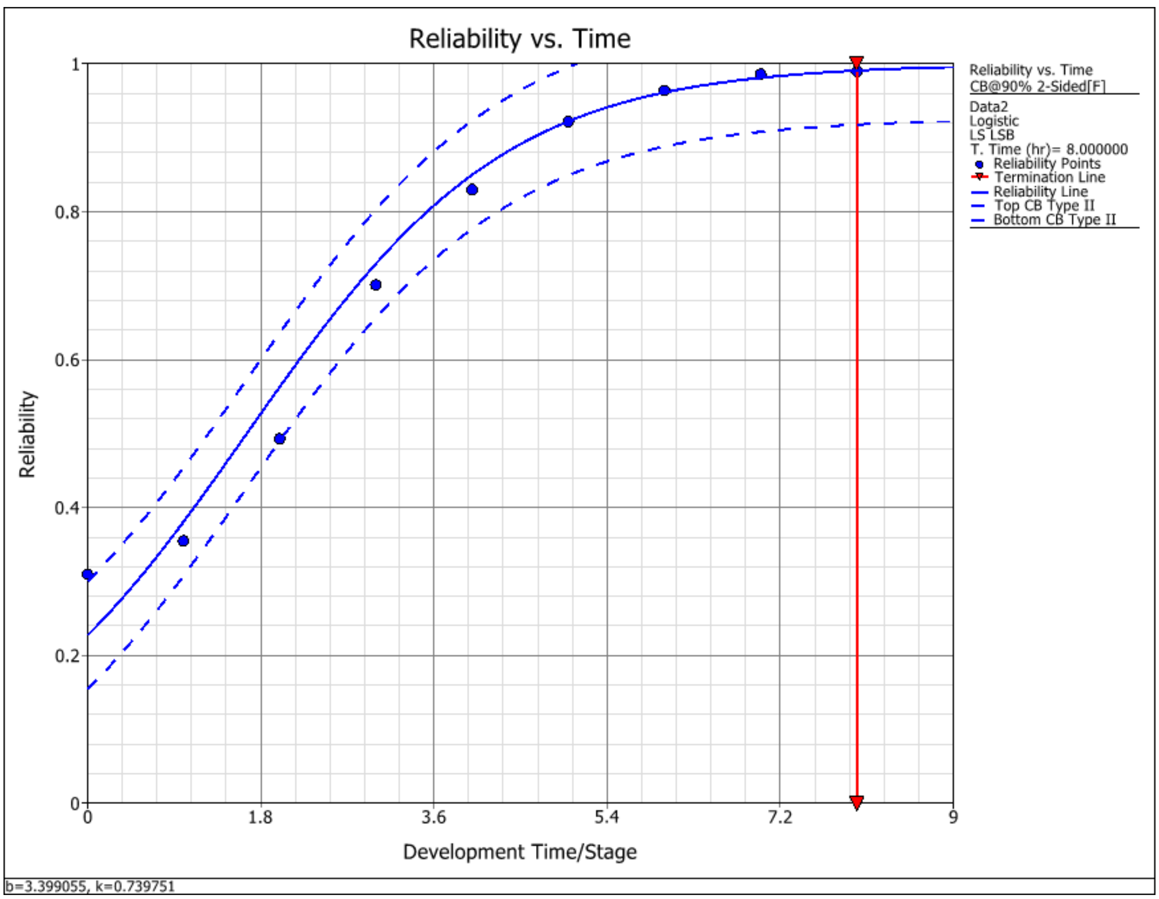
2. First, calculate the reliability estimation at month 5:

$$R_5 = \frac{1}{1 + be^{-5k}} = 0.9224$$

Thus, the 2-sided 90% confidence bounds on reliability at month 5 are:

$$[R_5]_{lower} = 0.8493$$
$$[R_5]_{upper} = 0.9955$$

The next figure shows a graph of the reliability plotted with 2-sided 90% confidence bounds.



More Examples

Auto Transmission Reliability Data

The following table presents the reliabilities observed monthly for an automobile transmission that was tested for one year.

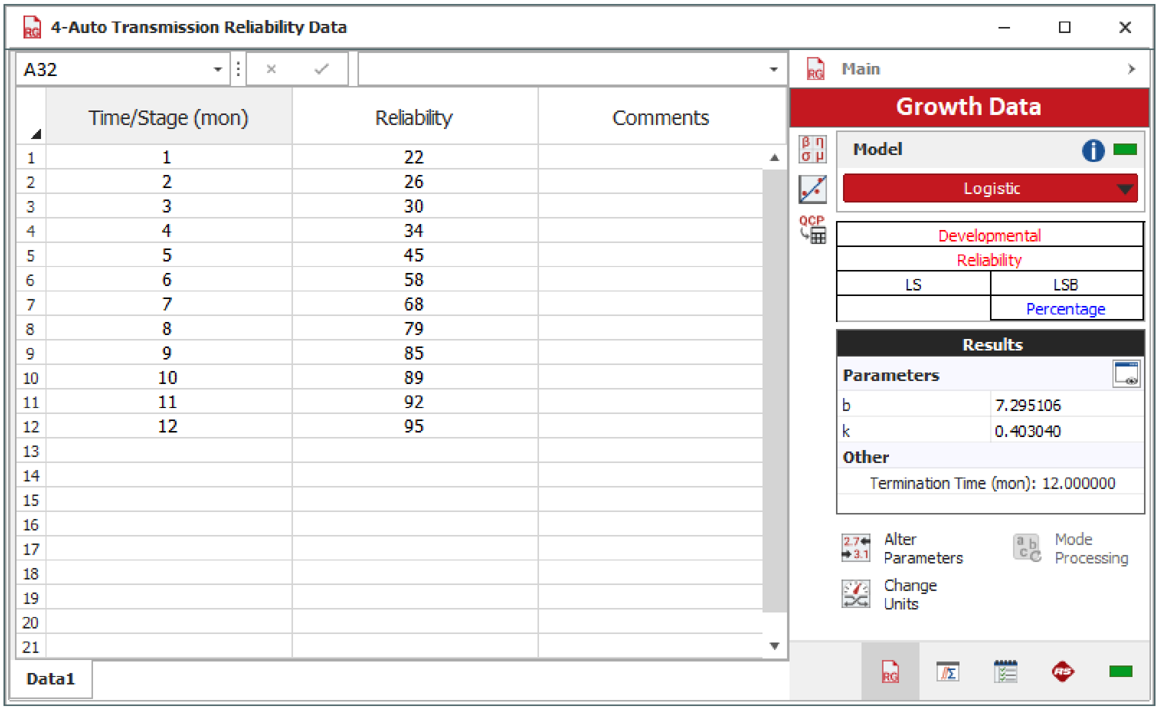
1. Find a Logistic reliability growth curve that best represents the data.
2. Plot it comparatively with the raw data.
3. If design changes continue to be incorporated and the testing continues, when will the reliability goal of 99% be achieved?
4. If design changes continue to be incorporated and the testing continues, what will be the attainable reliability at the end of January the following year?

Reliability Data

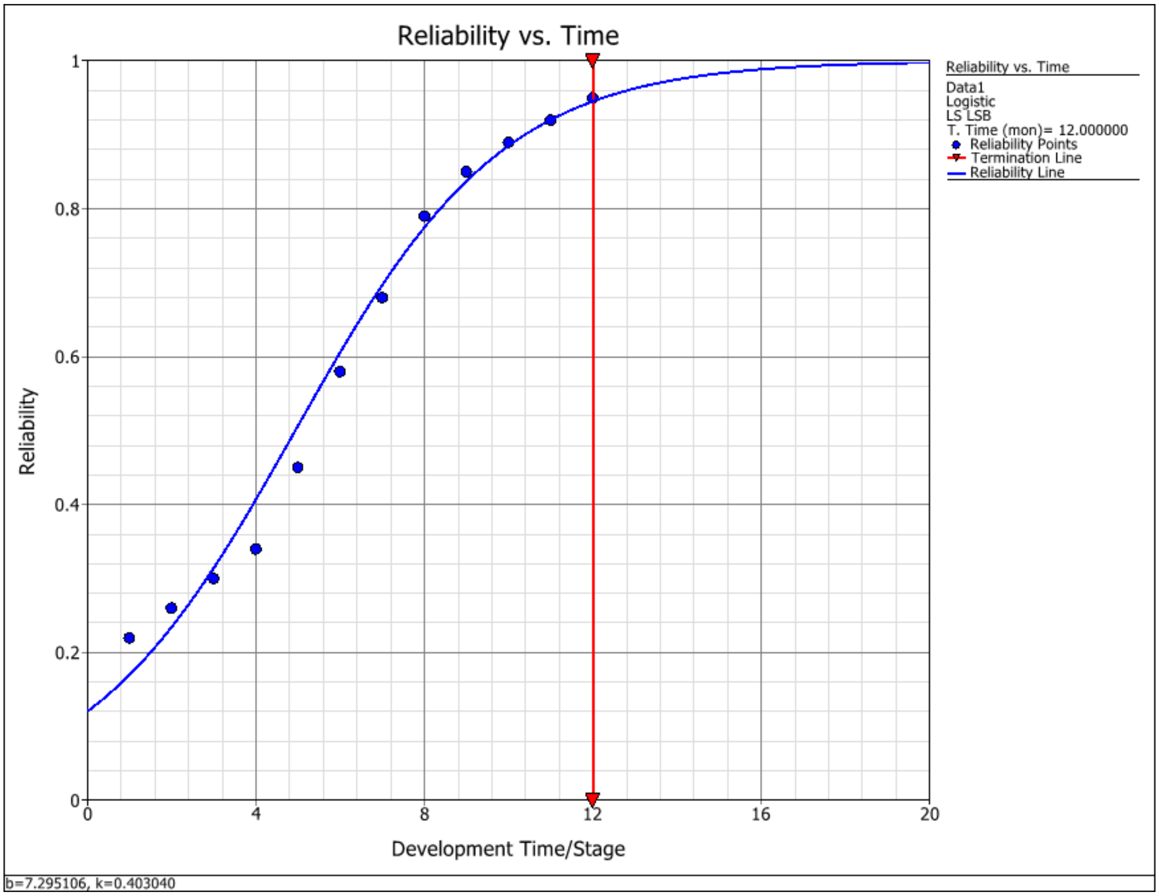
Month	Observed Reliability (%)
June	22
July	26
August	30
September	34
October	45
November	58
December	68
January	79
February	85
March	89
April	92
May	95

Solution

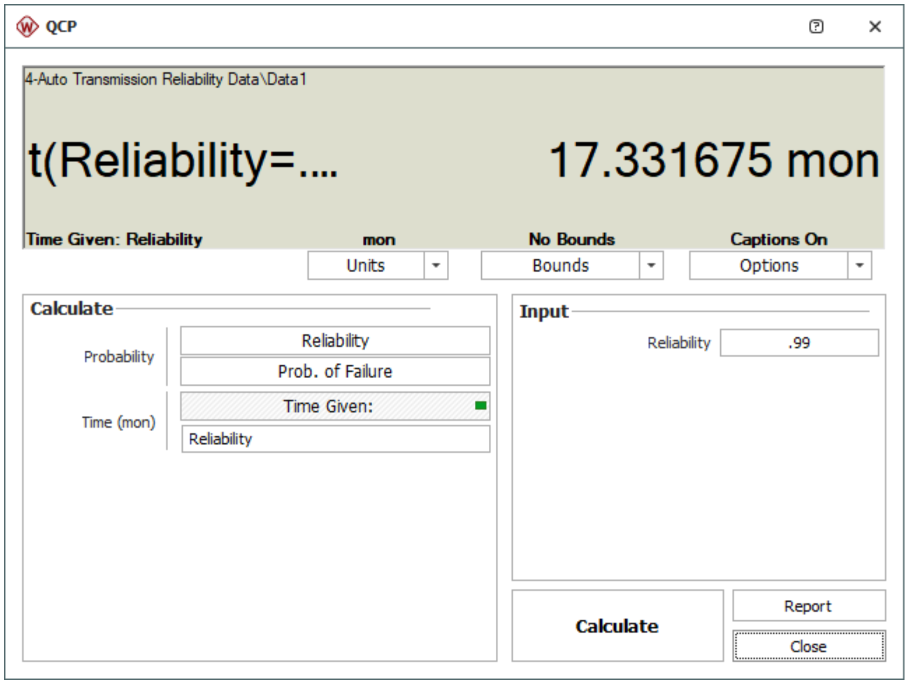
1. The next figure shows the entered data and the estimated parameters.



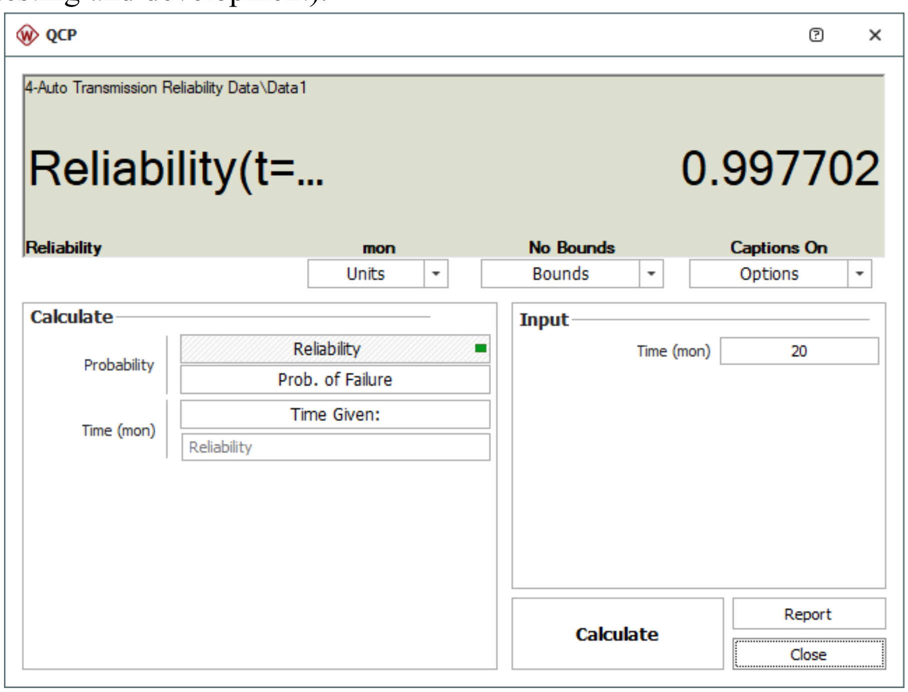
2. The next figure displays the Reliability vs. Time plot.



3. Using the QCP, the next figure displays, in months, when the reliability goal of 99% will be achieved.



4. The last figure shows the reliability at the end of January the following year (i.e., after 20 months of testing and development).



Sequential Data from Missile Launch Test

The following table presents the results for a missile launch test. The test consisted of 20 attempts. If the missile launched, it was recorded as a success. If not, it was recorded as a failure. Note that, at this development stage, the test did not consider whether or not the target was destroyed.

1. Find a Logistic reliability growth curve that best represents the data.
2. Plot it comparatively with the raw data.
3. If design changes continue to be incorporated and the testing continues, when will the reliability goal of 99.5% with a 90% confidence level be achieved?
4. If design changes continue to be incorporated and the testing continues, what will be the attainable reliability at the end of the 35th launch?

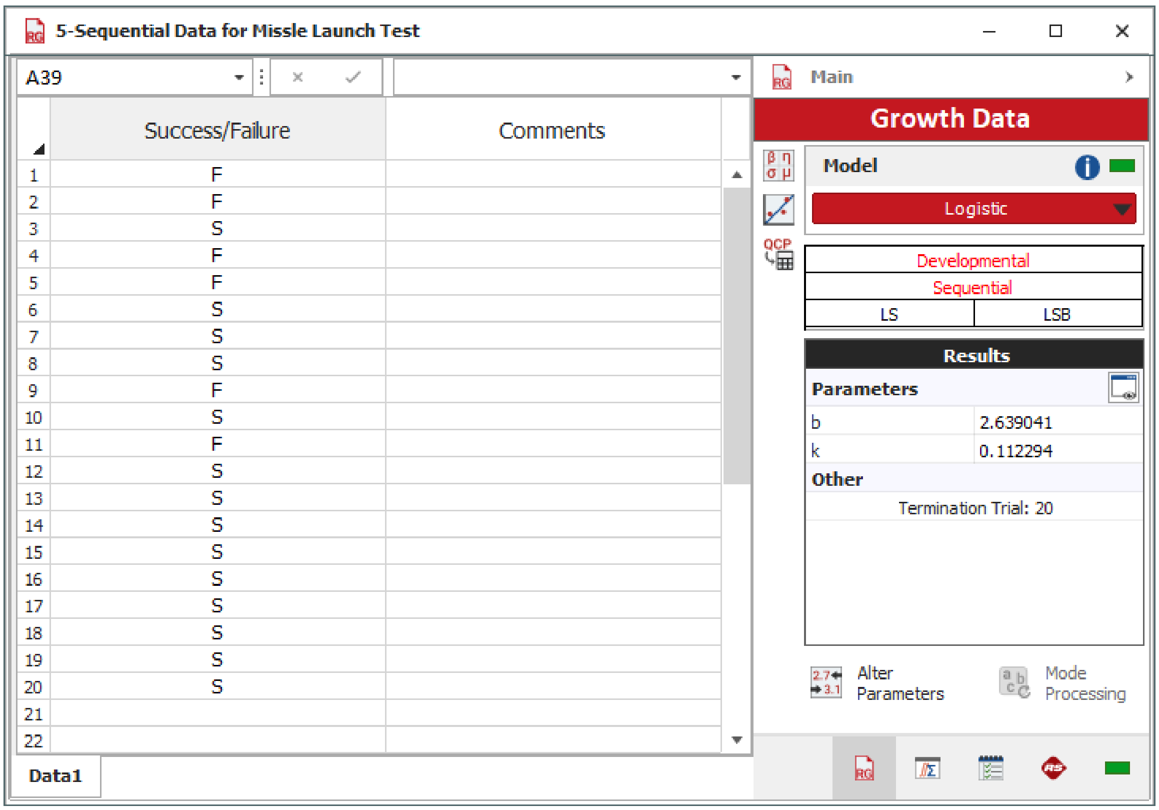
Sequential Success/Failure Data

Launch Number	Result
1	F
2	F
3	S
4	F
5	F
6	S
7	S
8	S
9	F
10	S
11	F
12	S
13	S
14	S
15	S

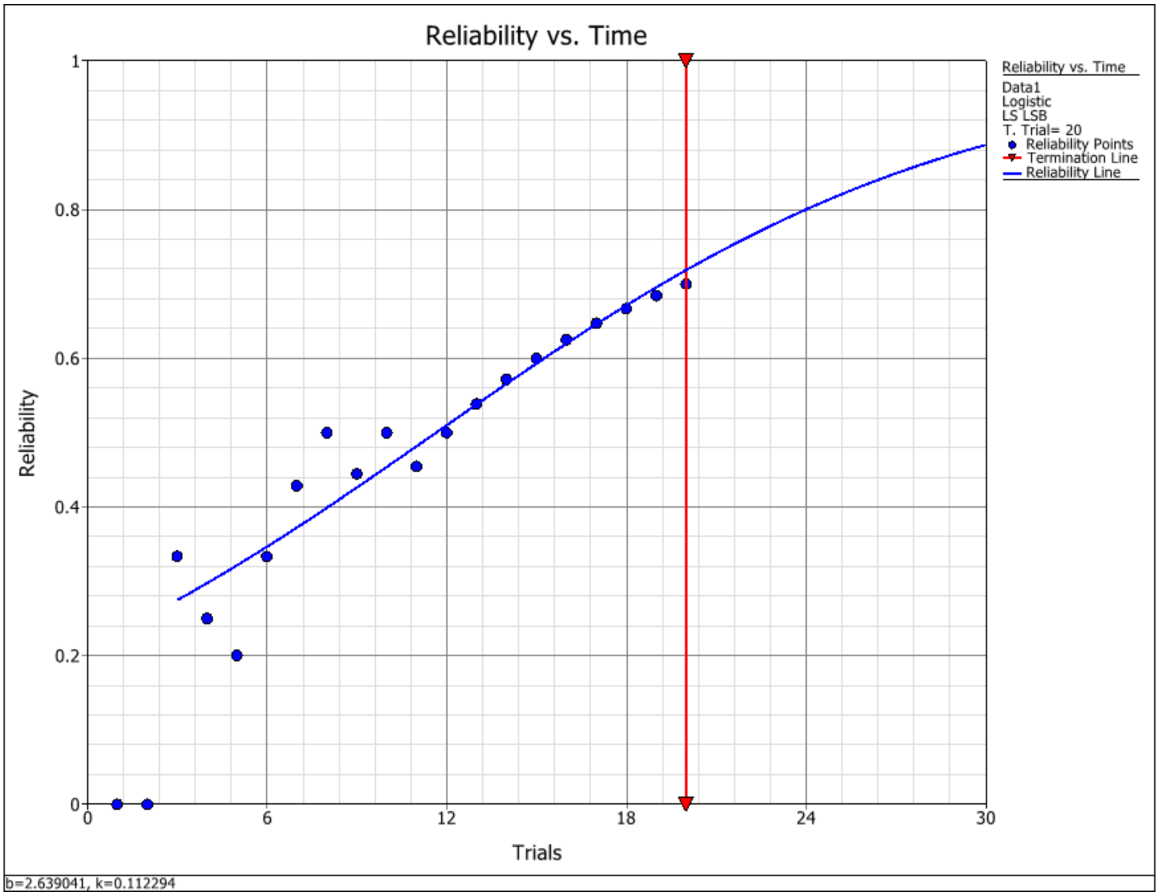
16	S
17	S
18	S
19	S
20	S

Solution

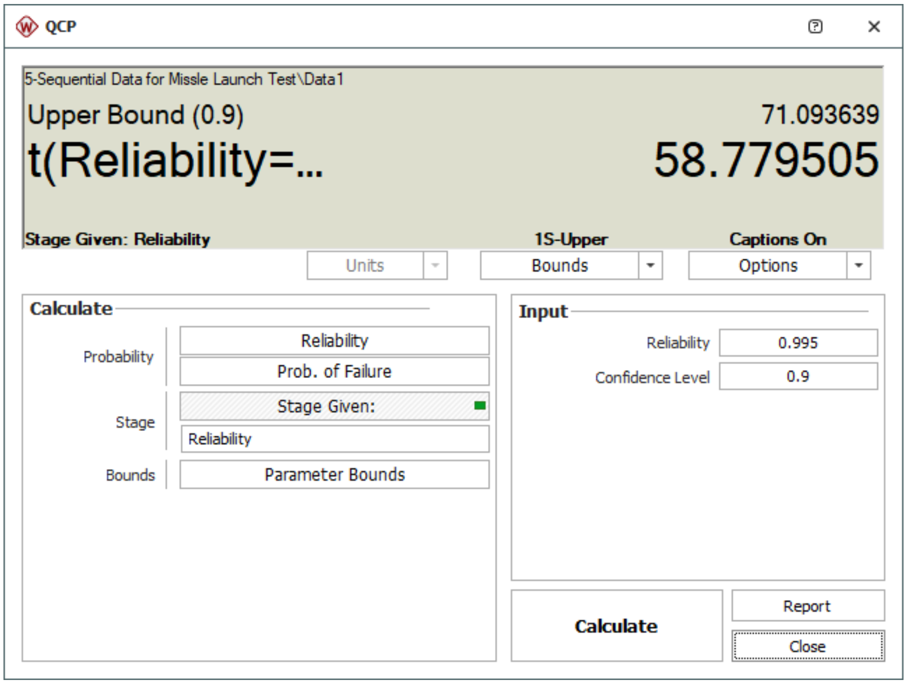
1. The next figure shows the entered data and the estimated parameters.



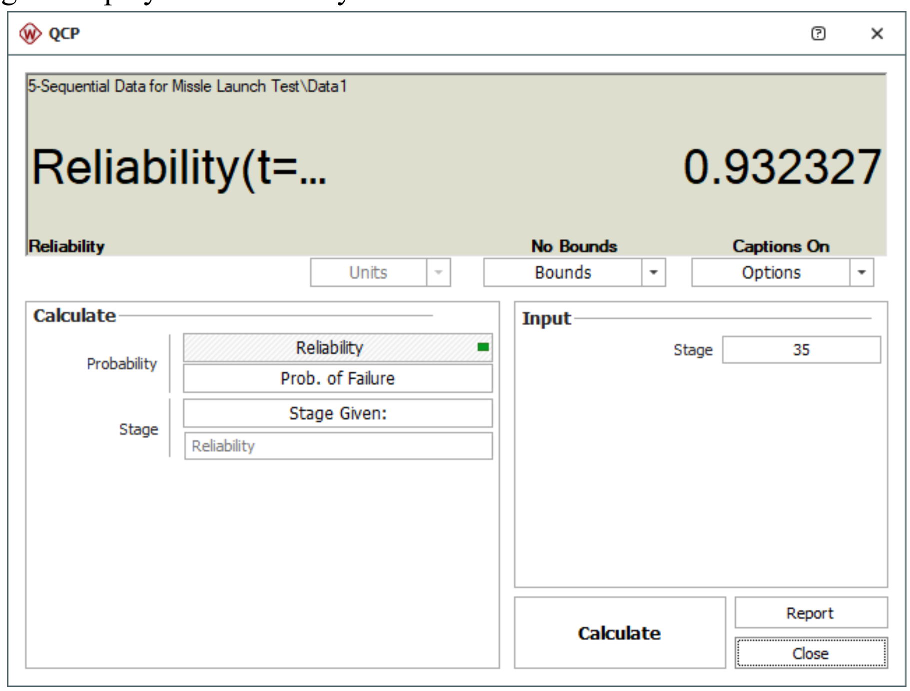
2. The next figure displays the Reliability vs. Time plot.



3. The next figure displays the number of launches before the reliability goal of 99.5% will be achieved with a 90% confidence level.



4. The next figure displays the reliability achieved after the 35th launch.



Sequential Data with Failure Modes

Consider the data given in the previous example. Now suppose that the engineers assigned failure modes to each failure and that the appropriate corrective actions were taken.

The table below presents the data.

1. Find the Logistic reliability growth curve that best represents the data.
2. Plot it comparatively with the raw data.
3. If design changes continue to be incorporated and the testing continues, when will the reliability goal of 99.50% be achieved?
4. If design changes continue to be incorporated and the testing continues, what will be the attainable reliability at the end of the 35th launch?

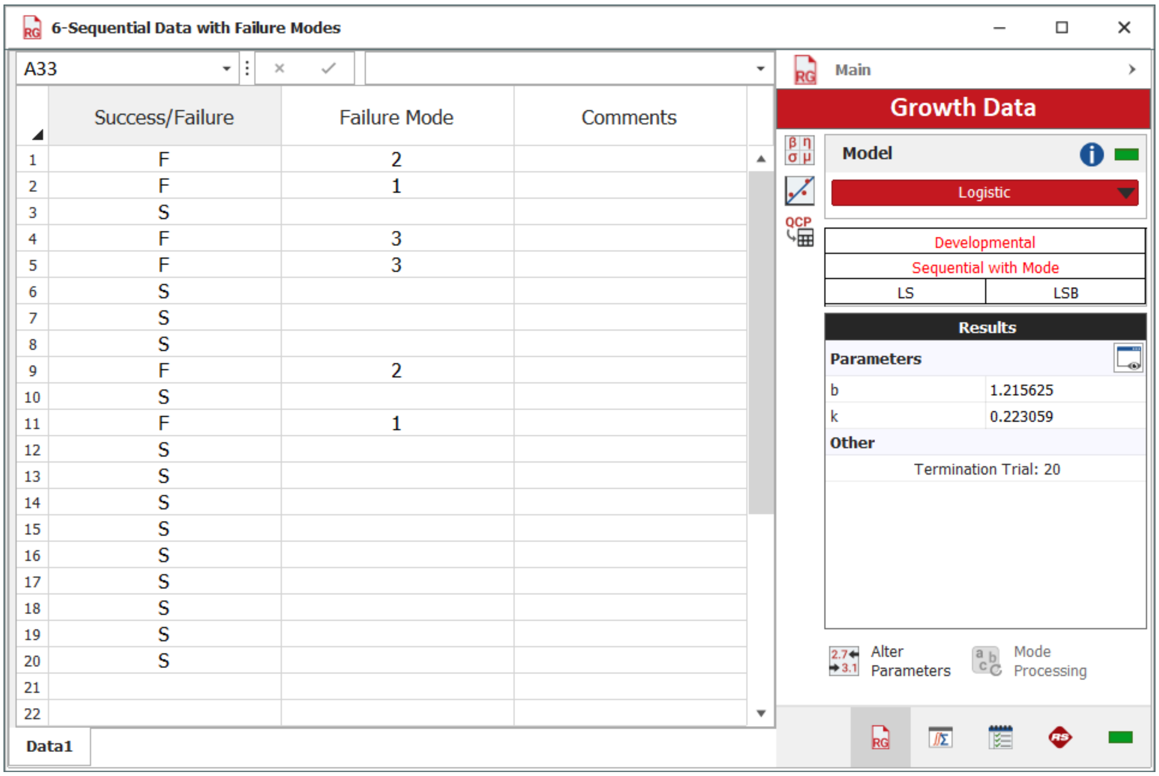
Sequential Success/Failure Data with Modes

Launch Number	Result	Mode
1	F	2
2	F	1
3	S	
4	F	3
5	F	3
6	S	
7	S	
8	S	
9	F	2
10	S	
11	F	1
12	S	
13	S	
14	S	
15	S	
16	S	
17	S	

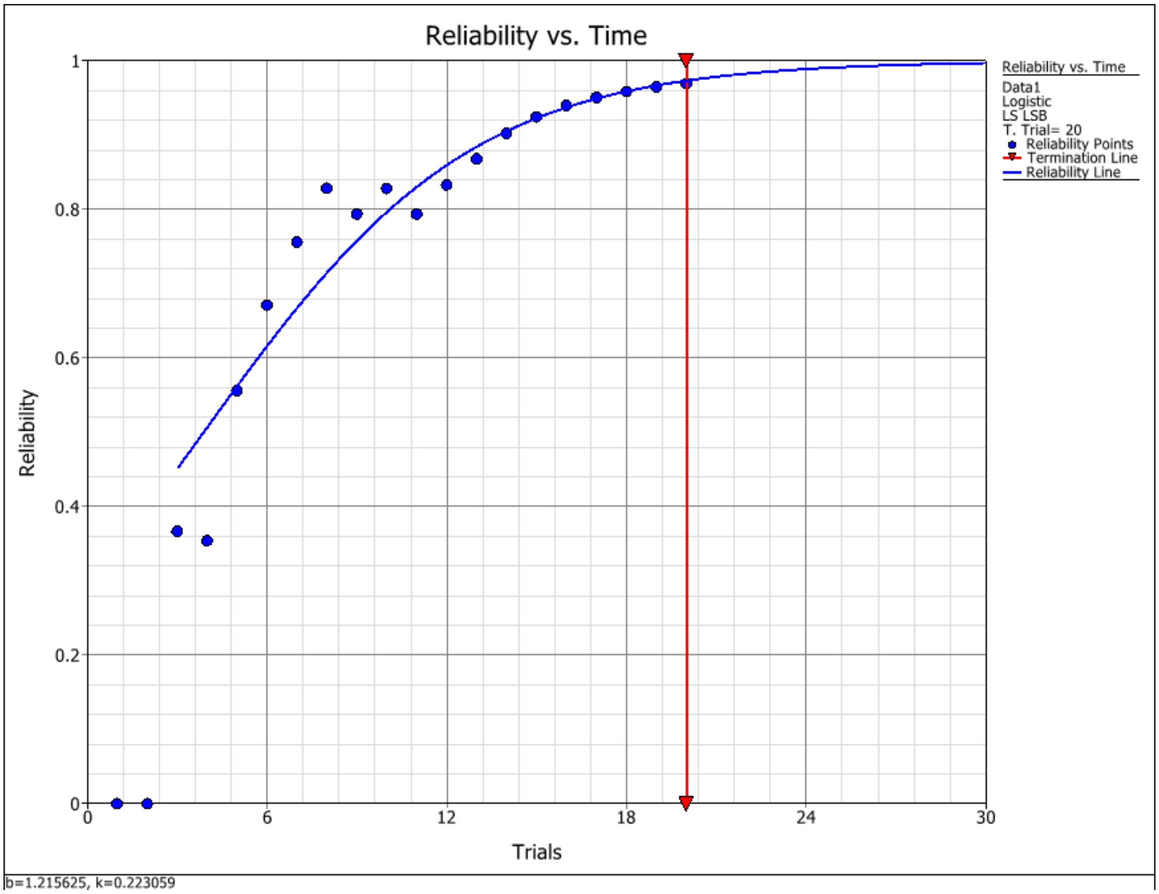
18	S	
19	S	
20	S	

Solution

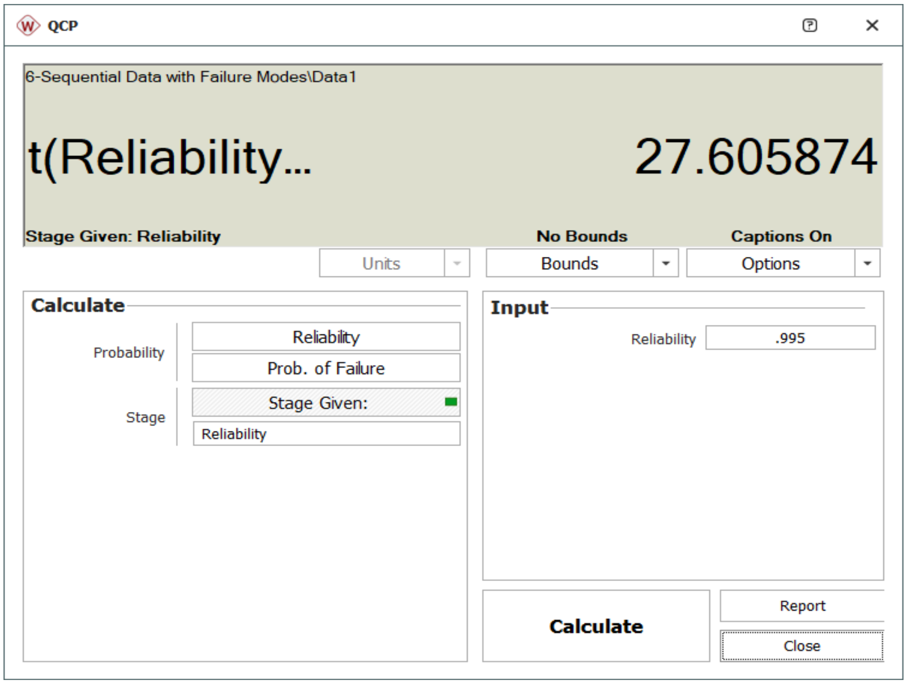
1. The next figure shows the entered data and the estimated parameters.



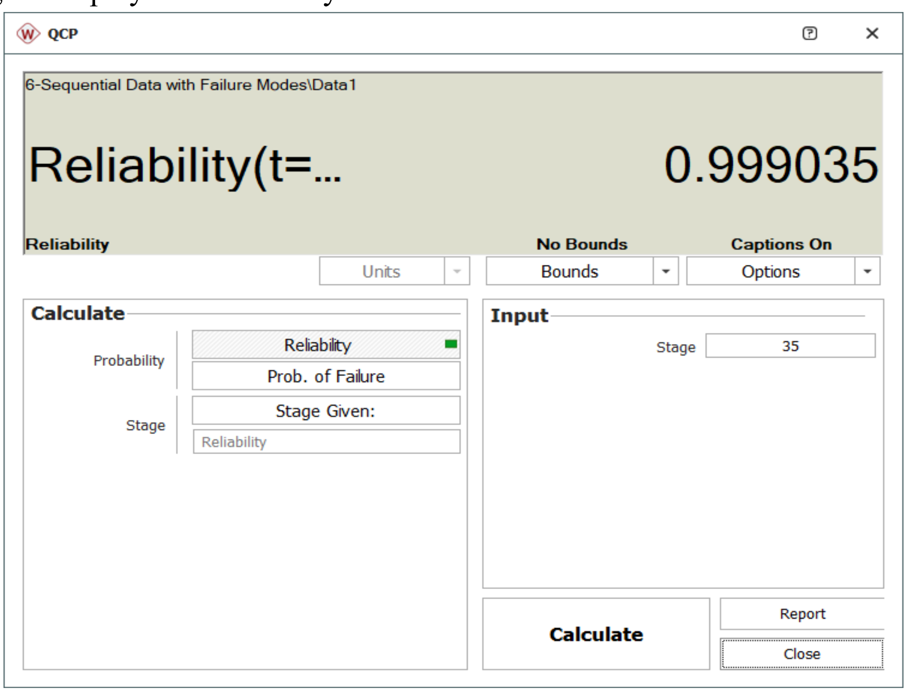
2. The next figure displays the Reliability vs. Time plot.



3. The next figure displays the number of launches before the reliability goal of 99.5% will be achieved.



4. The last figure displays the reliability after the 35th launch.



Reliability Growth Planning

IN THIS CHAPTER

In developmental reliability growth testing, the objective is to test a system, find problem failure modes, incorporate corrective actions and therefore increase the reliability of the system. This process is continued for the duration of the test time. If the corrective actions are effective then the system mean time between failures (MTBF) or mean trials between failures (MTrBF) will move from an initial low value to a higher value. Typically, the objective of reliability growth testing is not to just increase the MTBF/MTrBF, but to increase it to a particular value called the goal or requirement. Therefore, determining how much test time is needed for a particular system is generally of particular interest in reliability growth testing.

The Duane postulate is based on empirical observations, and it reflects a learning curve pattern for reliability growth. This learning curve pattern forms the basis of the Crow-AMSAA (NHPP) model. The Duane postulate is also reflected in the Crow Extended model in the form of the discovery function $h(t)$.

The discovery function is the rate in which new, distinct problems are being discovered during reliability growth development testing. The Crow-AMSAA (NHPP) model is a special case of the discovery function. Consider that when a new and distinct failure mode is first seen, the testing is stopped and a corrective action is incorporated before the testing is resumed. In addition, suppose that the corrective action is highly effective that the failure mode is unlikely to be seen again. In this case, the only failures observed during the reliability growth test are the first occurrences of the failure modes. Therefore, if the Crow-AMSAA (NHPP) model and the Duane postulate are accepted as the pattern for a test-fix-test reliability growth testing program, then the form of the Crow-AMSAA (NHPP) model must be the form for the discovery function, $h(t)$.

To be consistent with the Duane postulate and the Crow-AMSAA (NHPP) model, the discovery function must be of the same form. This form of the discovery function is an important property of the Crow extended model and its application in growth planning. As with the Crow-AMSAA (NHPP) model, this form of the discovery function ties the model directly to real-world data and experiences.

Growth Planning Models

There are two types of reliability growth planning models available in Weibull++:

- Continuous Reliability Growth Planning
- Discrete Reliability Growth Planning

Growth Planning Inputs

The following parameters are used in both the continuous and discrete reliability growth models.

Management Strategy Ratio & Initial Failure Intensity

When a system is tested and failure modes are observed, management can make one of two possible decisions, either to fix or to not fix the failure mode. Therefore, the management strategy places failure modes into two categories: A modes and B modes. The A modes are all failure modes such that, when seen during the test, no corrective action will be taken. This accounts for all modes for which management determines to be not economical or otherwise justified to take a corrective action. The B modes are either corrected during the test or the corrective action is delayed to a later time. The management strategy is defined by what portion of the failures will be fixed.

Let λ_I be the initial failure intensity of the system in test. λ_A is defined as the A mode's initial failure intensity and λ_B is defined as the B mode's initial failure intensity. λ_A is the failure intensity of the system that will not be addressed by corrective actions even if a failure mode is seen during testing. λ_B is the failure intensity of the system that will be addressed by corrective actions if a failure mode is seen during testing.

Then, the initial failure intensity of the system is:

$$\lambda_I = \lambda_A + \lambda_B$$

The initial system MTBF is:

$$M_I = \frac{1}{\lambda_I}$$

Based on the initial failure intensity definitions, the management strategy ratio is defined as:

$$msr = \frac{\lambda_B}{\lambda_A + \lambda_B}$$

The msr is the portion of the initial system failure intensity that will be addressed by corrective actions, if seen during the test.

The failure mode intensities of the type A and type B modes are:

$$\begin{aligned}\lambda_A =& (1 - msr) \cdot \lambda_I \\ \lambda_B =& msr \cdot \lambda_I\end{aligned}$$

Effectiveness Factor

When a delayed corrective action is implemented for a type B failure mode, in other words a BD mode, the failure intensity for that mode is reduced if the corrective action is effective. Once a BD mode is discovered, it is rarely totally eliminated by a corrective action. After a BD mode has been found and fixed, a certain percentage of the failure intensity will be removed, but a certain percentage of the failure intensity will generally remain. The fraction decrease in the BD mode failure intensity due to corrective actions, d, $(0 < d < 1)$, is called the *effect-iveness factor* (EF).

A study on EFs showed that an average EF, d, is about 70%. Therefore, about 30% (i.e., $100(1-d)\%$) of the BD mode failure intensity will typically remain in the system after all of the corrective actions have been implemented. However, individual EFs for the failure modes may be larger or smaller than the average. This average value of 70% can be used for planning purposes, or if such information is recorded, an average effectiveness factor from a previous reliability growth program can be used.

MTBF Goal

When putting together a reliability growth plan, a goal MTBF/MTrBF M_G (or goal failure intensity λ_G) is defined as the requirement or target for the product at the end of the growth program.

Growth Potential

The failure intensity remaining in the system at the end of the test will depend on the management strategy given by the classification of the type A and type B failure modes. The engineering effort applied to the corrective actions determines the effectiveness factors. In addition, the failure intensity depends on $h(t)$, which is the rate at which problem failure modes are being discovered during testing. The rate of discovery drives the opportunity to take corrective actions based on the seen failure modes, and it is an important factor in the overall reliability growth rate. The reliability growth potential is the limiting value of the failure intensity as time T increases. This limit is the maximum MTBF that can be attained with the current management strategy. The maximum MTBF/MTrBF will be attained when all type B modes have been observed and fixed.

If all the discovered type B modes are corrected by time T, that is, no deferred corrective actions at time T, then the growth potential is the maximum attainable with the type B designation of the failure modes and the corresponding assigned effectiveness factors. This is called the *nominal growth potential*. In other words, the nominal growth potential is the maximum attainable growth potential assuming corrective actions are implemented for every mode that is planned to be fixed. In reality, some corrective actions might be implemented at a later time due to schedule, budget, engineering, etc.

If some of the discovered type B modes are not corrected at the end of the current test phase, then the prevailing growth potential is below the maximum attainable with the type B designation of the failure modes and the corresponding assigned effectiveness factors.

If all type B failure modes are discovered and corrected with an average effectiveness factor, d, then the maximum reduction in the initial system failure intensity is the growth potential failure intensity:

$$\lambda_{GP} = \lambda_A + (1 - d)\,\lambda_B$$

The growth potential MTBF/MTrBF is:

$$M_{GP}=\frac{1}{\lambda_{GP}}$$

Note that based on the equations for the initial failure intensity and the management strategy ratio (given in the Management Strategy and Initial Failure Intensity section), the initial failure intensity is equal to:

$$\lambda_I=\frac{\lambda_{GP}}{1-d\cdot msr}$$

Growth Potential Design Margin

The Growth Potential Design Margin ($GPDM$) can be considered as a safety margin when setting target MTBF/MTrBF values for the reliability growth plan. It is common for systems to degrade in terms of reliability when a prototype product is going into full manufacturing. This is due to variations in materials, processes, etc. Furthermore, the in-house reliability growth testing usually overestimates the actual product reliability because the field usage conditions may not be perfectly simulated during testing. Typical values for the $GPDM$ are around 1.2. Higher values yield less risk for the program, but require a more rigorous reliability growth test plan. Lower values imply higher program risk, with less safety margin.

During the planning stage, the growth potential MTBF/MTrBF, M_{GP}, can be calculated based on the goal MTBF, M_G, and the growth potential design margin, $GPDM$.

$$M_{GP}=GPDM\cdot M_G$$

or in terms of failure intensity:

$$\lambda_{GP}=\frac{\lambda_G}{GPDM}$$

Continuous Reliability Growth Planning

The use of the Duane postulate as a reliability growth planning model poses two significant drawbacks: The first drawback is that the Duane postulate's MTBF is zero at time equal to zero. This was addressed in MIL-HDBK-189 by specifying a time T_i where growth starts after T_i and the Duane postulate applies [13]. However, determining T_i is subjective and is not a desirable property of the MIL-HDBK-189. The second drawback is that the MTBF for the Duane postulate increases indefinitely to infinity, which is not realistic.

Therefore, the desirable features of a planning model are:

1. The discovery function must have the form of the Crow-AMSAA (NHPP) model and the Duane postulate.
2. The start time T_i is not required as an input.
3. An upper bound on the system MTBF is specified in the model.

All of these desirable features are included in the planning model discussed in this chapter, which is based on the Crow extended model.

The Crow extended model for reliability growth planning is a revised and improved version of the MIL-HDBK-189 growth curve [13]. MIL-HDBK-189 can be considered as the growth curve based on the Crow-AMSAA (NHPP) model. Using MIL-HDBK-189 for reliability growth planning assumes that the corrective actions for the observed failure modes are incorporated during the test and at the specific time of failure. However, in actual practice, fixes may be delayed until after the completion of the test or some fixes may be implemented during the test while others are delayed and some are not fixed at all. The Crow extended model for reliability growth planning provides additional inputs that accounts for specific management strategies and delayed fixes with specified effectiveness factors.

Nominal Idealized Growth Curve

During developmental testing, management should expect that certain levels of reliability will be attained at various points in the program in order to have assurance that reliability growth is progressing at a sufficient rate to meet the product reliability requirement. The idealized curve portrays an overall characteristic pattern, which is used to determine and evaluate intermediate levels of reliability and construct the program planned growth curve. Note that growth profiles on previously developed, similar systems provide significant insight into the reliability growth process and are valuable in the construction of idealized growth curves.

The nominal idealized growth curve portrays a general profile for reliability growth throughout system testing. The idealized curve has the baseline value λ_I until an initialization time, t_0, when reliability growth occurs. From that time and until the end of testing, which can be a single or, most commonly, multiple test phases, the idealized curve increases steadily according to a learning curve pattern until it reaches the final reliability requirement, M_F . The slope of this curve on a log-log plot is the growth rate of the Crow extended model [13].

Nominal Failure Intensity Function

The nominal idealized growth curve failure intensity as a function of test time t is:

$$r_{NI}(t) = \lambda_A + (1-d)\lambda_B + d\lambda\beta t^{(\beta-1)} \text{ for } t \geq t_0$$

and:

$$r_{NI}(t) = \lambda_I \text{ for } t \leq t_0$$

where λ_I is the initial system failure intensity, t is test time and t_0 is the initialization time, which is discussed in the next section.

It can be seen that the first equation for $r_{NI}(t)$ is the failure intensity equation of the Crow extended model.

Initialization Time

Reliability growth can only begin after a type B failure mode occurs, which cannot be at a time equal to zero. Therefore, there is a need to define an initialization time that is different from zero. The nominal idealized growth curve failure intensity is initially set to be equal to the initial failure intensity, λ_I, until the initialization time, t_0 :

$$r_{NI}(t_0) = \lambda_A + (1-d)\lambda_B + d\lambda\beta t_0^{(\beta-1)}$$

Therefore:

$$\lambda_I = \lambda_A + (1-d)\lambda_B + d\lambda\beta t_0^{(\beta-1)}$$

Then:

$$t_0 = \left[\frac{\lambda_I - \lambda_A - (1-d)\lambda_B}{d\lambda\beta}\right]^{\frac{1}{\beta-1}}$$

Using the equation for initial failure intensity:

$$\lambda_I = \lambda_A + \lambda_B$$

we substitute λ_I to get:

$$t_0 = \left[\frac{\lambda_A + \lambda_B - \lambda_A - (1-d)\lambda_B}{d \cdot \lambda \cdot \beta}\right]^{\frac{1}{\beta-1}}$$

Then:

$$t_0 = \left(\frac{\lambda_B}{\lambda \cdot \beta}\right)^{\frac{1}{\beta-1}}$$

The initialization time, t_0, allows for growth to start after a type B failure mode has occurred.

Nominal Time to Reach Goal

Assuming that we have a target MTBF or failure intensity goal, we can solve the equation for the nominal failure intensity to find out how much test time, $t_{N,G}$, is required (based on the Crow extended model and the nominal idealized growth curve) to reach that goal:

$$t_{N,G} = \left[\frac{r_G - \lambda_A - (1-d)\lambda_B}{d \cdot \lambda \cdot \beta}\right]^{\frac{1}{\beta-1}}$$

Note that when $\lambda_I < r_G$ or, in other words, the initial failure intensity is lower than the goal failure intensity, then there is no need to solve for the nominal time to reach the goal because the goal is already met. In this case, no further reliability growth testing is needed.

Growth Rate for Nominal Idealized Curve

The growth rate for the nominal idealized curve is defined in the same context as the growth rate for the Duane postulate [8]. The nominal idealized curve has the same functional form for the growth rate as the Duane postulate and the Crow-AMSAA (NHPP) model.

For both the Duane postulate and the Crow-AMSAA (NHPP) model, the average failure intensity is given by:

$$C(t) = \frac{\lambda t^{\beta}}{t} = \lambda t^{(\beta-1)}$$

Also, for both the Duane postulate and the Crow-AMSAA (NHPP) model, the instantaneous failure intensity is given by:

$$r(t) = \lambda \beta t^{(\beta-1)}$$

Taking the difference, $D(t)$, between the average failure intensity, $C(t)$ and the instantaneous failure intensity, $r(t)$, yields:

$$D(t) = \lambda t^{(\beta-1)} - \lambda \beta t^{(\beta-1)}$$

Then:

$$D(t) = \lambda t^{(\beta-1)} [1-\beta]$$

For reliability growth to occur, $D(t)$ must be decreasing.

The growth rate for both the Duane postulate and the Crow-AMSAA (NHPP) model is the negative of the slope of $\log(D(t))$ as a function of $\log(t)$:

$$\log_e(D(t)) = \text{constant} - (1-\beta)\log_e(t)$$

The slope is negative under reliability growth and equals:

$$\text{slope} = -(1-\beta)$$

The growth rate for both the Duane postulate and the Crow-AMSAA (NHPP) model is equal to the negative of this slope:

$$\text{Growth Rate} = (1-\beta)$$

The instantaneous failure intensity for the nominal idealized curve is:

$$r_{NI}(t) = \lambda_A + (1-d)\lambda_B + d\lambda\beta(t)^{(\beta-1)}$$

The cumulative failure intensity for the nominal idealized curve is:

$$C_{NI}(t) = \lambda_A + (1-d)\lambda_B + d\lambda(t)^{(\beta-1)}$$

therefore:

$$D_{NI}(t) = [C_{NI}(t) - r_{NI}(t)] = \lambda t^{(\beta-1)}[1-\beta]$$

and:

$$\log_e(D_{NI}(t)) = \text{constant} - (1-\beta)\log_e(t)$$

Therefore, in accordance with the Duane postulate and the Crow-AMSAA (NHPP) model, $a = 1-\beta$ is the growth rate for the reliability growth plan.

Lambda - Beta Parameter Relationship

Under the Crow-AMSAA (NHPP) model, the time to first failure is a Weibull random variable. The MTTF of a Weibull distributed random variable with parameters β and η is:

$$MTTF = \eta \cdot \Gamma\left(1 + \frac{1}{\beta}\right)$$

The parameter lambda is defined as:

$$\lambda = \frac{1}{\eta^{\beta}}$$

Using the equation for lambda in the MTTF relationship, we have:

$$MTBF_B = \frac{\Gamma\left(1+\frac{1}{\beta}\right)}{\lambda^{\left(\frac{1}{\beta}\right)}}$$

or, in terms of failure intensity:

$$\lambda_B = \frac{\lambda^{\left(\frac{1}{\beta}\right)}}{\Gamma\left(1+\frac{1}{\beta}\right)}$$

Actual Idealized Growth Curve

The actual idealized growth curve differs from the nominal idealized curve in that it takes into account the average fix delay that might occur in each test phase. The actual idealized growth curve is continuous and goes through each of the test phase target MTBFs.

Fix Delays and Test Phase Target MTBF

Fix delays reflect how long it takes from the time a problem failure mode is discovered in testing, to the time the corrective action is incorporated into the system and reliability growth is realized. The consideration of the fix delay is often in terms of how much calendar time it takes to incorporate a corrective action fix after the problem is first seen. However, the impact of the delay on reliability growth is reflected in the average test time it takes between finding a problem failure mode and incorporating a corrective action. The fix delay is reflected in the actual idealized growth curve in terms of test time.

In other words, the average fix delay is calendar time converted to test hours. For example, say that we expect an average fix delay of two weeks. If in two weeks the total test time is 1,000 hours, then the average fix delay is 1,000 hours. If in the same two weeks the total test time is 2,000 hours (maybe there are more units available or more shifts), then the average fix delay is 2,000 hours.

There can be a constant fix delay across all test phases or, as a practical matter, each test phase can have a different fix delay time. In practice, the fix delay will generally be constant over a particular test phase. L_i denotes the fix delay for phase $i = 1, \ldots, P$, where P is the total number of phases in the test. The Weibull++ software allows for a maximum of ten test phases.

Actual Failure Intensity Function

Consider a test plan consisting of i phases. Taking into account the fix delay within each phase, we expect the actual failure intensity to be different (i.e., shifted) from the nominal failure intensity. This is because fixes are not incorporated instantaneously; thus, growth is realized at a later time compared to the nominal case.

Specifically, the actual failure intensity will be estimated as follows:

Test Phase 1

For the first phase of a test plan, the actual idealized curve failure intensity, $r_{AI}(t)$, is :

$$r_{AI}(t) = \lambda_A + (1-d)\lambda_B + d\lambda\beta\left[\left(\frac{T_1 - L_1}{T_1}\right)t\right]^{(\beta-1)} \text{ for } 0 < t \leq T_1$$

Note that the end time of Phase 1, T_1, must be greater than $L_1 + t_0$. That is, $T_1 > L_1 + t_0$.

The actual idealized curve initialization time for Phase 1, T_0^{AIC}, is calculated from:

$$r_{AI}(T_0^{AIC}) = \lambda_A + (1-d)\lambda_B + d\lambda\beta\left[\left(\frac{T_1 - L_1}{T_1}\right)T_0^{AIC}\right]^{(\beta-1)}$$

where $r_{AI}(T_0^{AIC}) = r_I$.

Therefore, using the equation for the initialization time, we have:

$$\lambda_A + (1-d)\lambda_B + d\lambda\beta\left[\left(\frac{T_1 - L_1}{T_1}\right)T_0^{AIC}\right]^{(\beta-1)} = \lambda_A + (1-d)\lambda_B + d\lambda\beta t_0^{(\beta-1)}$$

By obtaining the initial failure intensity for T_0^{AIC}, we get:

$$T_0^{AIC} = \frac{t_0}{\left(\frac{T_1 - L_1}{T_1}\right)}$$

Test Phase i

For any test phase i, the actual idealized curve failure intensity is given by:

$$r_{AI}(t) = \lambda_A + (1-d)\lambda_B + d\lambda\beta\left[T_{i-1} - L_{i-1} + \left(\frac{T_i - L_i - T_{i-1} + L_{i-1}}{T_i - T_{i-1}}\right)(t - T_{i-1})\right]^{(\beta-1)}$$

where $T_{i-1} \leq t \leq T_i$ and T_i is the test time of each corresponding test phase.

The actual idealized curve MTBF is:

$$M_{AI} = \frac{1}{r_{AI}(t)}$$

Actual Time to Reach Goal

The actual time to reach the target MTBF or failure intensity goal, $t_{AC,G}$, can be found by solving for the actual idealized curve failure intensity:

$$r_{AI}(t_{AC,G}) = \lambda_A + (1-d)\lambda_B + d\lambda\beta\left[T_{i-1} - L_{i-1} + \left(\frac{T_i - L_i - T_{i-1} + L_{i-1}}{T_i - T_{i-1}}\right)(t_{AC,G} - T_{i-1})\right]^{(\beta-1)}$$

Since the actual idealized growth curve depends on the phase durations and average fix delays, there are three different cases that need to be treated differently in order to determine the actual time to reach the MTBF goal. The cases depend on when the actual MTBF that can be reached within the specific phase durations and fix delays becomes equal to the MTBF goal. This can be determined by solving for the actual idealized curve failure intensity for phases 1 through i, and then solving in terms of actual idealized curve MTBF for each phase and finding the phase during which the actual MTBF becomes equal to the goal MTBF. The three cases are presented next.

Case 1: MTBF goal is met during the last phase

If T_F indicates the cumulative end phase time for the last phase, and L_F indicates the fix delay for the last phase, then we have:

$$\begin{aligned} r_G =& \lambda_A + (1-d)\lambda_B \\ &+ d\lambda\beta\left[T_{F-1} - L_{F-1} + \left(\frac{T_F - L_F - T_{F-1} + L_{F-1}}{T_F - T_{F-1}}\right)(t_{AC,G} - T_{F-1})\right]^{(\beta-1)} \end{aligned}$$

Starting to solve for $t_{AC,G}$ yields:

$$\left[\frac{r_G - \lambda_A - (1-d)\lambda_B}{d\lambda\beta}\right]^{\frac{1}{\beta-1}} = T_{F-1} - L_{F-1} + \left(\frac{T_F - L_i - T_{F-1} + L_{F-1}}{T_F - T_{F-1}}\right)(t_{AC,G} - T_{F-1})$$

We can substitute the left term by solving for the nominal time to reach the goal; thus, we have:

$$t_{N,G} = T_{F-1} - L_{F-1} + \left(\frac{T_F - L_F - T_{F-1} + L_{F-1}}{T_F - T_{F-1}}\right)(t_{AC,G} - T_{i-1})$$

therefore:

$$t_{AC,G} = \frac{t_{N,G} - T_{F-1} + L_{F-1}}{\left(\frac{T_F - L_F - T_{F-1} + L_{F-1}}{T_F - T_{F-1}}\right)} + T_{F-1}$$

Case 2: MTBF goal is met before the last phase

The equation for $t_{AC,G}$ that was derived for case 1 still applies, but in this case T_F and L_F are the time and fix delay of the phase during which the goal is met.

Case 3: MTBF goal is met after the final phase

If the goal MTBF, M_G, is met after the final test phase, then the actual time to reach the goal is not calculated since additional phases have to be added with specific duration and fix delays. The reliability growth program needs to be re-evaluated with the following options:

- Add more phase(s) to the program.
- Re-examine the phase duration of the existing phases.
- Investigate whether there are potential process improvements in the program that can reduce the average fix delay for the phases.

Other alternative routes for consideration would be to investigate the rest of the inputs in the model:

- Change the management strategy.
- Consider if further program risk can be acceptable, and if so, reduce the growth potential design margin.
- Consider if it is feasible to increase the effectiveness factors of the delayed fixes by using more robust engineering redesign methods.

Note that each change of input variables into the model can significantly influence the results.

With that in mind, any alteration in the input parameters should be justified by actionable decisions that will influence the reliability growth program. For example, increasing the average effectiveness factor value should be done only when there is proof that the program will pursue a different, more effective path in terms of addressing fixes.

Examples

Growth Plan for 3 Phases

A complex system is under design. The reliability team wants to create an overall reliability growth plan based on the Crow Extended model. The inputs to the model are the following:

- The requirement or goal MTBF is $M_G = 56$ hours.
- The growth potential design margin factor is $GPDM = 1.35$.

- The average effectiveness factor is $d = 0.7$.
- The management strategy is $msr = 0.95$.
- The beta parameter for the discovery function, $h(t)$, of the type B failure modes is $\beta = 0.7$.
- The test will be conducted in three phases. The cumulative phase end times are $T_1 = 1500, T_2 = 3500$ and $T_3 = 4500$ hours. The average fix delays for each phase are $L_1 = 500$, $L_2 = 700$ and $L_3 = 1000$ test hours, respectively.

Determine the following:

1. The growth potential MTBF and failure intensity.
2. The initial failure intensity.
3. The type A and type B initial failure intensity.
4. The parameter λ of the Crow extended model.
5. The type B failure mode discovery function.
6. The initialization time, t_0, for the nominal failure intensity function.
7. The nominal idealized failure intensity function.
8. The nominal time to reach the MTBF goal.
9. The MTBF that can be reached at the end of the last test phase based on the nominal idealized growth curve.
10. The actual initialization time for phase 1, T_0^{AIC}.
11. The MTBF that can be reached at the end of the last test phase based on the actual idealized growth curve.
12. The actual time to reach the MTBF goal.
13. The actual time to reach the MTBF goal if the cumulative end phase time for the third (last) phase is $T_3 = 6000$ hours.
14. Use the Weibull++ software to generate the results for this example. Consider the cumulative last phase as $T_3 = 6000$ hours, as given in question 13.

Solution

1. The growth potential MTBF is:

$$\begin{aligned} M_{GP} &= GPDM \cdot M_G \\ &= 56 \cdot 1.35 \\ &= 75.6 \end{aligned}$$

   The growth potential failure intensity is:

$$\begin{aligned} \lambda_{GP} &= \frac{1}{M_{GP}} \\ &= \frac{1}{75.6} \\ &= 0.0132 \end{aligned}$$

2. The initial failure intensity is given by:

$$\begin{aligned} \lambda_I &= \frac{\lambda_{GP}}{1 - d \cdot msr} \\ &= \frac{0.0132}{(1 - 0.7 \cdot 0.95)} \\ &= 0.0394 \end{aligned}$$

3. The type A failure mode intensity, λ_A, is:

$$\begin{aligned} \lambda_A &= (1 - msr)\lambda_I \\ &= (1 - 0.95)0.375 \\ &= 0.0019 \end{aligned}$$

   The type B failure mode intensity, λ_B, is:

$$\begin{aligned} \lambda_B &= msr \cdot \lambda_I \\ &= 0.95 \cdot 0.0394 \\ &= 0.0375 \end{aligned}$$

4. Using the inputs of λ_B and β in the failure intensity equation for the Crow-AMSAA (NHPP) model, we have:

$$0.0375 = \frac{\lambda^{\left(\frac{1}{0.7}\right)}}{\Gamma\left(1 + \frac{1}{0.7}\right)}$$

   or:

$$0.0375 = \frac{\lambda^{1.428}}{1.2658}$$

   or:

$$\begin{aligned} \lambda &= (0.0375 \cdot 1.2658)^{\left(\frac{1}{1.428}\right)} \\ &= 0.1184 \end{aligned}$$

5. The type B failure mode discovery function is:

$$h(t) = \lambda\beta t^{\beta-1}$$

Since we know the λ and β parameters, we have:

$$\begin{aligned} h(t) &= 0.1183 \cdot 0.7 t^{0.7-1} \\ &= 0.0828 t^{-0.3} \end{aligned}$$

6. The initialization time, t_0, is:

$$\begin{aligned} t_0 &= \left[\frac{\lambda_I - \lambda_A - (1-d)\lambda_B}{d\lambda\beta}\right]^{\frac{1}{\beta-1}} \\ &= \left[\frac{0.0394 - 0.001974 - (1-0.7)0.0375}{0.7 \cdot 0.1183 \cdot 0.7}\right]^{\frac{1}{0.7-1}} \end{aligned}$$

or:

$$t_0 = 14.07$$

7. The nominal idealized growth curve is given by the following equation:

$$r_{NI}(t) = \left\{ \begin{array}{cc} \lambda_I & t \le t_0 \\ r_{IT}(t) = \lambda_A + (1-d)\lambda_B + d\lambda\beta t^{(\beta-1)} & t > t_0 \end{array} \right\}$$

or:

$$r_{NI}(t) = \left\{ \begin{array}{cc} 0.0394 & t \le 14.07 \\ = 0.0013 + 0.058 \cdot t^{-0.3} & t > 14.07 \end{array} \right\}$$

8. The nominal time to reach the MTBF goal is:

$$t_{N,goal} = \left[\frac{r_G - \lambda_A - (1-d)\lambda_B}{d\lambda\beta}\right]^{\frac{1}{\beta-1}}$$

For the goal failure intensity we have:

$$\begin{aligned} r_G &= \frac{1}{M_G} \\ &= \frac{1}{56} \\ &= 0.01785 \end{aligned}$$

So the nominal time to reach the MTBF goal is:

$$\begin{aligned} t_{N,G} &= \left[\frac{0.01785 - 0.001974 - (1-0.7)0.0375}{0.7 \cdot 0.1184 \cdot 0.7}\right]^{\frac{1}{\beta-1}} \\ &= 4578 \text{ hours} \end{aligned}$$

9. Using the equation for the nominal failure intensity, we find the failure intensity that can be reached at the end of the last test phase based on the nominal idealized growth curve, $r_{NI,final}$. The total (cumulative) test time is $T=4500$ hours. Therefore we have:

$$\begin{aligned} r_{NI,final} =& \lambda_A + (1-d)\lambda_B + d\lambda\beta T^{(\beta-1)} \\ =& 0.0019 + (1-0.7)\,0.0375 + 0.7\cdot 0.1184\cdot 0.7\cdot (4500)^{0.7-1} \\ =& 0.01788 \end{aligned}$$

So the nominal final MTBF that can be reached at 4500 hours of test time is:

$$\begin{aligned} M_{NI,final} =& \frac{1}{r_{NI,final}} \\ =& \frac{1}{0.01788} \\ =& 55.92 \end{aligned}$$

10. We can find the actual initialization time by using:

$$T_0^{AIC} = \frac{t_0}{\left(\frac{T_1-L_1}{T_1}\right)}$$

For question 6, we had found that $t_0 = 14.07$. So we have:

$$\begin{aligned} T_0^{AIC} =& \frac{14.07}{\left(\frac{1500-500}{1500}\right)} \\ =& 21.10887 \end{aligned}$$

11. The failure intensity that can be reached at the end of the last test phase based on the actual idealized growth curve for $T=4500$ hours is given by:

$$\begin{aligned} r_{AI,final}(T) =& \lambda_A + (1-d)\lambda_B \\ &+ d\lambda\beta\left[T_2 - L_2 + \left(\frac{T_3-L_3-T_2+L_2}{T_3-T_2}\right)(T-T_2)\right]^{(\beta-1)} \end{aligned}$$

Then:

$$\begin{aligned} r_{AI,final}(T) =& 0.0019 + (1-0.7)\,0.0375 + 0.7\cdot 0.1184\cdot 0.7 \\ &\cdot\left[3500-700+\left(\frac{4500-1000-3500+700}{4500-3500}\right)(4500-3500)\right]^{(0.7-1)} \end{aligned}$$

Therefore:

$$r_{AI,final}(T) = 0.0182$$

So the actual final MTBF that can be reached at 4500 hours of test time is:

$$\begin{aligned}M_{AI,final} &= \frac{1}{r_{AI,final}}\\ &= \frac{1}{0.0182}\\ &= 54.80\end{aligned}$$

12. From question 11, it is shown that the actual final MTBF that can be reached at 4,500 hours of test time is less than the MTBF goal for the program. So, in accordance with the third case that is described in Actual Time to Reach Goal section, this is the scenario where the actual time to meet the MTBF goal will not be calculated since the reliability growth program needs to be redesigned.

13. The failure intensity that can be reached at the end of the last test phase based on the actual idealized growth curve for $T = T_3 = 6000$ hours is given by:

$$\begin{aligned}r_{AI,final}(T) =& \lambda_A + (1-d)\lambda_B\\ &+ d\lambda\beta\left[T_2 - L_2 + \left(\frac{T_3 - L_3 - T_2 + L_2}{T_3 - T_2}\right)(T - T_2)\right]^{(\beta-1)}\end{aligned}$$

Then:

$$\begin{aligned}r_{AI,final}(T) =& 0.0019 + (1-0.7)\,0.0375 + 0.7\cdot 0.1184\cdot 0.7\\ &\cdot\left[3500 - 700 + \left(\frac{6000 - 1000 - 3500 + 700}{6000 - 3500}\right)(6000 - 3500)\right]^{(0.7-1)}\end{aligned}$$

Therefore:

$$r_{AI,final}(T) = 0.0177$$

So, the actual final MTBF that can be reached at 6000 hours of test time is:

$$\begin{aligned}M_{AI,final} &= \frac{1}{r_{AI,final}}\\ &= \frac{1}{0.0177}\\ &= 56.38\end{aligned}$$

In this case, the actual MTBF goal becomes higher than the target MTBF sometime during the last phase. Therefore, we can determine the actual time to reach the MTBF goal by using:

$$t_{AC,G} = \frac{t_{N,G} - T_{F-1} + L_{F-1}}{\left(\frac{T_F - L_F - T_{F-1} + L_{F-1}}{T_F - T_{F-1}}\right)} + T_{F-1}$$

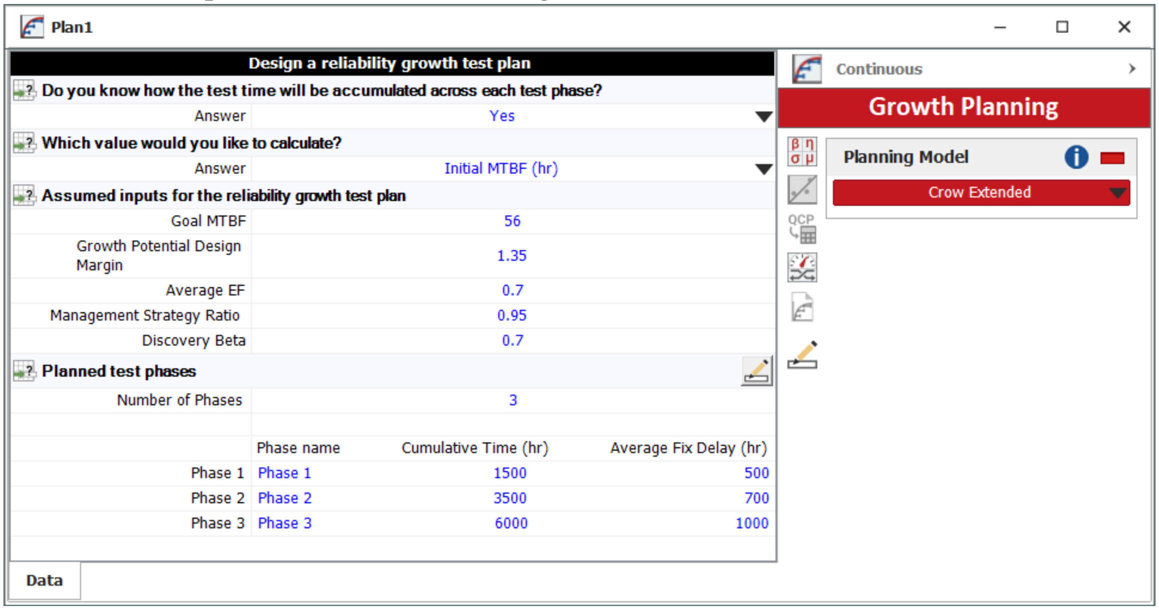
therefore:

$$t_{AC,G} = \frac{4578 - 3500 + 700}{\left(\frac{6000 - 1000 - 3500 + 700}{6000 - 3500}\right)} + 3500$$

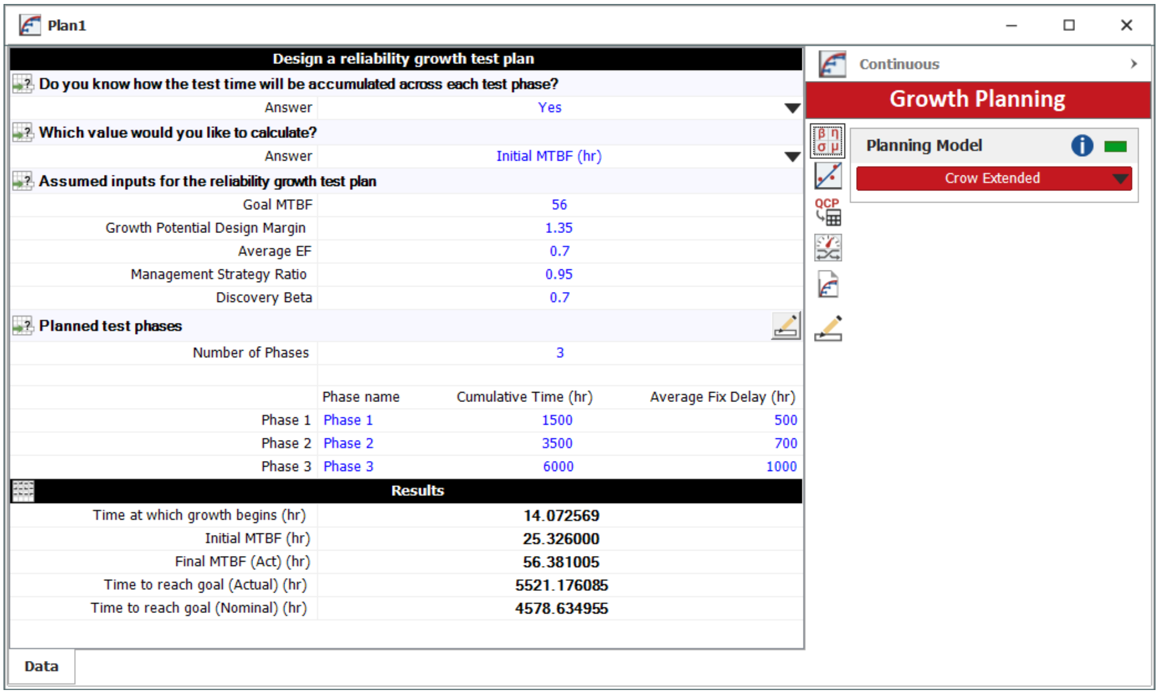
then:

$$t_{AC,G} = 5521$$

14. To use Weibull++ to obtain the results for this example, first we create a growth planning folio by choosing **Home > Insert > Continuous Growth Planning**. In the growth planning folio, specify the number of phases, the cumulative phase time and average fix delays for each of the three phases, as shown in the figure below.

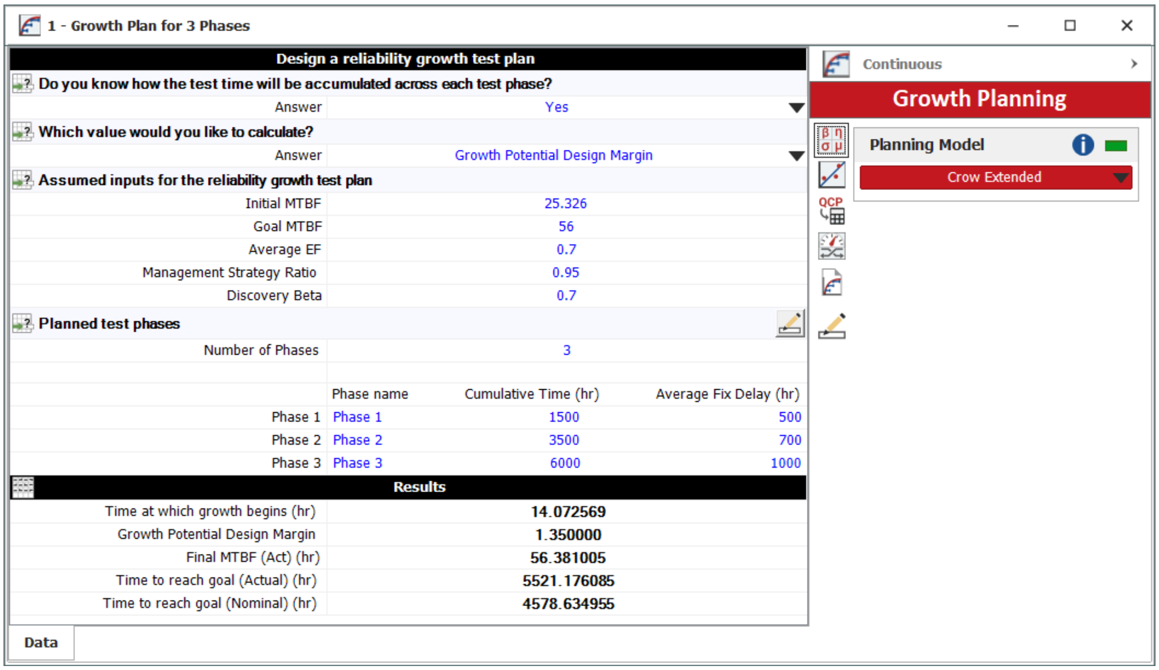


We then enter the remaining input needed by the Crow Extended model, as shown in the following figure.

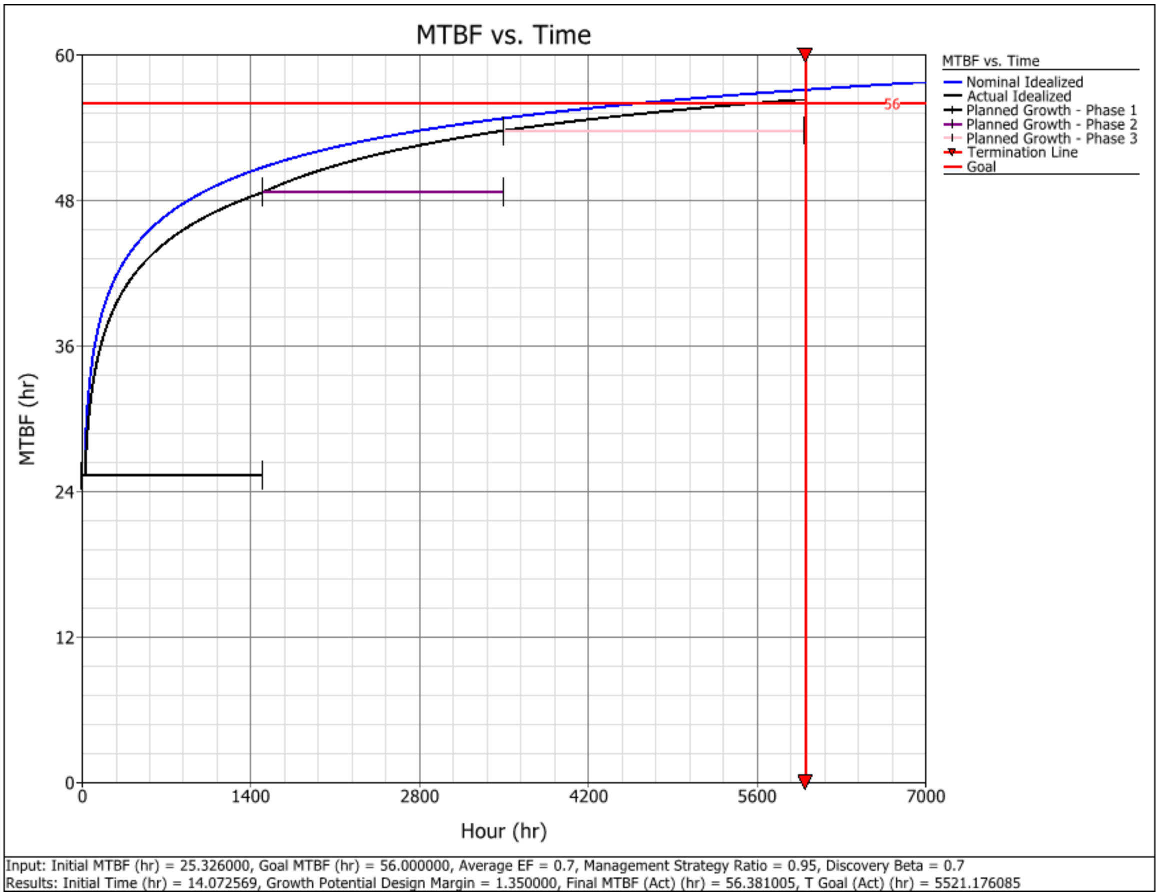


The input provided is the required MTBF goal of 56, the growth potential design margin of 1.35, the management strategy of 0.95 and the discovery beta of 0.7. The results show the initialization time, t_o , the initial MTBF, the final actual MTBF that can be reached for this growth program and the actual time when the MTBF goal is met. Note that in this case we provided the goal MTBF and solved for the initial MTBF.

Different calculation options are available depending on the desired input and output. For example, we could provide the initial and goal MTBF to solve for the growth potential design margin, as shown in the figure below.



Click the **Plot** icon to generate a plot that shows the nominal and actual idealized growth curve, the test termination time, the MTBF goal and the planned growth for each phase, as shown in the figure below.



Growth Plan for 7 Phases

The reliability team of a product manufacturer is putting together a reliability growth plan for one of the new products under design. The team wants to create an overall reliability growth plan based on the Crow Extended model. The inputs to the model are the following:

- The requirement or goal MTBF is $M_G = 100$ hours.
- The growth potential design margin factor is $GPDM = 1.15$.
- The average effectiveness factor is $d = 0.6$, based on historical information for similar products developed by the company.
- The management strategy is $msr = 0.90$.
- The beta parameter for the discovery function, $h(t)$, of the type B failure modes is $\beta = 0.71$, based on a data analysis of the previous product development project that was of similar nature.
- The test will be conducted in seven phases. The cumulative phase end times are $T_1 = 4000, T_2 = 8000$,

$T_3 = 12000, T_4 = 20000, T_5 = 25000, T_6 = 30000$ and $T_7 = 40000$ hours. The average fix delay for each phase is one week.

- For the first two phases, the plan is to test 1,000 hours per week, so the average fix delay in terms of test hours for phases 1 and 2 is $L_1 = 1000$ and $L_2 = 1000$ hours.
- For phases 3 and 4, the prototype test units are planned to be doubled, so the average fix delay in terms of test hours within one week is also going to be doubled, so we have: $L_3 = 2000$ and $L_4 = 2000$ hours.
- For phases 5 and 6, a second shift is going to be added, so the amount of test hours within a week is going to be doubled again. Therefore, the average fix delay in terms of test hours within one week is also going to be doubled: $L_5 = 4000$ and $L_6 = 4000$ hours. ***For the last phase, more units and weekend shifts are planned to be added, so the total test hours within one week are going to be 6,000 hours. In accordance, the average fix delay in terms of test hours will be $L_7 = 6000$ hours.

Determine the following: Construct a reliability growth plan using the Weibull++ software and make sure that the goal MTBF can be met within the total test hours allocated for growth testing. If not, make necessary changes in the phase durations in order to meet the goal.

Solution

The following figure shows the planning inputs in Weibull++'s continuous growth planning folio.

Design a reliability growth test plan			
Do you know how the test time will be accumulated across each test phase?			
Answer	Yes		
Which value would you like to calculate?			
Answer	Initial MTBF (hr)		
Assumed inputs for the reliability growth test plan			
Goal MTBF	100		
Growth Potential Design Margin	1.15		
Average EF	0.6		
Management Strategy Ratio	0.9		
Discovery Beta	0.72		
Planned test phases			
Number of Phases	7		
	Phase name	Cumulative Time (hr)	Average Fix Delay (hr)
Phase 1	Phase 1	4000	1000
Phase 2	Phase 2	8000	1000
Phase 3	Phase 3	12000	2000
Phase 4	Phase 4	20000	2000
Phase 5	Phase 5	25000	4000
Phase 6	Phase 6	30000	4000
Phase 7	Phase 7	40000	6000

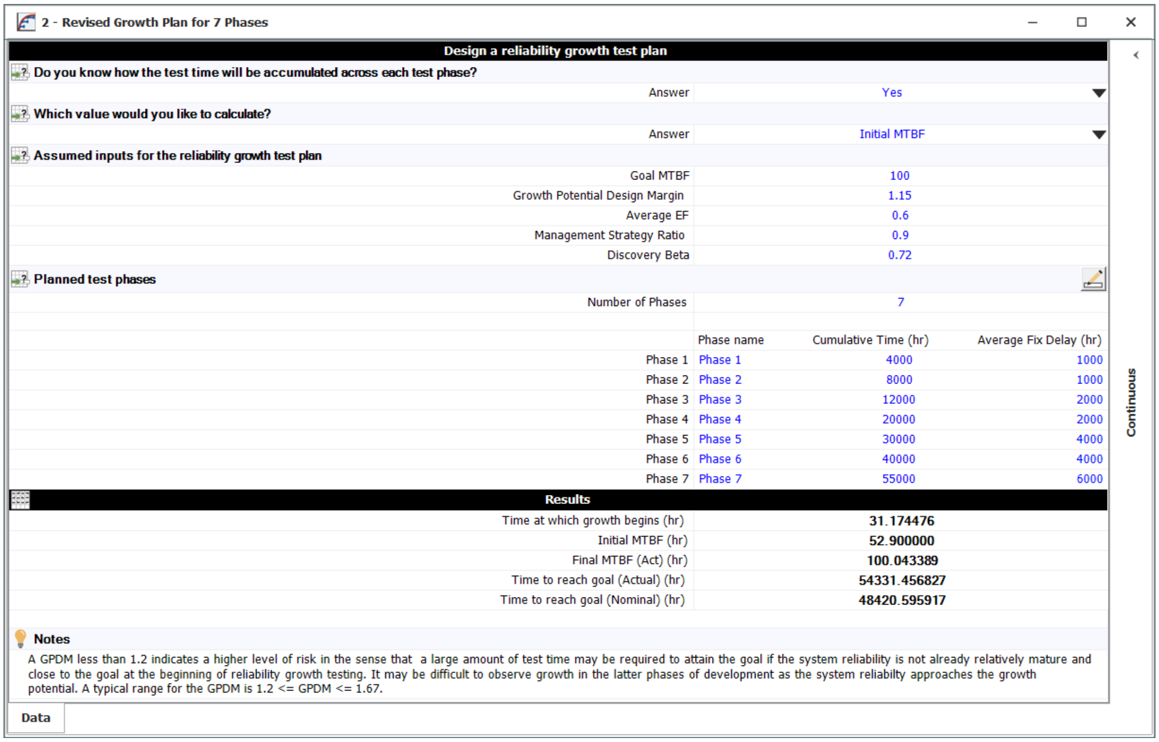
The next figure shows the planning calculations based on the goal MTBF of 100 hours and the rest of the known inputs into the model. As seen by the planning calculation results, the final actual MTBF at the end of the seventh phase is 98.66 hours, which is less than the MTBF goal of 100 hours.

Design a reliability growth test plan			
Do you know how the test time will be accumulated across each test phase?			
Answer	Yes		
Which value would you like to calculate?			
Answer	Initial MTBF		
Assumed inputs for the reliability growth test plan			
Goal MTBF	100		
Growth Potential Design Margin	1.15		
Average EF	0.6		
Management Strategy Ratio	0.9		
Discovery Beta	0.72		
Planned test phases			
Number of Phases	7		
	Phase name	Cumulative Time (hr)	Average Fix Delay (hr)
Phase 1	Phase 1	4000	1000
Phase 2	Phase 2	8000	1000
Phase 3	Phase 3	12000	2000
Phase 4	Phase 4	20000	2000
Phase 5	Phase 5	30000	4000
Phase 6	Phase 6	40000	4000
Phase 7	Phase 7	55000	6000
Results			
Time at which growth begins (hr)	31.174476		
Initial MTBF (hr)	52.900000		
Final MTBF (Act) (hr)	100.043389		
Time to reach goal (Actual) (hr)	54331.456827		
Time to reach goal (Nominal) (hr)	48420.595917		
Notes			
A GPDM less than 1.2 indicates a higher level of risk in the sense that a large amount of test time may be required to attain the goal if the system reliability is not already relatively mature and close to the goal at the beginning of reliability growth testing. It may be difficult to observe growth in the latter phases of development as the system reliabilty approaches the growth potential. A typical range for the GPDM is 1.2 <= GPDM <= 1.67.			

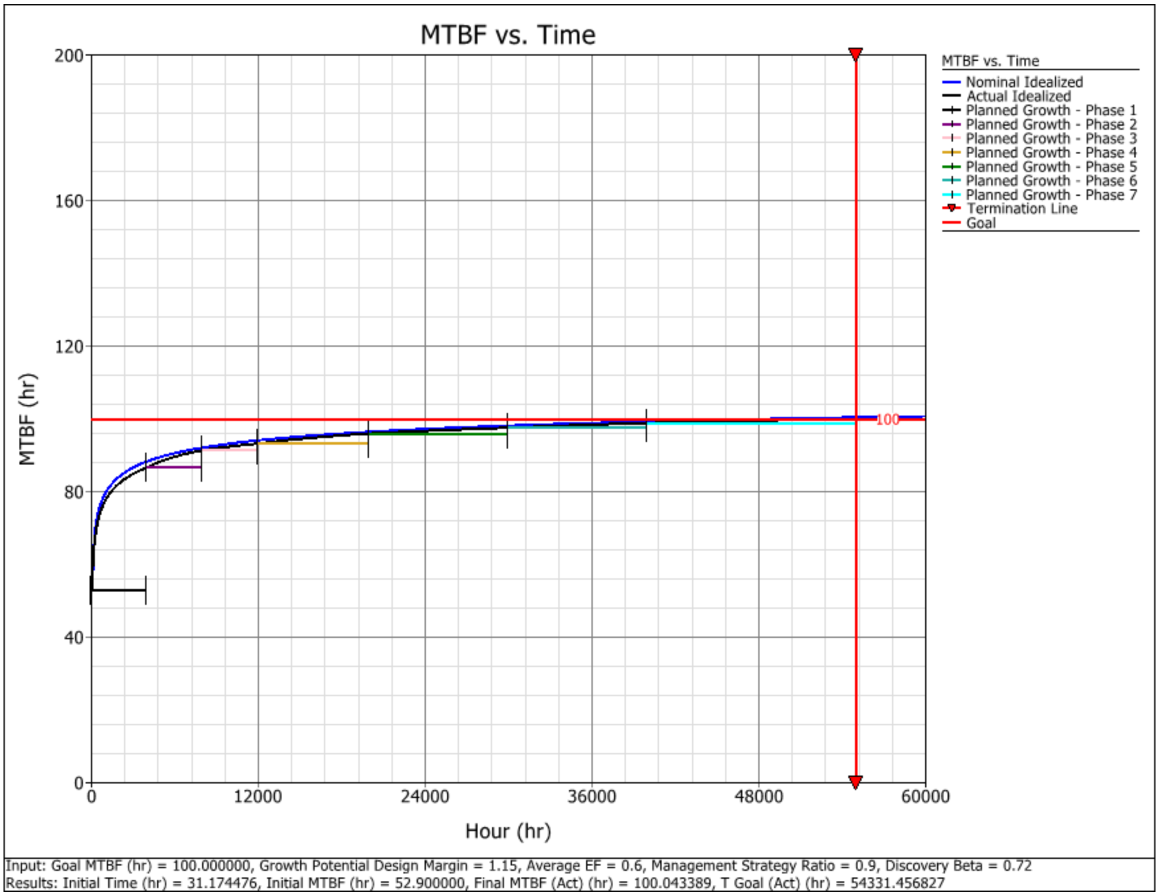
Also, the next figure shows that the nominal time to meet the MTBF goal is 48,420 hours, which is higher than the total test time that is currently planned to be allocated for reliability growth.

Results	
Time at which growth begins (hr)	31.174476
Initial MTBF (hr)	52.900000
Final MTBF (Act) (hr)	100.043389
Time to reach goal (Actual) (hr)	54331.456827
Time to reach goal (Nominal) (hr)	48420.595917

A new plan needs to be considered. The reliability team decides to increase the test time for phases 5 and 6 to be 10,000 hours each, instead of the 5,000 hours that were considered in the initial plan. Also, the team decides to increase the duration of the 7th phase to 15,000 hours. Since this is going to be achieved by testing for more calendar time in phases 5, 6 and 7, the average fix delay during those phases is the same as the one in the initial plan. The test time per week is not going to be affected, only more weeks will be added to the schedule. The next figure shows the revised plan, together with the calculated results. The actual time to meet the MTBF goal is now 54,331 hours. The final MTBF that can be achieved at the end of the 55,000 hours of growth testing is 100.04, which is slightly higher than the goal MTBF of 100 hours. The reliability team considers this plan as acceptable, since the MTBF goal will be met.



The next plot shows the overall reliability growth plan, with the nominal and actual growth curves, the MTBF goal line and the planned MTBF for each of the seven phases.



Growth Plan for 4 Phases

The reliability team of a product manufacturer has put together a reliability growth plan, based on the Crow Extended model, for one of their new products. The reliability growth model was constructed with the following inputs:

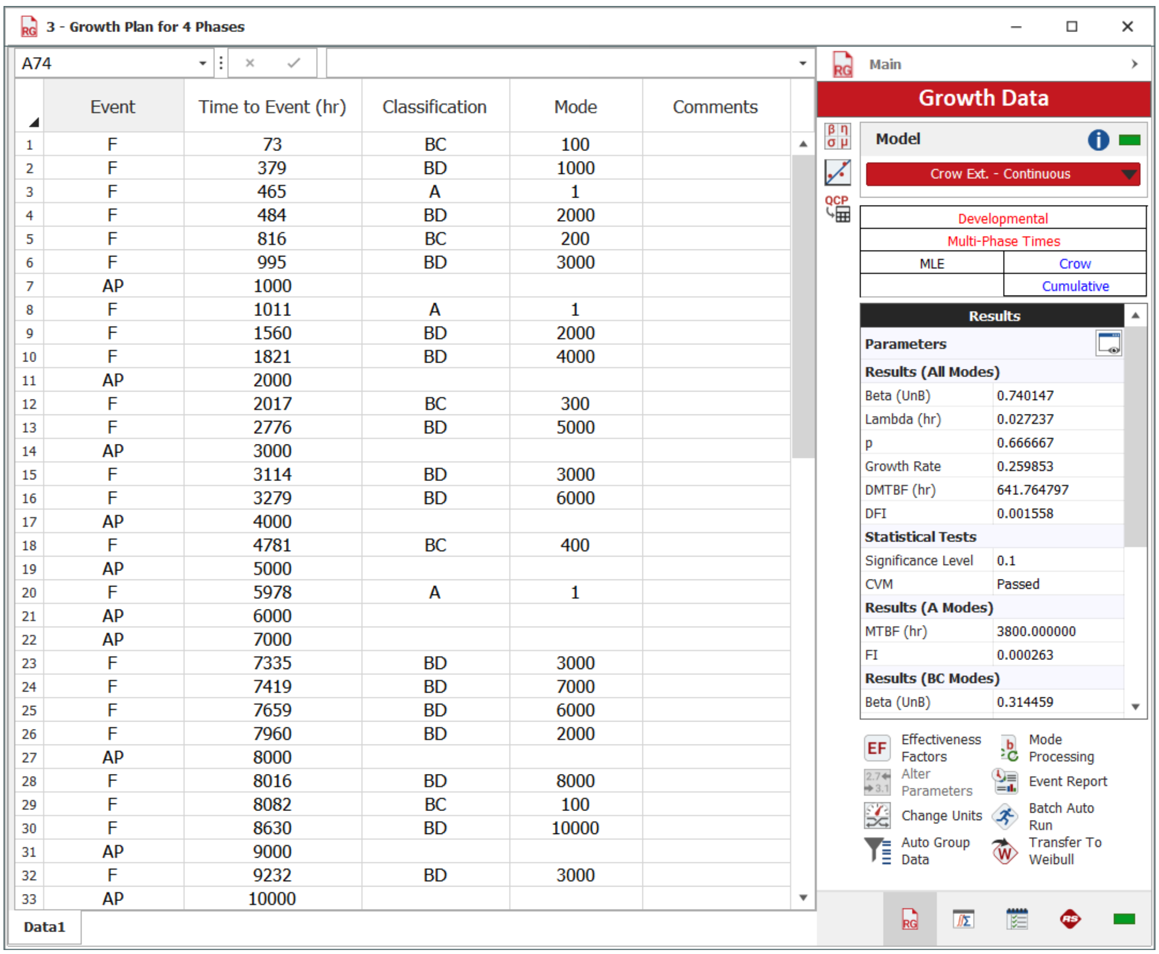
- The requirement or goal MTBF is $M_G = 910$ hours.
- The growth potential design margin factor is $GPDM = 1.35$.
- The average effectiveness factor is $d \quad = 0.7$.
- The management strategy is $msr = 0.95$.
- The beta parameter for the discovery function, $h(t)$, of the type B failure modes is $\beta = 0.70$.
- The test is planned to be conducted in four phases. The cumulative phase end times are $T_1 = 19000, T_2 = 38000$, $T_3 = 70000$ and $T_4 = 100000$. The average fix delay in terms of test hours for each phase is: $L_1 = 10000$, $L_2 = 15000, L_3 = 20000$ and $L_4 = 25000$ hours.

Determine the following:

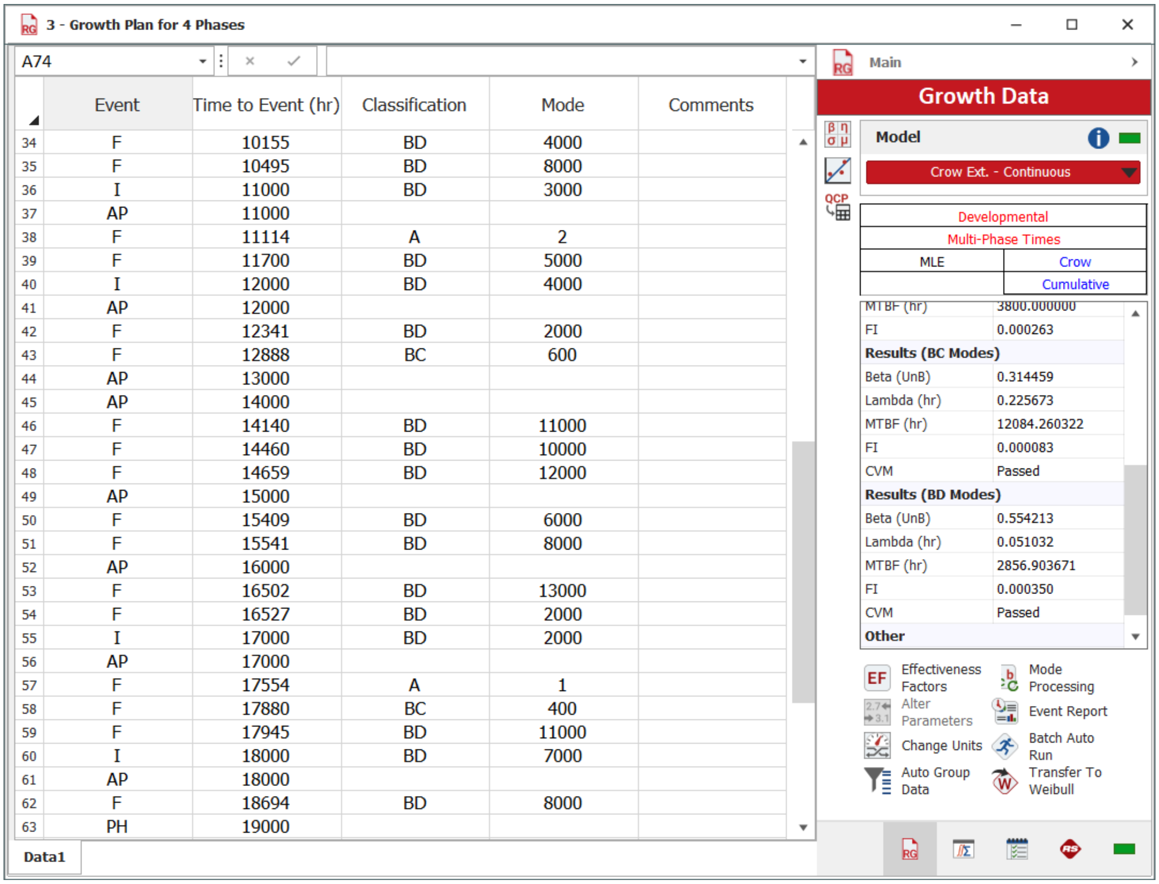
1. Plot the nominal and actual reliability growth curves for this program, using the Weibull++ software.

2. The reliability program was initiated and actual test data from phase 1 is now available. The test data were analyzed using the Crow Extended - Continuous Evaluation model and the following figure shows the results.

Note that the *I* events, which represent the times for implementation of fixes for BD modes, occur at least 10,000 hours after the first occurrence of the specific BD mode. This is a reflection of the average fix delay being equal to 10,000 hours for the first test phase. Of course we could have some exceptions that could have been fixed before 10,000 hours after the first occurrence (i.e., discovery) of the mode, but on average, they should be fixed after 10,000 hours.



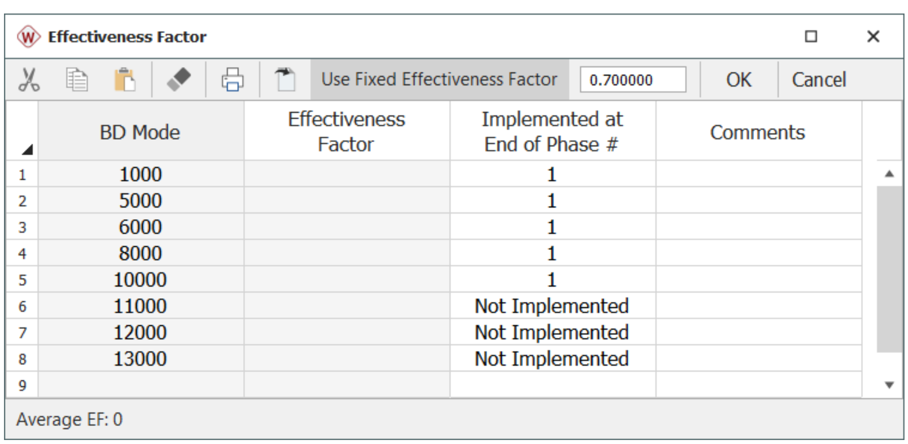
	Event	Time to Event (hr)	Classification	Mode	Comments
1	F	73	BC	100	
2	F	379	BD	1000	
3	F	465	A	1	
4	F	484	BD	2000	
5	F	816	BC	200	
6	F	995	BD	3000	
7	AP	1000			
8	F	1011	A	1	
9	F	1560	BD	2000	
10	F	1821	BD	4000	
11	AP	2000			
12	F	2017	BC	300	
13	F	2776	BD	5000	
14	AP	3000			
15	F	3114	BD	3000	
16	F	3279	BD	6000	
17	AP	4000			
18	F	4781	BC	400	
19	AP	5000			
20	F	5978	A	1	
21	AP	6000			
22	AP	7000			
23	F	7335	BD	3000	
24	F	7419	BD	7000	
25	F	7659	BD	6000	
26	F	7960	BD	2000	
27	AP	8000			
28	F	8016	BD	8000	
29	F	8082	BC	100	
30	F	8630	BD	10000	
31	AP	9000			
32	F	9232	BD	3000	
33	AP	10000			



	Event	Time to Event (hr)	Classification	Mode	Comments
34	F	10155	BD	4000	
35	F	10495	BD	8000	
36	I	11000	BD	3000	
37	AP	11000			
38	F	11114	A	2	
39	F	11700	BD	5000	
40	I	12000	BD	4000	
41	AP	12000			
42	F	12341	BD	2000	
43	F	12888	BC	600	
44	AP	13000			
45	AP	14000			
46	F	14140	BD	11000	
47	F	14460	BD	10000	
48	F	14659	BD	12000	
49	AP	15000			
50	F	15409	BD	6000	
51	F	15541	BD	8000	
52	AP	16000			
53	F	16502	BD	13000	
54	F	16527	BD	2000	
55	I	17000	BD	2000	
56	AP	17000			
57	F	17554	A	1	
58	F	17880	BC	400	
59	F	17945	BD	11000	
60	I	18000	BD	7000	
61	AP	18000			
62	F	18694	BD	8000	
63	PH	19000			

The same applies to the delayed fixes at the end of phase 1. On average there should be at least 10,000 test hours between the discovery of the failure mode and the implementation of a fix for that mode, so failure modes discovered after 9,000 hours of testing in the first phase cannot be implemented at the end of the first phase.

The figure below shows the effectiveness factor window. The record shows that the fixes for failure modes BD 11000, BD 12000 and BD 13000 are not implemented at the end of the first phase due to the fix delay. The effectiveness factor used is in accordance with the planning model and has been fixed to 0.7.

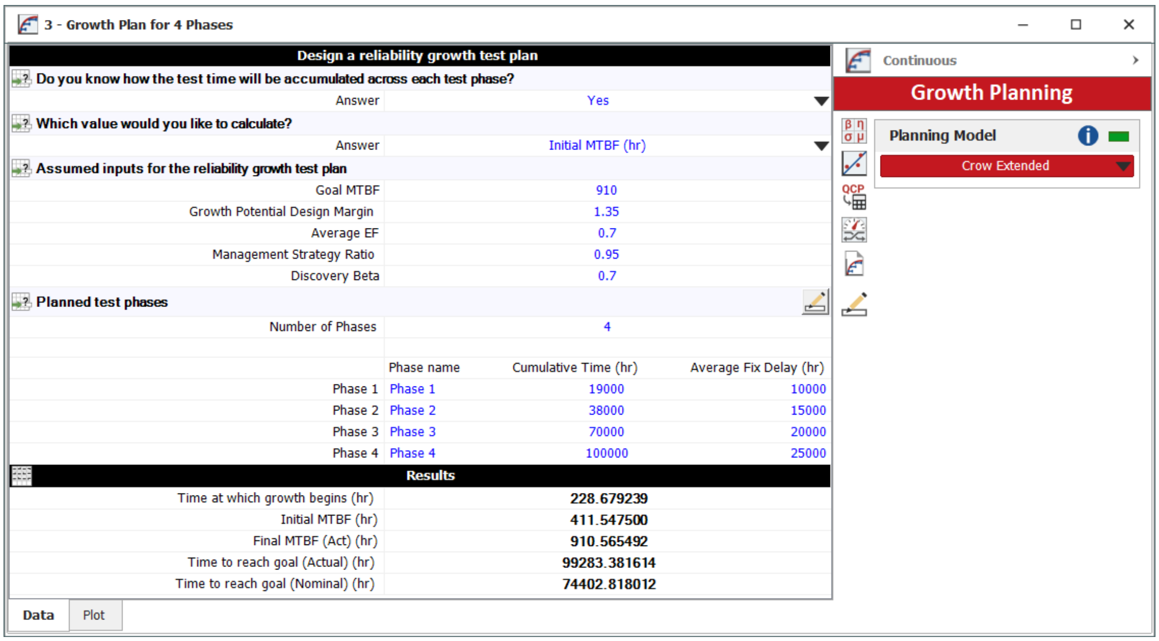


	BD Mode	Effectiveness Factor	Implemented at End of Phase #	Comments
1	1000		1	
2	5000		1	
3	6000		1	
4	8000		1	
5	10000		1	
6	11000		Not Implemented	
7	12000		Not Implemented	
8	13000		Not Implemented	
9				

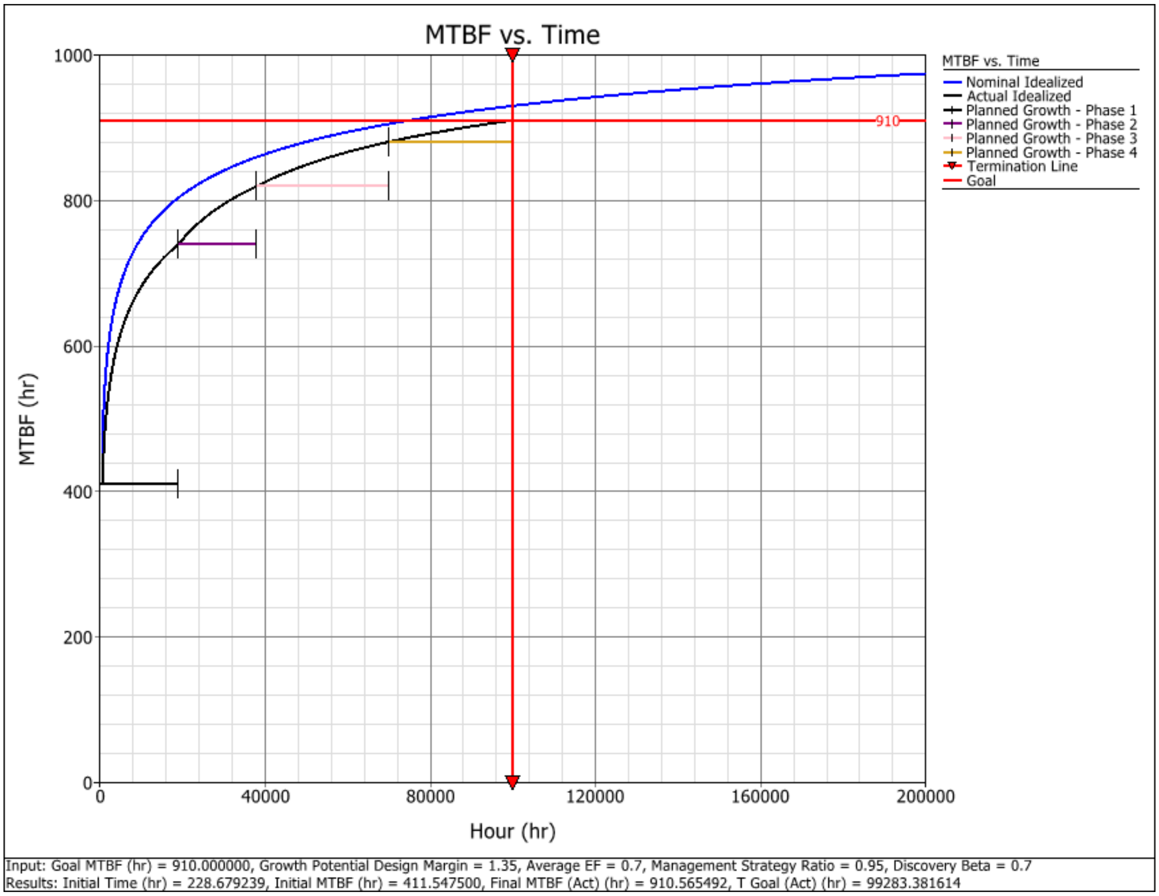
Construct a multi-phase graph that shows the analysis points for the test data in phase 1, as compared to the nominal and actual idealized growth curves for this reliability growth plan. Is the program on track, so far, as compared to the plan?

Solution

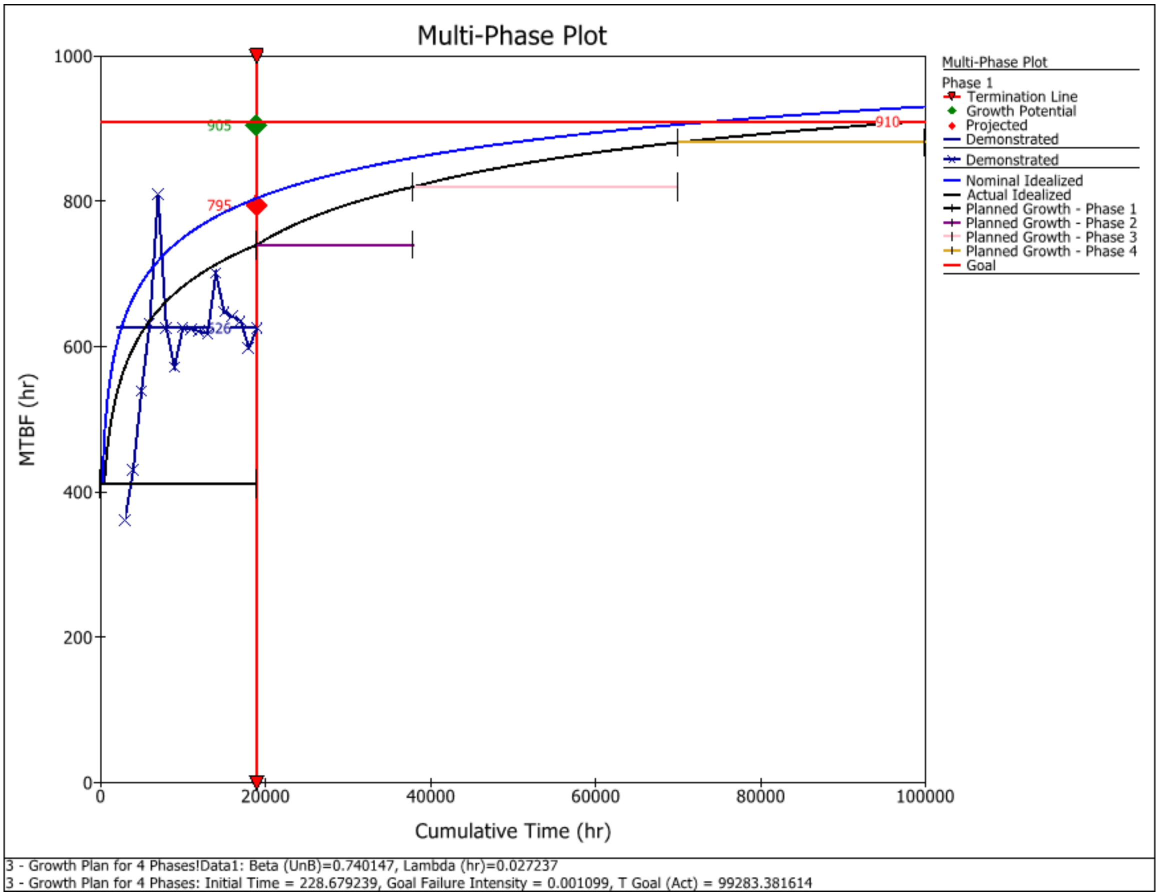
1. The following figures show the plan inputs, phase durations and average fix delays as entered in Weibull++'s continuous growth planning folio, together with the results of the growth model.



The next figure shows the nominal and actual idealized growth curves for this program, together with the program MTBF goal.



2. The next step is to associate the reliability growth plan with the data from phase 1. To do that, right-click the **Multiplots** heading in the current project explorer and choose **Add Multi-Phase Plot** from the shortcut menu, then follow the wizard to specify the data sheet that contains the phase 1 data with the folio that defines the growth plan. The figure below shows the generated multi-phase plot, which brings together the reliability growth planning model with the actual test results. This plot shows the demonstrated MTBF for each analysis point during the first phase of testing. Analysis points for projected and growth potential MTBF can also be plotted. For this example, it can be said that the program is on track since the demonstrated MTBF of the first phase is higher than the planned MTBF for that phase.



The multi-phase plot can be updated continuously as the growth program progresses and more test data become available. In that manner, the reliability team can assess the actual reliability performance against the established goals.

Discrete Reliability Growth Planning

To be consistent with the Duane postulate and the Crow (AMSAA) model, the discovery function must be of the same form. This form of the discovery function is an important property of the Crow Extended model and Crow Extended Discrete Planning model. As with the Crow (AMSAA) model, this form of the discovery function ties the model directly to real-world data and experiences. Therefore, a desirable feature of a discrete reliability growth planning model is that expected number of distinct failure modes over $(0,t) = \lambda(t)^{\beta}$. This implies that the probability, f , of a new distinct failure mode occurring at trial t is given by:

$$f = \lambda t^{\beta} - \lambda(t-1)^{\beta}$$

Let f_A be the Type A initial failure probability and let f_B be the Type B initial failure probability.

A system failure occurs at the first event of a Type A mode or a Type B mode. Only one event causes a system failure. That is, Type A failures and Type B failures are disjointed.

The initial system failure probability f_I is:

$$f_I = f_A + f_B$$

The Type A failure probability is the failure probability in that part of the system that will not be addressed by corrective actions even if a failure mode is seen during test. The Type B failure probability is the failure probability in that part of the system that will be addressed by corrective actions if a failure mode is seen during test.

When a failure mode in the Type B part of the system is seen during test, a corrective action will be implemented. For discrete trials, the corrective action is always implemented at a later time after the failure mode is first seen.

When a corrective action is implemented for a Type B failure mode, the failure probability for that mode is reduced if the corrective action is effective. The fraction decrease in the Type B failure modes due to corrective actions is d, where the Average Effectiveness Factor (EF) is d and the corrective action may be implemented before the next trial or at a later date.

The management strategy ratio is:

$$msr = \frac{f_B}{f_A + f_B}$$

The msr is the fraction of the initial system failure probability that will be addressed by corrective actions, if seen during the test.

If all Type B failure modes are seen and corrected with an average EF d, then the maximum reduction in the initial system failure probability is the Growth Potential failure probability:

$$f_{GP} = f_A + (1 - d)\, f_B$$

The initial system Mean Trials Between Failure (MTrBF) is:

$$M_I = \frac{1}{f_I}$$

The Growth Potential MTrBF is:

$$M_{GP} = \frac{1}{f_{GP}}$$

Nominal Idealized Growth Curve

The nominal idealized growth curve portrays a general profile for reliability growth throughout system testing. It represents a best case scenario where fix delay is not taken into account. The Crow Extended Discrete Planning Model Nominal Idealized Growth Curve failure probability as a function of test trials t is:

$$f_{NI}(t) = f_A + (1-d)f_B + d\left[\lambda t^{\beta} - \lambda(t-1)^{\beta}\right] \text{ for } t \geq t_0$$

$$f_{NI}(t) = f_I \text{ for } t < t_0$$

where t_0 is the initialization time, and t is the number of test trials.

It is virtually impossible to justify with any model that reliability growth starts at time zero because growth can only start when a Type B failure mode occurs. The Extended Discrete Planning Model Nominal Idealized Growth Curve failure probability is initially set equal to the initial failure probability f_I until the time t_0 when:

$$f_{NI}(t_0) = f_A + (1-d)f_B + d\left[\lambda t^{\beta} - \lambda(t_0-1)^{\beta}\right] = f_I$$

The Initialization time t_0 allows for the probability of a Type B failure mode to be greater than zero before growth starts. t_0 is then solved numerically.

The mean time to the first failure mode is given by the mean of a sequence of independent Bernoulli Trials, each trial with a different probability of a new failure mode given by:

$$g_i = \lambda i^{\beta} - \lambda(i-1)^{\beta} \text{ for } i = 1, 2, \ldots$$

Then the mean trial to the first Type B mode is given by:

$$MTrBF_B = \sum_{k=1}^{\infty} k \cdot g_k \cdot \left[\prod_{j=1}^{k-1} P_j\right]$$

where:

$$MTrBF_B = \frac{1}{f_B}$$

$$g_k = \lambda\left[\overset{\beta}{k} - (k-1)^{\beta}\right]$$

$$P_k = 1 - g_k \text{ such that } k = 1, 2, \ldots, \infty$$

λ is then solved numerically.

Nominal Time to Reach Goal

The nominal time to reach the goal is calculated by setting the nominal idealized failure intensity, f_{NI} , equal to the failure intensity goal, f_G . Therefore,

$$f_G = f_A + (1-d)f_B + d\left[\lambda {t_{N,G}}^{\beta} - \lambda(t_{N,G} - 1)^{\beta}\right]$$

$t_{N,G}$ is solved numerically.

Actual Idealized Growth Curve

The Actual Idealized Growth Curve is a continuous function that incorporates the test phase average fix delay times and goes through each of the test phase target MTrBF.

Fix Delay

The fix delay reflects how long it takes from the time a problem failure mode is discovered in test to the time the corrective action is incorporated into the system and reliability growth is realized. The consideration of the fix delay is often in terms of how much calendar time it takes to incorporate a corrective action fix after the problem is first seem. However, the impact of the fix delay on reliability growth is reflected in the average test time it takes between finding a problem failure mode and incorporating a corrective action.

There can be a constant fix delay across all test phases or, as a practical matter, each test phase can have a different fix delay time. In practice, the fix delay will be considered constant over a test phase. L_i denotes the fix delay for phase $i = 1, \ldots, P,$ where P is the total number of phases in the test. The Weibull++ software allows for a maximum of ten test phases.

Actual Failure Intensity Function

Test Phase 1

The actual failure intensity function for test phase 1 is given by:

$$f_{AI}(t) = f_A + (1-d)f_B + d\lambda\left[\left(\left(\frac{T_1 - L_1}{T_1}\right)t\right)^{\beta} - \left(\left(\frac{T_1 - L_1}{T_1}\right)t - 1\right)^{\beta}\right] \text{ for } 0 < t \leq T_1$$

Note that the end time of Phase 1, $T_1,$ must be greater than $L_1 + t_0$. That is, $T_1 > L_1 + t_0$.

The actual idealized curve initialization time for Phase 1, T_0^{AIC} , is calculated from:

$$f_{AI}(T_0^{AIC}) = \lambda_A + (1-d)\lambda_B + d\lambda\left[\left(\left(\frac{T_1 - L_1}{T_1}\right)T_0^{AIC}\right)^{\beta} - \left(\left(\frac{T_1 - L_1}{T_1}\right)T_0^{AIC} - 1\right)^{\beta}\right]$$

By obtaining the initial failure intensity for T_0^{AIC} :

$$T_0^{AIC} = \frac{t_0}{\left(\frac{T_1 - L_1}{T_1}\right)}$$

Test Phase i

For any test phase i , the actual idealized curve failure intensity is given by:

$$f_{AI}(t) = f_A + (1-d)f_B + d\lambda\left[\left(T_{i-1} - L_{i-1} + \left(\frac{T_i - L_i - T_{i-1} + L_{i-1}}{T_i - T_{i-1}}\right)(t - T_{i-1})\right)^{\beta} - \left(T_{i-1} - L_{i-1} + \left(\frac{T_i - L_i - T_{i-1} + L_{i-1}}{T_i - T_{i-1}}\right)(t - T_{i-1}) - 1\right)^{\beta}\right]$$

where $T_{i-1} \leq t \leq T_i$ and T_i is the test time of each corresponding test phase.

Actual Time to Reach Goal

The actual time to reach the goal is calculated using the actual failure intensity equation by solving for $t_{AC,G}$ such that the $f_{AI}(t)$ is equal to the failure intensity goal, f_G . Therefore:

$$f_G = f_A + (1-d)f_B + d\lambda\left[\left(T_{i-1} - L_{i-1} + \left(\frac{T_i - L_i - T_{i-1} + L_{i-1}}{T_i - T_{i-1}}\right)(t_{AC,G} - T_{i-1})\right)^{\beta} - \left(T_{i-1} - L_{i-1} + \left(\frac{T_i - L_i - T_{i-1} + L_{i-1}}{T_i - T_{i-1}}\right)(t_{AC,G} - T_{i-1}) - 1\right)^{\beta}\right].$$

$t_{AC,G}$ is solved numerically where T_F and L_F correspond to the phase in which the goal is met.

If the goal, M_G , is met after the final test phase, then the actual time to reach the goal is not calculated since additional phases have to be added with specific duration and fix delays. The reliability growth program can be re-evaluated with the following options:

- Add more phase(s) to the program.
- Re-examine the phase duration of the existing phases.
- Investigate whether there are potential process improvements in the program that can reduce the average fix delay for the phases.

Other alternative routes for consideration would be to investigate the rest of the inputs in the model:

- Change the management strategy.
- Consider if further program risk can be acceptable, and if so, reduce the growth potential design margin.
- Consider if it is feasible to increase the effectiveness factors of the delayed fixes by using more robust engineering redesign methods.

Note that each change of input variables into the model can significantly influence the results.

With that in mind, any alteration in the input parameters should be justified by actionable decisions that will influence the reliability growth program. For example, increasing the average effectiveness factor value should be done only when there is proof that the program will pursue a different, more effective path in terms of addressing fixes.

Example

A missile system will be going through a reliability growth development program. The goal for the system is to reach a reliability requirement of 0.9 at the end of the program. This corresponds to a goal MTrBF, R_G , of 10 trials. The new missile system is similar to an existing missile system, but this will not be an upgrade. Therefore, to help design the reliability growth plan for the new system, parameters from the existing version will be used.

The previous missile system met its reliability goal of $R_G = 0.95$ which corresponds to a MTrBF = 20. There were over 600 trials of this missile system during developmental testing. The trials were used for testing and training. In addition, given the complexity of the technology that was utilized, additional trials were needed. The system also had an initial reliability of $R_I = 0.84$. Additional parameters associated with the previous system are given below:

- Average Effectiveness Factor is $EF = 0.85$
- Discovery Beta is $\beta = 0.70$
- Management Strategy Ratio is $msr = 0.972$
- Growth Potential Design Margin is $GPDM = 1.8$

For the new program, a $msr = 0.97$ will be used, along with an average $EF = 0.8$. For initial planning purposes, the average fix delay will be assumed to be equal to zero. Therefore, the nominal idealized curve will equal the actual idealized curve.

Determine the following:

1. The growth potential MTrBF and failure intensity.
2. The initial failure intensity and reliability.
3. The growth potential reliability.
4. The type A and type B initial failure intensity.
5. The parameter λ of the Crow extended model.
6. The type B failure mode discovery function.
7. The initialization time, t_0, for the nominal failure intensity function.
8. The nominal idealized failure intensity function.
9. The nominal time to reach the MTrBF goal.
10. Use Weibull++ to generate the reliability growth plan for the new missile design.

Solution

1. The growth potential MTrBF is:

$$\begin{aligned} MTrBF_{GP} &= MTrBF_G \cdot GPDM \\ &= 10 \cdot 1.8 \\ &= 18 \end{aligned}$$

2. The growth potential failure intensity is equal to:

$$\begin{aligned} f_{GP} &= \frac{1}{M_{GP}} \\ &= \frac{1}{18} \\ &= 0.0556 \end{aligned}$$

Therefore, the initial failure intensity is:

$$\begin{aligned} f_I &= \frac{f_{GP}}{1 - d \cdot msr} \\ &= \frac{0.0556}{(1 - 0.85 \cdot 0.97)} \\ &= 0.2480 \end{aligned}$$

The initial reliability is then:

$$\begin{aligned} R_I &= 1 - f_I \\ &= 1 - 0.2480 \\ &= 0.7520 \end{aligned}$$

3. The growth potential reliability is equal to:

$$\begin{aligned} R_{GP} &= 1 - f_{GP} \\ &= 1 - 0.0556 \\ &= 0.9444 \end{aligned}$$

4. The type A failure mode intensity, f_A, is:

$$\begin{aligned} f_A &= (1 - msr) f_I \\ &= (1 - 0.97)0.2480 \\ &= 0.0074 \end{aligned}$$

The type B failure mode intensity, f_B, is:

$$\begin{aligned} f_B &= msr \cdot f_I \\ &= 0.97 \cdot 0.2480 \\ &= 0.2406 \end{aligned}$$

5. The initial MTrBF is:

$$\begin{aligned} MTrBF_{GP} &= \frac{1}{f_I} \\ &= \frac{1}{0.2480} \\ &= 4.032 \end{aligned}$$

λ is estimated such that:

$$\begin{aligned} MTrBF_B &= \sum_{k=1}^{\infty} k \cdot g_k \cdot \left[\prod_{j=1}^{k-1} P_j \right] \\ &= 4.0322 \end{aligned}$$

where:

$$g_k = \lambda \left[k^{\beta} - (k-1)^{\beta} \right]$$

Therefore: $\lambda = 0.4145$

6. The type B failure mode discovery function is:

$$h(t) = \lambda \beta t^{\beta - 1}$$

Since λ and β are known:

$$\begin{aligned} h(t) &= 0.4145 \cdot 0.7 t^{0.7-1} \\ &= 0.2901 t^{-0.3} \end{aligned}$$

7. The initialization time, t_0, is calculated using the nominal idealized equation and setting this equal to initial failure intensity.

$$f_{NI}(t_0) = f_A + (1-d)f_B + d\left[\lambda t^{\beta} - \lambda(t_0 - 1)^{\beta}\right] = f_I$$
$$= 0.2480$$

Then, using numerical methods:

$$t_0 = 2.3964$$

8. The nominal idealized growth curve is given by the following equation:

$$f_{NI}(t) = \begin{Bmatrix} \lambda_I & t \leq t_0 \\ f_A + (1-d)f_B + d\left[\lambda t^{\beta} - \lambda(t-1)^{\beta}\right] & t > t_0 \end{Bmatrix}$$

or:

$$f_{NI}(t) = \begin{Bmatrix} 0.2480 & t \leq 2.3964 \\ = 0.0556 + 0.8\left[0.4145 t^{0.7} - 0.4145(t-1)^{0.7}\right] & t > 2.3964 \end{Bmatrix}$$

9. The nominal time to reach the MTrBF goal is estimated by solving for the value of t such that the nominal idealized curve f_{NI} is equal to f_I. Therefore, solve for t such that:

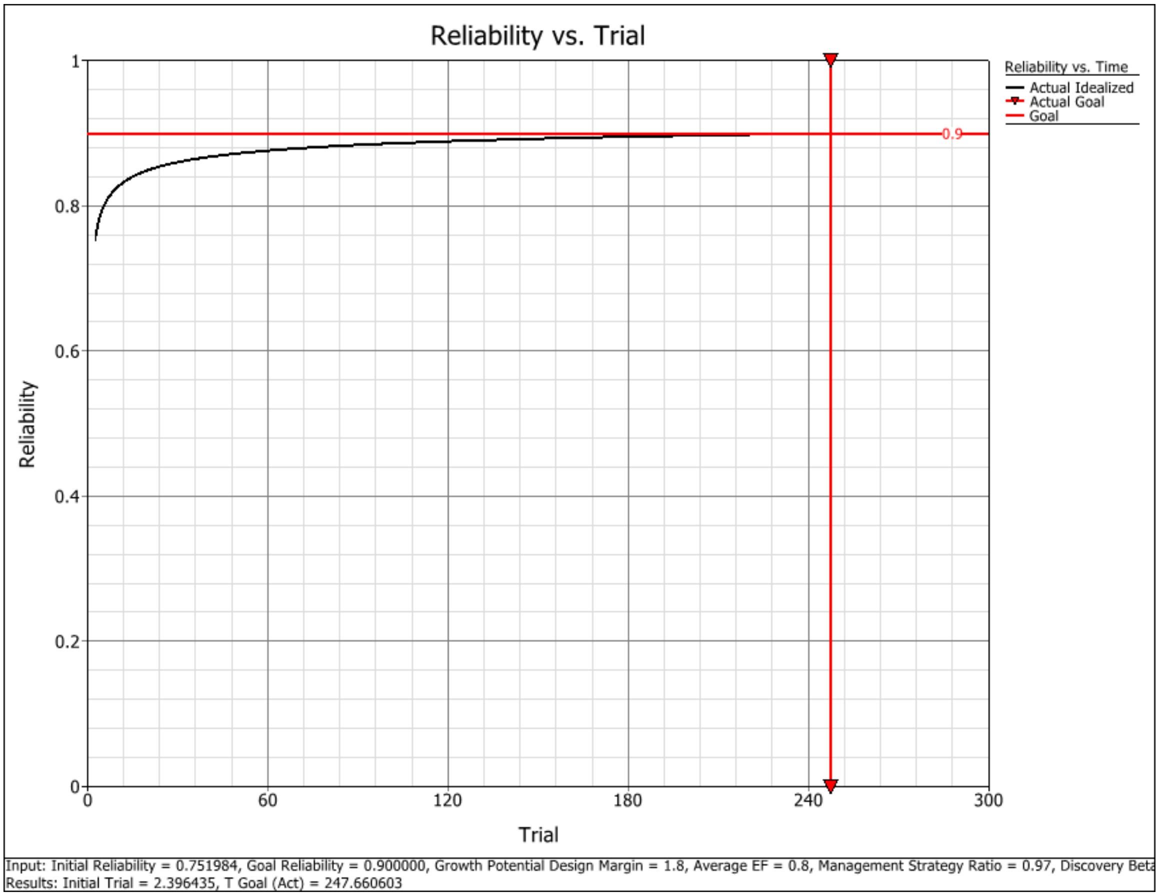
$$f_{NI}(t_0) = f_A + (1-d)f_B + d\left[\lambda t^{\beta} - \lambda(t_0 - 1)^{\beta}\right] = 0.2480$$

Using numerical methods, the nominal time to reach the goal $t = 247.6606$ trials.

10. The reliability growth plan in Weibull++ is shown next.

Design a reliability growth test plan	
Do you know how the test time will be accumulated across each test phase?	
Answer	No
Which value would you like to calculate?	
Answer	Initial Reliability
Assumed inputs for the reliability growth test plan	
Goal Reliability	0.9
Growth Potential Design Margin	1.8
Effectiveness Factor	0.8
Management Strategy Ratio	0.97
Discovery Beta	0.7
Average Fix Delay	0
Results	
Trial at which growth begins	2.396435
Initial Reliability	0.751984
Initial MTrBF	4.032000
Growth Potential MTrBF	18.000000
Num. trials to reach goal (Actual)	247.660603
Num. trials to reach goal (Nominal)	247.660603

The plot of the reliability growth plan is shown below.


Reliability vs. Trial
Reliability
Trial
Reliability vs. Time
Actual Idealized
Actual Goal
Goal
0.9
Input: Initial Reliability = 0.751984, Goal Reliability = 0.900000, Growth Potential Design Margin = 1.8, Average EF = 0.8, Management Strategy Ratio = 0.97, Discovery Beta
Results: Initial Trial = 2.396435, T Goal (Act) = 247.660603

Operational Mission Profile Testing

IN THIS CHAPTER

It is common practice for systems to be subjected to operational testing during a development program. The objective of this testing is to evaluate the performance of the system, including reliability, under conditions that represent actual use. Because of budget, resources, schedule and other considerations, these operational tests rarely match exactly the actual use conditions. Usually, stated mission profile conditions are used for operational testing. These mission profile conditions are typically general statements that guide testing on an average basis. For example, a copier might be required to print 3,000 pages by time T=10 days and 5,000 pages by time T=15 days. In addition the copier is required to scan 200 documents by time T=10 days, 250 documents by time T=15 days, etc.

Because of practical constraints, these full mission profile conditions are typically not repeated one after the other during testing. Instead, the elements that make up the mission profile conditions are tested under varying schedules with the intent that, on average, the mission profile conditions are met. In practice, reliability corrective actions are generally incorporated into the system as a result of this type of testing.

Because of a lack of structure for managing the elements that make up the mission profile, it is difficult to have an agreed upon methodology for estimating the system's reliability. Many systems fail operational testing because key assessments such as growth potential and projections cannot be made in a straightforward manner so that management can take appropriate action. The Weibull++ software addresses this issue by incorporating a systematic mission profile methodology for operational reliability testing and reliability growth assessments.

Introduction

Operational testing is an attempt to subject the system to conditions close to the actual environment that is expected under customer use. Often this is an extension of reliability growth testing where operation induced failure modes and corrective actions are of prime interest. Sometimes the stated intent is for a demonstration test where corrective actions are not the prime objective. However, it is not unusual for a system to fail the demonstration test, and the management issue is what to do next. In both cases, important and valid key parameters are needed to properly assess this situation and make cost-effective and timely decisions. This is often difficult in practice.

For example, a system may be required to:

- Conduct a specific task a fixed number times for each hour of operation (task 1).
- Move a fixed number of miles under a specific operating condition for each hour of operation (task 2).
- Move a fixed number of miles under another operating condition for each hour of operation (task 3).

During operational testing, these guidelines are met individually as averages. For example, the actual as-tested profile for task 1 may not be uniform relative to the stated mission guidelines during the testing. What is often the case is that some of the tasks (for example task 1) could be operated below the stated guidelines. This can mask a major reliability problem. In other cases during testing, tasks 1, 2 and 3 might never meet their stated averages, except perhaps at the end of the test. This becomes an issue because an important aspect of effective reliability risk management is to not wait until the end of the test to have an assessment of the reliability performance.

Because the elements of the mission profile during the testing will rarely, if ever, balance continuously to the stated averages, a common analysis method is to piece the reliability assessments together by evaluating each element of the profile separately. This is not a well-defined methodology and does not account for improvement during the testing. It is therefore not unusual for two separate organizations (e.g., the customer and the developer) to analyze the same data and obtain different MTBF numbers. In addition, this method does not address the delayed corrective actions that are to be incorporated at the end of the test nor does it estimate growth potential or interaction effects. Therefore, to reduce this risk there is a need for a rigorous methodology for reliability during operational testing that does not rely on piecewise analysis and avoids the issues noted above.

The Weibull++ software incorporates a new methodology to manage system reliability during operational mission profile testing. This methodology draws information from particular plots of the operational test data and inserts key information into a growth model. The improved methodology does not piece the analysis together, but gives a direct MTBF mission profile estimate of the system's reliability that is directly compared to the MTBF requirement. The methodology will reflect any reliability growth improvement during the test, and will also give management a higher projected MTBF for the system mission profile reliability after delayed corrected actions are incorporated at the end of the test. In addition, the methodology also gives an estimate of the system's growth potential, and provides management metrics to evaluate whether changes in the program need to be made. A key advantage is that the methodology is well-defined and all organizations will arrive at the same reliability assessment with the same data.

Testing Methodology

The methodology described here will use the Crow extended model for data analysis. In order to have valid Crow extended model assessments, it is required that the operational mission profile be conducted in a structured manner. Therefore, this testing methodology involves convergence and stopping points during the testing. A stopping point is when the testing is stopped for the expressed purpose of incorporating the type BD delayed corrective actions. There may be more than one stopping point during a particular testing phase. For simplicity, the methodology with only one stopping point will be described; however, the methodology can be extended to the case of more than one stopping point. A convergence point is a time during the test when all the operational mission profile tasks meet their expected averages or fall within an acceptable range. At least three convergence points are required for a well-balanced test. The end of the test, time T, must be a convergence point. The test times between the convergence points do not have to be the same.

The objective of having the convergence points is to be able to apply the Crow extended model directly in such a way that the projection and other key reliability growth parameters can be estimated in a valid fashion. To do this, the grouped data methodology is applied. Note that the methodology can also be used with the Crow-AMSAA (NHPP) model for a simpler analysis without the ability to estimate projected and growth potential reliability. See the Grouped Data for the Crow-AMSAA (NHPP) model or for the Crow extended model.

Example - Mission Profile Testing

Consider the test-fix-find-test data set that was introduced in the Crow Extended model chapter and is shown again in the table below. The total test time for this test is 400 hours. Note that for this example we assume one stopping point at the end of the test for the incorporation of the delayed fixes. Also, suppose that the data set represents a military system with:

- Task 1 = firing a gun.
- Task 2 = moving under environment E1.
- Task 3 = moving under environment E2.

For every hour of operation, the operational profile states that the system operates in the E1 environment for 70% of the time and in the E2 environment for 30% of the time. In addition, for each hour of operation, the gun must be fired 10 times.

Test-Fix-Find-Test Data					
i	X_i	**Mode**	i	X_i	**Mode**
1	0.7	BC17	29	192.7	BD11
2	3.7	BC17	30	213	A
3	13.2	BC17	31	244.8	A
4	15	BD1	32	249	BD12
5	17.6	BC18	33	250.8	A
6	25.3	BD2	34	260.1	BD1
7	47.5	BD3	35	263.5	BD8
8	54	BD4	36	273.1	A
9	54.5	BC19	37	274.7	BD6
10	56.4	BD5	38	282.8	BC27
11	63.6	A	39	285	BD13
12	72.2	BD5	40	304	BD9
13	99.2	BC20	41	315.4	BD4
14	99.6	BD6	42	317.1	A
15	100.3	BD7	43	320.6	A

16	102.5	A	44	324.5	BD12
17	112	BD8	45	324.9	BD10
18	112.2	BC21	46	342	BD5
19	120.9	BD2	47	350.2	BD3
20	121.9	BC22	48	355.2	BC28
21	125.5	BD9	49	364.6	BD10
22	133.4	BD10	50	364.9	A
23	151	BC23	51	366.3	BD2
24	163	BC24	52	373	BD8
25	164.7	BD9	53	379.4	BD14
26	174.5	BC25	54	389	BD15
27	177.4	BD10	55	394.9	A
28	191.6	BC26	56	395.2	BD16

In general, it is difficult to manage an operational test so that these operational profiles are continuously met throughout the test. However, the operational mission profile methodology requires that these conditions be met on average at the convergence points. In practice, this almost always can be done with proper program and test management. The convergence points are set for the testing, often at interim assessment points. The process for controlling the convergence at these points involves monitoring a graph for each of the tasks.

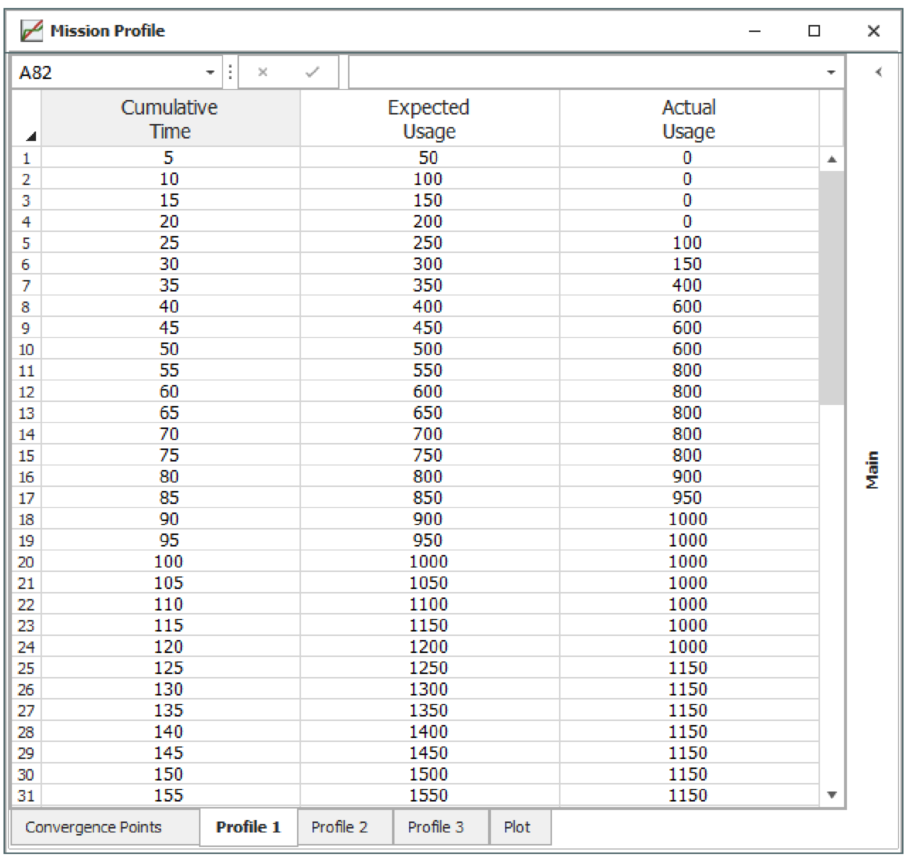
The following table shows the expected and actual results for each of the operational mission profiles.

Expected and Actual Results for Profiles 1, 2, 3						
	Profile 1(gun firings)		Profile 2 (E1)		Profile 3 (E2)	
Time	Expected	Actual	Expected	Actual	Expected	Actual
5	50	0	3.5	5	1.5	0
10	100	0	7	10	3	0

15	150	0	10.5	15	4.5	0
20	200	0	14	20	6	0
25	250	100	17.5	25	7.5	0
30	300	150	21	30	9	0
35	350	400	24.5	30	10.5	5
40	400	600	28	30	12	10
45	450	600	31.5	30	13.5	15
50	500	600	35	30	15	20
55	550	800	38.5	35	16.5	20
60	600	800	42	40	18	20
65	650	800	45.5	45	19.5	20
70	700	800	49	50	21	20
75	750	800	52.5	55	22.5	20
80	800	900	56	55	24	25
85	850	950	59.5	55	25.5	30
90	900	1000	63	60	27	30
95	950	1000	66.5	65	28.5	30
100	1000	1000	70	70	30	30
105	1050	1000	73.5	70	31.5	35
...	...	...	...	...	...	
...	...	...	...	...	...	
355	3550	3440	248.5	259	106.5	96
360	3600	3690	252	264	108	96
365	3650	3690	255.5	269	109.5	96

370	3700	3850	259	274	111	96
375	3750	3850	262.5	279	112.5	96
380	3800	3850	266	280	114	100
385	3850	3850	269.5	280	115.5	105
390	3900	3850	273	280	117	110
395	3950	4000	276.5	280	118.5	115
400	4000	4000	280	280	120	120

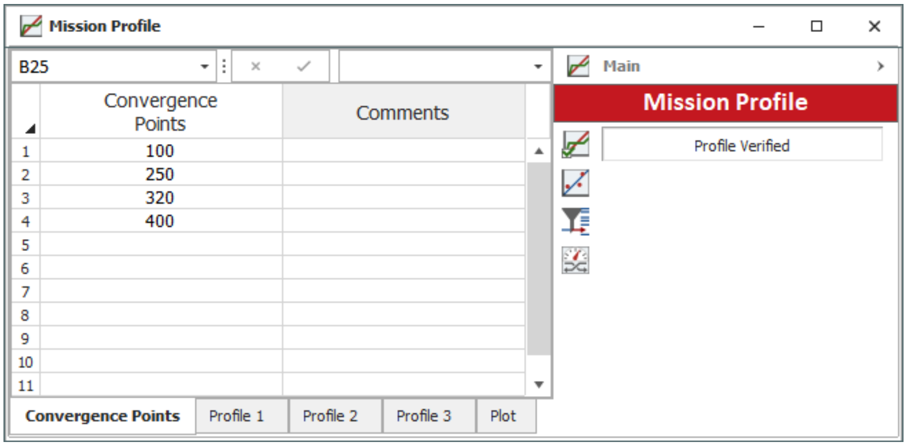
The next figure shows a portion of the expected and actual results for mission profile 1, as entered in the Weibull++ software.



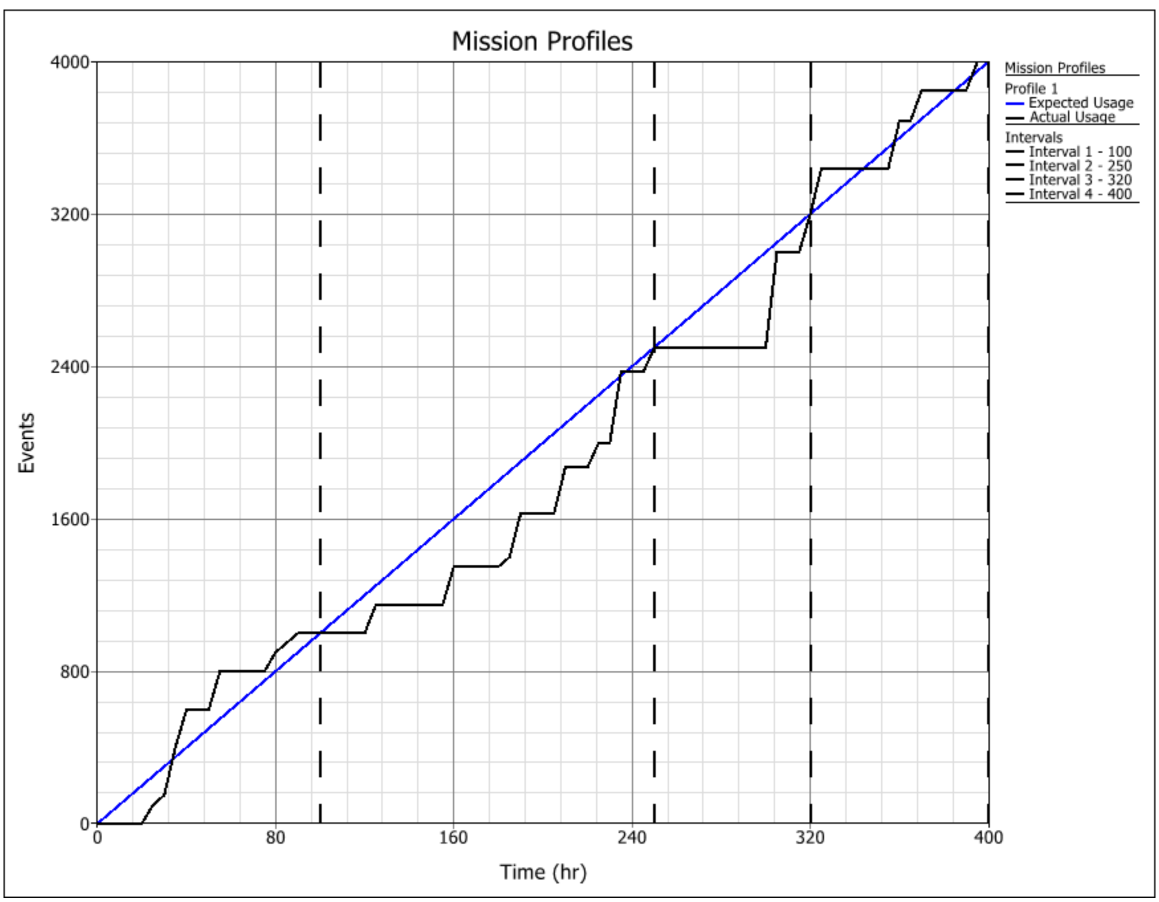
	Cumulative Time	Expected Usage	Actual Usage
1	5	50	0
2	10	100	0
3	15	150	0
4	20	200	0
5	25	250	100
6	30	300	150
7	35	350	400
8	40	400	600
9	45	450	600
10	50	500	600
11	55	550	800
12	60	600	800
13	65	650	800
14	70	700	800
15	75	750	800
16	80	800	900
17	85	850	950
18	90	900	1000
19	95	950	1000
20	100	1000	1000
21	105	1050	1000
22	110	1100	1000
23	115	1150	1000
24	120	1200	1000
25	125	1250	1150
26	130	1300	1150
27	135	1350	1150
28	140	1400	1150
29	145	1450	1150
30	150	1500	1150
31	155	1550	1150

A graph exists for each of the three tasks in this example. Each graph has a line with the expected average as a function of hours, and the corresponding actual value. When the actual value for a task meets the expected value then it is a convergence for that task. A convergence point occurs when all of the tasks converge at the same time. At least three convergence points are required, one of which is the stopping point T. In our example, the total test time is 400 hours.

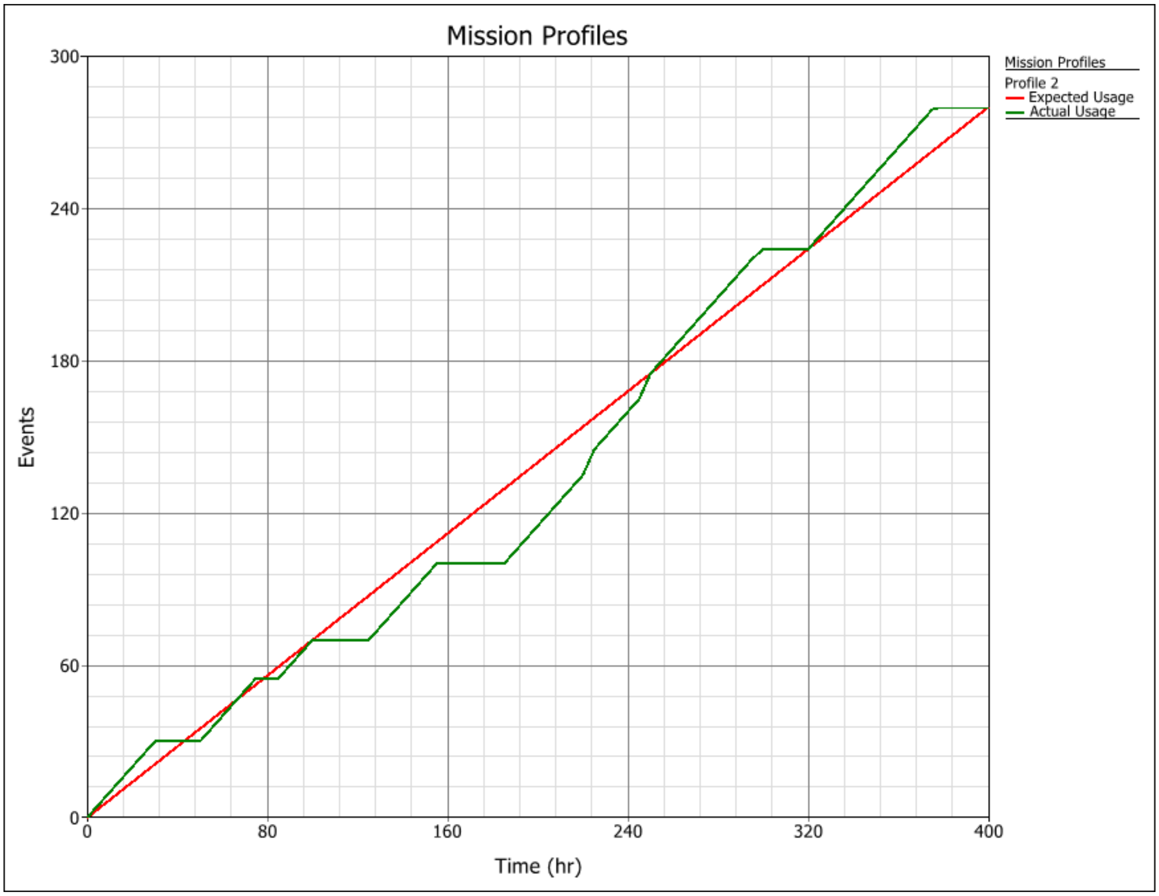
The convergence points were chosen to be at 100, 250, 320 and 400 hours. The next figure shows the data sheet that contains the convergence points in the Weibull++ software.

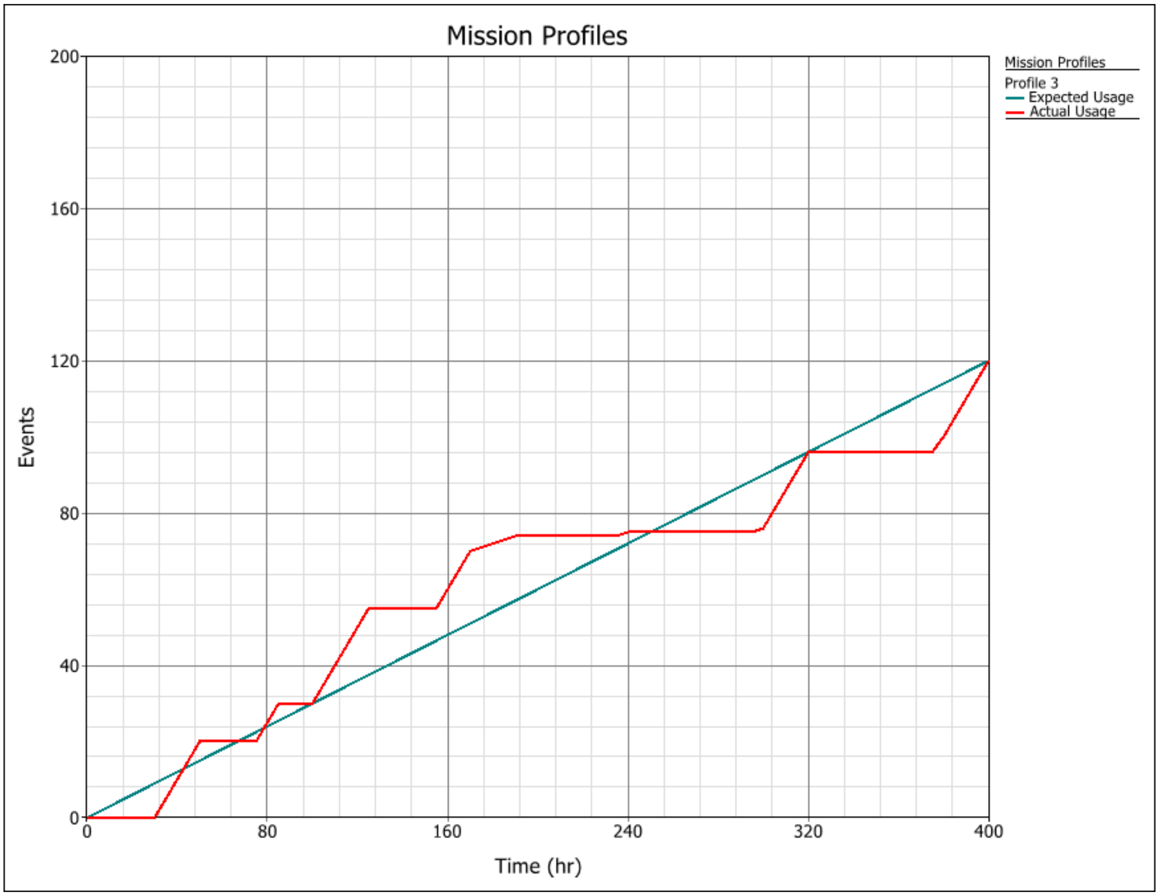


The testing profiles are managed so that at these times the actual operational test profile equals the expected values for the three tasks or falls within an acceptable range. The next graph shows the expected and actual gun firings.

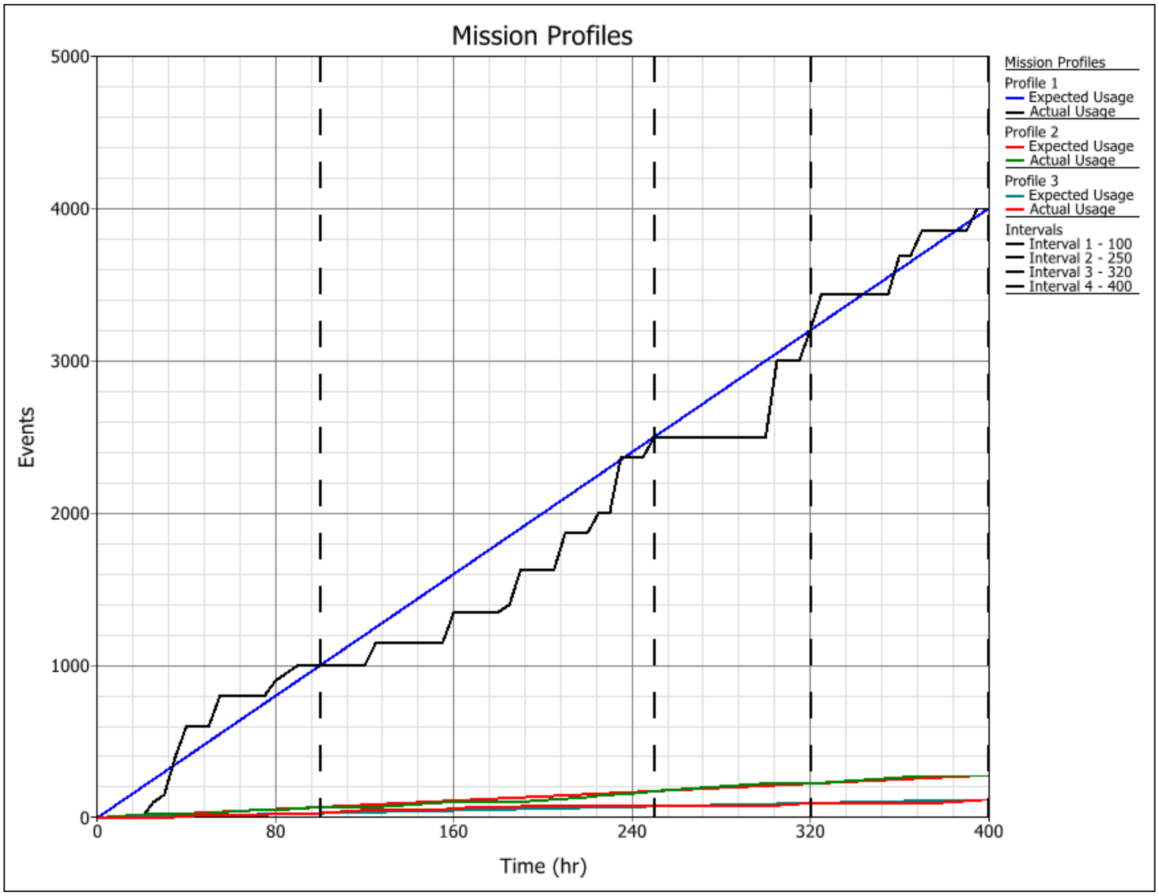


While the next two graphs show the expected and actual time in environments E1 and E2, respectively.





The objective of having the convergence points is to be able to apply the Crow extended model directly in such a way that the projection and other key reliability growth parameters can be estimated in a valid fashion. To do this, grouped data is applied using the Crow extended model. For reliability growth assessments using grouped data, only the information between time points in the testing is used. In our application, these time points are the convergence points 100, 250, 320, and 400. The next figure shows all three mission profiles plotted in the same graph, together with the convergence points.



The following table gives the grouped data input corresponding to the original data set.

Grouped Data at Convergence Points 100, 250, 320 and 400 Hours								
Number at Event	Time to Event	Classification	Mode		Number at Event	Time to Event	Classification	Mode
3	100	BC	17		1	250	BC	26
1	100	BD	1		1	250	BD	11
1	100	BC	18		1	250	BD	12
1	100	BD	2		3	320	A	
1	100	BD	3		1	320	BD	1
1	100	BD	4		1	320	BD	8
1	100	BC	19		1	320	BD	6

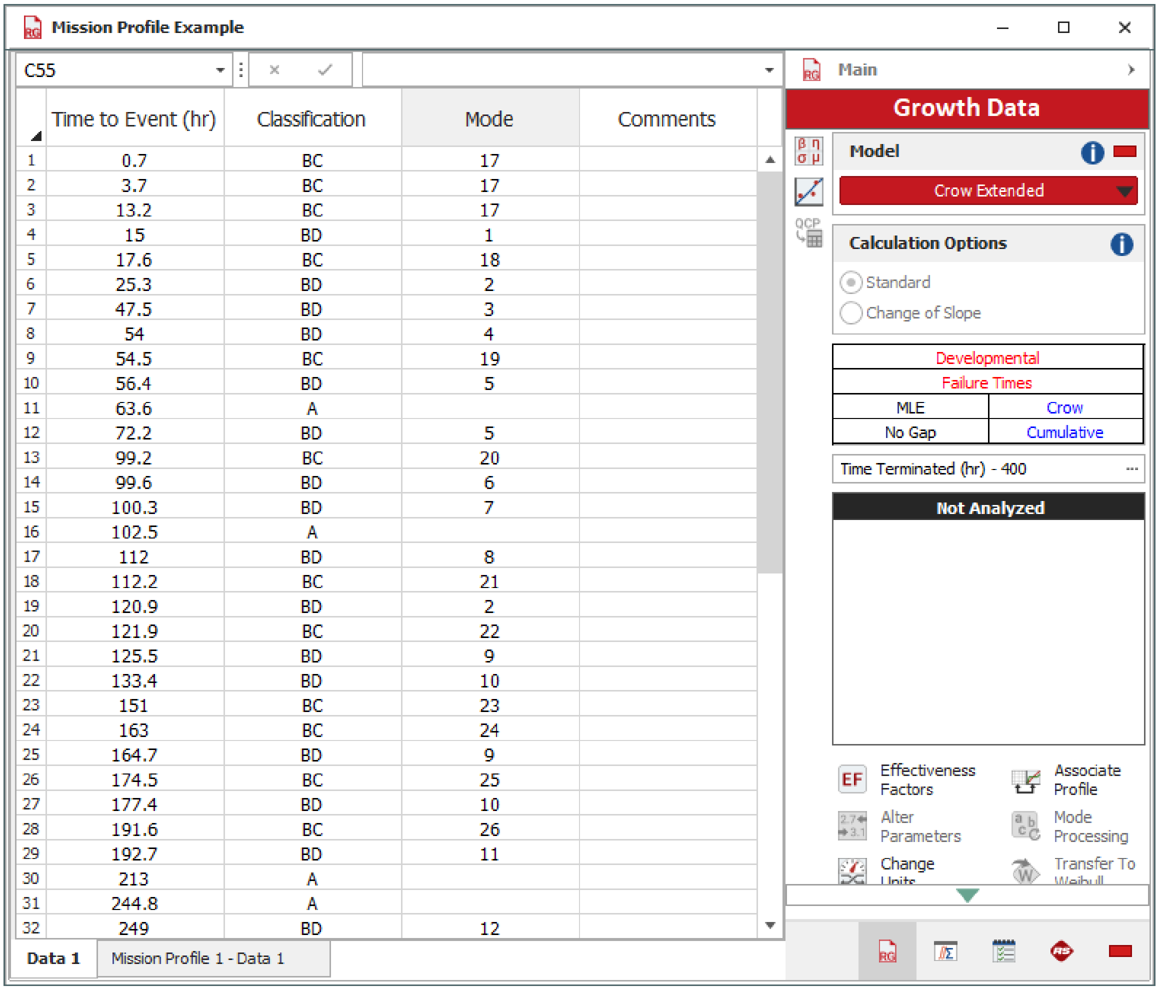
2	100	BD	5		1	320	BC	27
1	100	A			1	320	BD	13
1	100	BC	20		1	320	BD	9
1	100	BD	6		1	320	BD	4
1	250	BD	7		3	400	A	
3	250	A			1	400	BD	12
1	250	BD	8		2	400	BD	10
1	250	BC	21		1	400	BD	5
1	250	BD	2		1	400	BD	3
1	250	BC	22		1	400	BC	28
2	250	BD	9		1	400	BD	2
2	250	BD	10		1	400	BD	8
1	250	BC	23		1	400	BD	14
1	250	BC	24		1	400	BD	15
1	250	BC	25		1	400	BD	16

The parameters of the Crow extended model for grouped data are then estimated, as explained in the Grouped Data section of the Crow Extended chapter. The following table shows the effectiveness factors (EFs) for the BD modes.

Effectiveness Factors for Delayed Fixes	
Mode	**Effectiveness Factor**
1	0.67
2	0.72
3	0.77
4	0.77

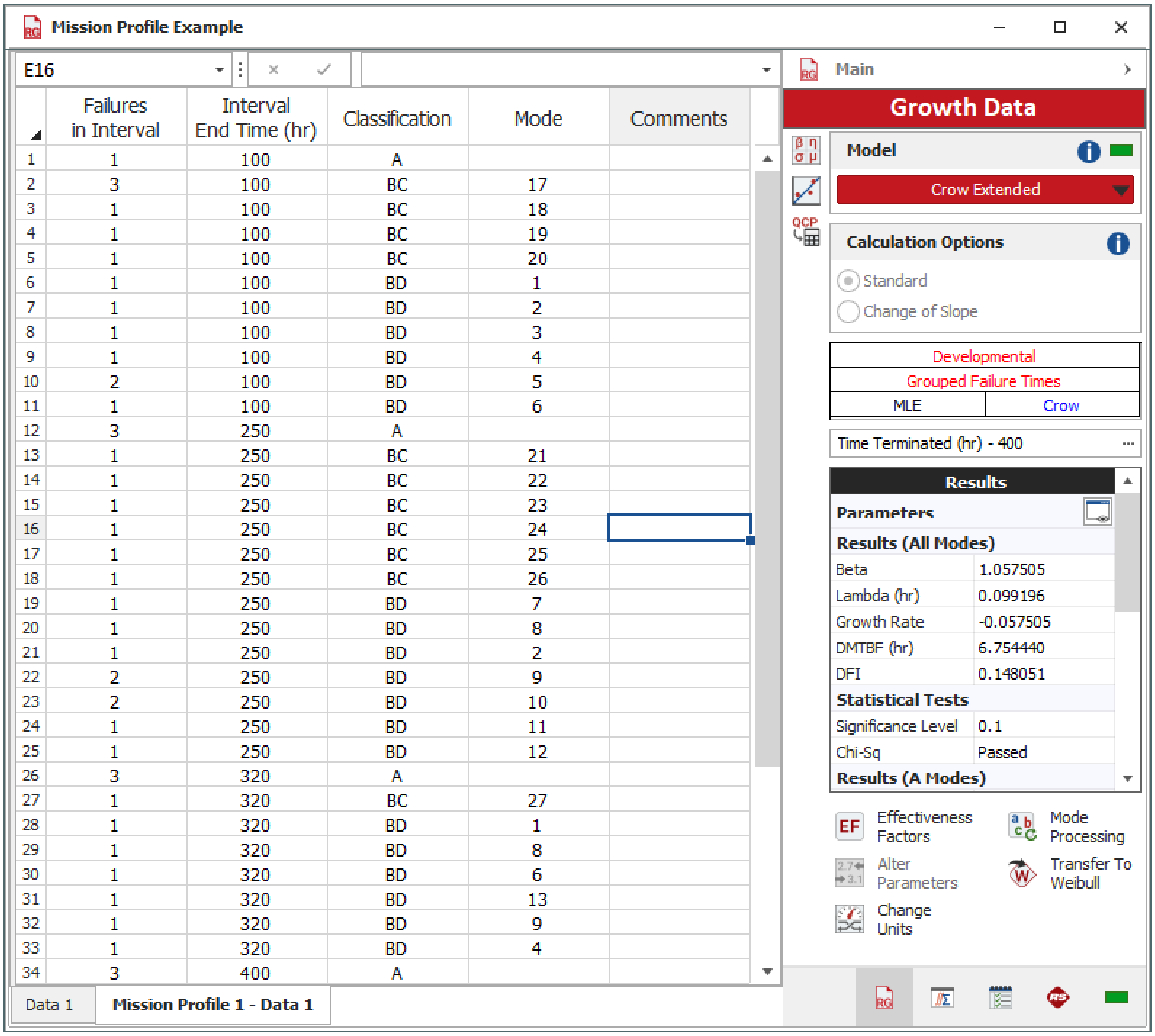
5	0.87
6	0.92
7	0.50
8	0.85
9	0.89
10	0.74
11	0.70
12	0.63
13	0.64
14	0.72
15	0.69
16	0.46

Using the failure times data sheet shown next, we can analyze this data set based on a specified mission profile. This will group the failure times data into groups based on the convergence points that have already been specified when constructing the mission profile.



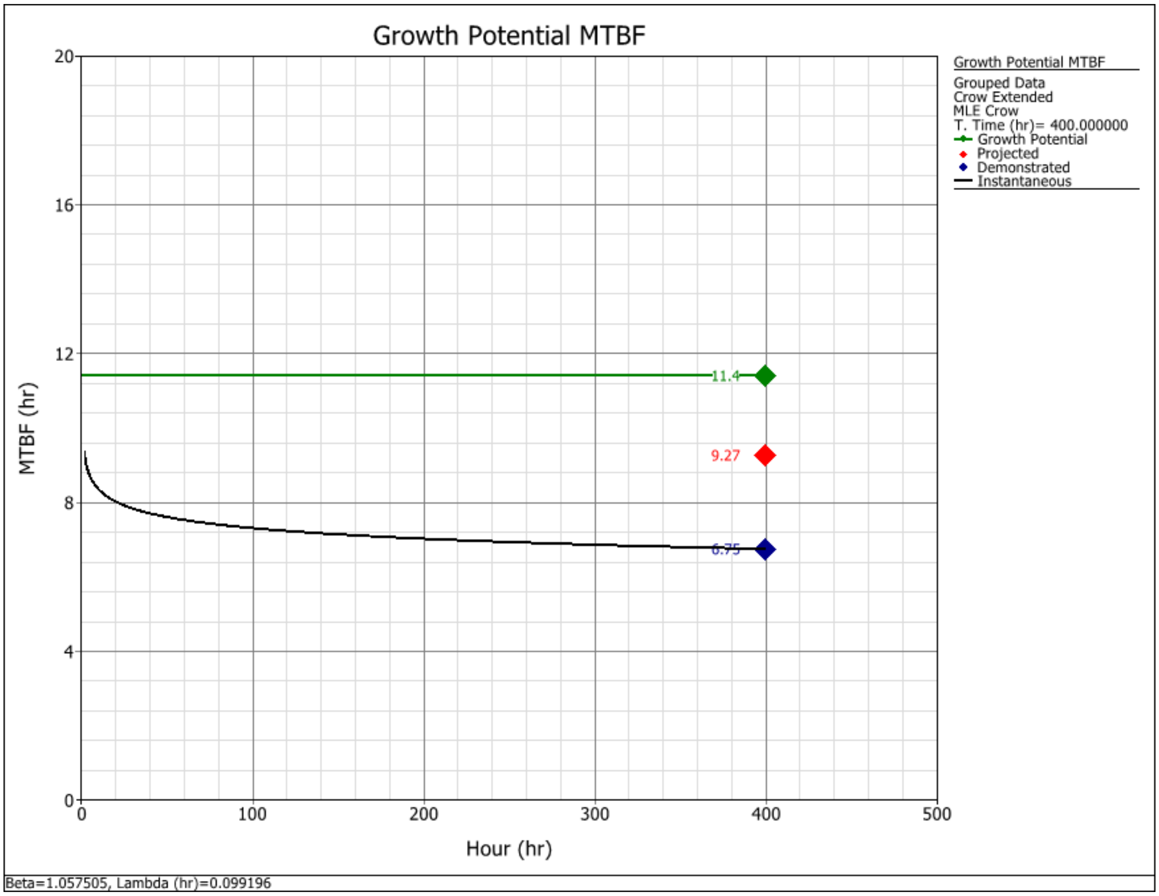
	Time to Event (hr)	Classification	Mode	Comments
1	0.7	BC	17	
2	3.7	BC	17	
3	13.2	BC	17	
4	15	BD	1	
5	17.6	BC	18	
6	25.3	BD	2	
7	47.5	BD	3	
8	54	BD	4	
9	54.5	BC	19	
10	56.4	BD	5	
11	63.6	A		
12	72.2	BD	5	
13	99.2	BC	20	
14	99.6	BD	6	
15	100.3	BD	7	
16	102.5	A		
17	112	BD	8	
18	112.2	BC	21	
19	120.9	BD	2	
20	121.9	BC	22	
21	125.5	BD	9	
22	133.4	BD	10	
23	151	BC	23	
24	163	BC	24	
25	164.7	BD	9	
26	174.5	BC	25	
27	177.4	BD	10	
28	191.6	BC	26	
29	192.7	BD	11	
30	213	A		
31	244.8	A		
32	249	BD	12	

A new data sheet with the grouped data is created, as shown in the figure below and the calculated results based on the grouped data are as follows:



	Failures in Interval	Interval End Time (hr)	Classification	Mode	Comments
1	1	100	A		
2	3	100	BC	17	
3	1	100	BC	18	
4	1	100	BC	19	
5	1	100	BC	20	
6	1	100	BD	1	
7	1	100	BD	2	
8	1	100	BD	3	
9	1	100	BD	4	
10	2	100	BD	5	
11	1	100	BD	6	
12	3	250	A		
13	1	250	BC	21	
14	1	250	BC	22	
15	1	250	BC	23	
16	1	250	BC	24	
17	1	250	BC	25	
18	1	250	BC	26	
19	1	250	BD	7	
20	1	250	BD	8	
21	1	250	BD	2	
22	2	250	BD	9	
23	2	250	BD	10	
24	1	250	BD	11	
25	1	250	BD	12	
26	3	320	A		
27	1	320	BC	27	
28	1	320	BD	1	
29	1	320	BD	8	
30	1	320	BD	6	
31	1	320	BD	13	
32	1	320	BD	9	
33	1	320	BD	4	
34	3	400	A		

The following plot shows the instantaneous, demonstrated, projected and growth potential MTBF for the grouped data, based the mission profile grouping with intervals at the specified convergence points of the mission profile.


Growth Potential MTBF
MTBF (hr)
Hour (hr)
Growth Potential MTBF
Grouped Data
Crow Extended
MLE Crow
T. Time (hr)= 400.000000
Growth Potential
Projected
Demonstrated
Instantaneous
11.4
9.27
6.75
Beta=1.057505, Lambda (hr)=0.099196

Fielded Systems

IN THIS CHAPTER

When a complex system with new technology is fielded and subjected to a customer use environment, there is often considerable interest in assessing its reliability and other related performance metrics, such as availability. This interest in evaluating the system reliability based on actual customer usage failure data may be motivated by a number of factors. For example, the reliability that is generally measured during development is typically related to the system's inherent reliability capability. This inherent capability may differ from actual use experience because of different operating conditions or environment, different maintenance policies, different levels of experience of maintenance personnel, etc. Although operational tests are conducted for many systems during development, it is generally recognized that in many cases these tests may not yield complete data representative of an actual use environment. Moreover, the testing during development is typically limited by the usual cost and schedule constraints, which prevent obtaining a system's reliability profile over an extended portion of its life. Other interests in measuring the reliability of a fielded system may center on, for example, logistics and maintenance policies, quality and manufacturing issues, burn-in, wearout, mission reliability or warranties.

Most complex systems are repaired, not replaced, when they fail. For example, a complex communication system or a truck would be repaired upon failure, not thrown away and replaced by a new system. A number of books and papers in literature have stressed that the usual non-repairable reliability analysis methodologies, such as the Weibull distribution, are not appropriate for repairable system reliability analyses and have suggested the use of non-homogeneous Poisson process models instead.

The homogeneous process is equivalent to the widely used Poisson distribution, and exponential times between system failures can be modeled appropriately when the system's failure intensity is not affected by the system's age. However, to realistically consider burn-in, wearout, useful life, maintenance policies, warranties, mission reliability, etc., the analyst will often require an approach that recognizes that the failure intensity of these systems may not be constant over the operating life of interest but may change with system age. A useful, and generally practical, extension of the homogeneous Poisson process, is the non-homogeneous Poisson process, which allows for the system failure intensity to change with system age. Typically, the reliability analysis of a repairable system under customer use will involve data generated by multiple systems. Crow [17] proposed the Weibull process or power law non-homogeneous Poisson process for this type of analysis, and developed appropriate statistical procedures for maximum likelihood estimation, goodness-of-fit and confidence bounds.

Applicable Analyses

The following chapters contain additional information on each of the analyses that can be used with data from fielded systems:

- Repairable Systems Analysis
- Operational Testing
- Fleet Data Analysis

Repairable Systems Analysis

Data from systems in the field can be analyzed in the Weibull++ software. This type of data is called *fielded systems data* and is analogous to warranty data. Fielded systems can be categorized into two basic types: one-time or non-repairable systems, and reusable or repairable systems. In the latter case, under continuous operation, the system is repaired, but not replaced after each failure. For example, if a water pump in a vehicle fails, the water pump is replaced and the vehicle is repaired.

This chapter presents repairable systems analysis, where the reliability of a system can be tracked and quantified based on data from multiple systems in the field. The next chapter will present fleet analysis, where data from multiple systems in the field can be collected and analyzed so that reliability metrics for the fleet as a whole can be quantified.

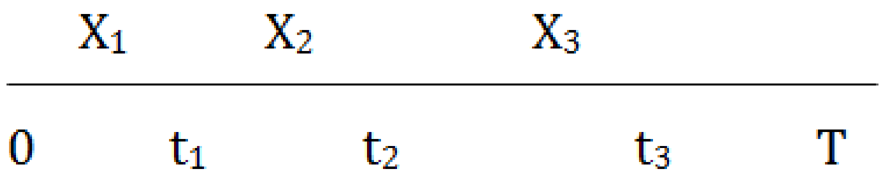
Background

Most complex systems, such as automobiles, communication systems, aircraft, printers, medical diagnostics systems, helicopters, etc., are repaired and not replaced when they fail. When these systems are fielded or subjected to a customer use environment, it is often of considerable interest to determine the reliability and other performance characteristics under these conditions. Areas of interest may include assessing the expected number of failures during the warranty period, maintaining a minimum mission reliability, evaluating the rate of wearout, determining when to replace or overhaul a system and minimizing life cycle costs. In general, a lifetime distribution, such as the Weibull distribution, cannot be used to address these issues. In order to address the reliability characteristics of complex repairable systems, a process is often used instead of a distribution. The most popular process model is the Power Law model. This model is popular for several reasons. One is that it has a very practical foundation in terms of minimal repair, which is a situation where the repair of a failed system is just enough to get the system operational again. Second, if the time to first failure follows the Weibull distribution, then each succeeding failure is governed by the Power Law model as in the case of minimal repair. From this point of view, the Power Law model is an extension of the Weibull distribution.

Sometimes, the Crow Extended model, which was introduced in a previous chapter for analyzing developmental data, is also applied for fielded repairable systems. Applying the Crow Extended model on repairable system data allows analysts to project the system MTBF after reliability-related issues are addressed during the field operation. Projections are calculated based on the mode classifications (A, BC and BD). The calculation procedure is the same as the one for the developmental data, and is not repeated in this chapter.

Distribution Example

Visualize a socket into which a component is inserted at time 0. When the component fails, it is replaced immediately with a new one of the same kind. After each replacement, the socket is put back into an *as good as new* condition. Each component has a time-to-failure that is determined by the underlying distribution. It is important to note that a distribution relates to a single failure. The sequence of failures for the socket constitutes a random process called a *renewal process*. In the illustration below, the component life is X_j , and t_j is the system time to the j^{th} failure.



Each component life X_j in the socket is governed by the same distribution $F(x)$.

A distribution, such as Weibull, governs a single lifetime. There is only one event associated with a distribution. The distribution $F(x)$ is the probability that the life of the component in the socket is less than x . In the illustration above, X_1 is the life of the first component in the socket. $F(x)$ is the probability that the first component in the socket fails in time x . When the first component fails, it is replaced in the socket with a new component of the same type. The probability that the life of the second component is less than x is given by the same distribution function, $F(x)$. For the Weibull distribution:

$$F(x) = 1 - e^{-\lambda x^{\beta}}$$

A distribution is also characterized by its density function, such that:

$$f(x) = \frac{d}{dx}F(x)$$

The density function for the Weibull distribution is:

$$f(x) = \lambda\beta x^{\beta-1} \cdot e^{-\lambda\beta x}$$

In addition, an important reliability property of a distribution function is the failure rate, which is given by:

$$r(x) = \frac{f(x)}{1 - F(x)}$$

The interpretation of the failure rate is that for a small interval of time Δx, $r(x)\Delta x$ is approximately the probability that a component in the socket will fail between time x and time $x+\Delta x$, given that the component has not failed by time x. For the Weibull distribution, the failure rate is given by:

$$r(x)=\lambda\beta x^{\beta-1}$$

It is important to note the condition that the component has not failed by time x. Again, a distribution deals with one lifetime of a component and does not allow for more than one failure. The socket has many failures and each failure time is individually governed by the same distribution. In other words, the failure times are independent of each other. If the failure rate is increasing, then this is indicative of component wearout. If the failure rate is decreasing, then this is indicative of infant mortality. If the failure rate is constant, then the component failures follow an exponential distribution. For the Weibull distribution, the failure rate is increasing for $\beta>1$, decreasing for $\beta<1$ and constant for $\beta=1$. Each time a component in the socket is replaced, the failure rate of the new component goes back to the value at time 0. This means that the socket is as good as new after each failure and each subsequent replacement by a new component. This process is continued for the operation of the socket.

Process Example

Now suppose that a system consists of many components with each component in a socket. A failure in any socket constitutes a failure of the system. Each component in a socket is a renewal process governed by its respective distribution function. When the system fails due to a failure in a socket, the component is replaced and the socket is again as good as new. The system has been repaired. Because there are many other components still operating with various ages, the system is not typically put back into a like new condition after the replacement of a single component. For example, a car is not as good as new after the replacement of a failed water pump. Therefore, distribution theory does not apply to the failures of a complex system, such as a car. In general, the intervals between failures for a complex repairable system do not follow the same distribution. Distributions apply to the components that are replaced in the sockets, but not at the system level. At the system level, a distribution applies to the very first failure. There is one failure associated with a distribution. For example, the very first system failure may follow a Weibull distribution.

For many systems in a real world environment, a repair may only be enough to get the system operational again. If the water pump fails on the car, the repair consists only of installing a new water pump. Similarly, if a seal leaks, the seal is replaced but no additional maintenance is done. This is the concept of *minimal repair*. For a system with many failure modes, the repair of a single failure mode does not greatly improve the system reliability from what it was just before the failure. Under minimal repair for a complex system with many failure modes, the system reliability after a repair is the same as it was just before the failure. In this case, the sequence of failures at the system level follows a non-homogeneous Poisson process (NHPP).

The system age when the system is first put into service is time 0. Under the NHPP, the first failure is governed by a distribution $F(x)$ with failure rate $r(x)$. Each succeeding failure is governed by the intensity function $u(t)$ of the process. Let t be the age of the system and Δt is very small. The probability that a system of age t fails between t and $t+\Delta t$ is given by the intensity function $u(t)\Delta t$. Notice that this probability is not conditioned on not having any system failures up to time t, as is the case for a failure rate. The failure intensity $u(t)$ for the NHPP has the same functional form as the failure rate governing the first system failure. Therefore, $u(t)=r(t)$, where $r(t)$ is the failure rate for the distribution function of the first system failure. If the first system failure follows the Weibull distribution, the failure rate is:

$$r(x)=\lambda\beta x^{\beta-1}$$

Under minimal repair, the system intensity function is:

$$u(t)=\lambda\beta t^{\beta-1}$$

This is the Power Law model. It can be viewed as an extension of the Weibull distribution. The Weibull distribution governs the first system failure, and the Power Law model governs each succeeding system failure. If the system has a constant failure intensity $u(t)=\lambda$, then the intervals between system failures follow an exponential distribution with failure rate λ. If the system operates for time T, then the random number of failures $N(T)$ over 0 to T is given by the Power Law mean value function.

$$E[N(T)]=\lambda T^{\beta}$$

Therefore, the probability $N(T)=n$ is given by the Poisson probability.

$$\frac{(\lambda T)^{n}e^{-\lambda T}}{n!};\ n=0,1,2\ldots$$

This is referred to as a *homogeneous Poisson process* because there is no change in the intensity function. This is a special case of the Power Law model for $\beta=1$. The Power Law model is a generalization of the homogeneous Poisson process and allows for change in the intensity function as the repairable system ages. For the Power Law model, the failure intensity is increasing for $\beta>1$ (wearout), decreasing for $\beta<1$ (infant mortality) and constant for $\beta=1$ (useful life).

Power Law Model

The Power Law model is often used to analyze the reliability of complex repairable systems in the field. The system of interest may be the total system, such as a helicopter, or it may be subsystems, such as the helicopter transmission or rotator blades. When these systems are new and first put into operation, the start time is 0. As these systems are operated, they accumulate age (e.g., miles on automobiles, number of pages on copiers, flights of helicopters). When these systems fail, they are repaired and put back into service.

Some system types may be overhauled and some may not, depending on the maintenance policy. For example, an automobile may not be overhauled but helicopter transmissions may be overhauled after a period of time. In practice, an overhaul may not convert the system reliability back to where it was when the system was new. However, an overhaul will generally make the system more reliable. Appropriate data for the Power Law model is over cycles. If a system is not overhauled, then there is only one cycle and the zero time is when the system is first put into operation. If a system is overhauled, then the same serial number system may generate many cycles. Each cycle will start a new zero time, the beginning of the cycle. The age of the system is from the beginning of the cycle. For systems that are not overhauled, there is only one cycle and the reliability characteristics of a system as the system ages during its life is of interest. For systems that are overhauled, you are interested in the reliability characteristics of the system as it ages during its cycle.

For the Power Law model, a data set for a system will consist of a starting time S, an ending time T and the accumulated ages of the system during the cycle when it had failures. Assume that the data exists from the beginning of a cycle (i.e., the starting time is 0), although non-zero starting times are possible with the Power Law model. For example, suppose data has been collected for a system with 2,000 hours of operation during a cycle. The starting time is $S=0$ and the ending time is $T=2000$. Over this period, failures occurred at system ages of 50.6, 840.7, 1060.5, 1186.5, 1613.6 and 1843.4 hours. These are the accumulated operating times within the cycle, and there were no failures between 1843.4 and 2000 hours. It may be of interest to determine how the systems perform as part of a fleet. For a fleet, it must be verified that the systems have the same configuration, same maintenance policy and same operational environment. In this case, a random sample must be gathered from the fleet. Each item in the sample will have a cycle starting time $S=0$, an ending time T for the data period and the accumulated operating times during this period when the system failed.

There are many ways to generate a random sample of K systems. One way is to generate K random serial numbers from the fleet. Then go to the records corresponding to the randomly selected systems. If the systems are not overhauled, then record when each system was first put into service. For example, the system may have been put into service when the odometer mileage equaled zero. Each system may have a different amount of total usage, so the ending times, T, may be different. If the systems are overhauled, then the records for the last completed cycle will be needed. The starting and ending times and the accumulated times to failure for the K systems constitute the random sample from the fleet. There is a useful and efficient method for generating a random sample for systems that are overhauled. If the overhauled systems have been in service for a considerable period of time, then each serial number system in the fleet would go through many overhaul cycles. In this case, the systems coming in for overhaul actually represent a random sample from the fleet. As K systems come in for overhaul, the data for the current completed cycles would be a random sample of size K.

In addition, the warranty period may be of particular interest. In this case, randomly choose K serial numbers for systems that have been in customer use for a period longer than the warranty period. Then check the warranty records. For each of the K systems that had warranty work, the ages corresponding to this service are the failure times. If a system did not have warranty work, then the number of failures recorded for that system is zero. The starting times are all equal to zero and the ending time for each of the K systems is equal to the warranty operating usage time (e.g., hours, copies, miles).

In addition to the intensity function $u(t)$ and the mean value function, which were given in the section above, other relationships based on the Power Law are often of practical interest. For example, the probability that the system will survive to age $t+d$ without failure is given by:

$$R(t) = e^{-\left[\lambda(t+d)^{\beta} - \lambda t^{\beta}\right]}$$

This is the mission reliability for a system of age t and mission length d.

Parameter Estimation

Suppose that the number of systems under study is K and the q^{th} system is observed continuously from time S_q to time T_q, $q = 1, 2, \ldots, K$. During the period $[S_q, T_q]$, let N_q be the number of failures experienced by the q^{th} system and let $X_{i,q}$ be the age of this system at the i^{th} occurrence of failure, $i = 1, 2, \ldots, N_q$. It is also possible that the times S_q and T_q may be the observed failure times for the q^{th} system. If $X_{N_q,q} = T_q$, then the data on the q^{th} system is said to be failure terminated, and T_q is a random variable with N_q fixed. If $X_{N_q,q} < T_q$, then the data on the q^{th} system is said to be time terminated with N_q a random variable. The maximum likelihood estimates of λ and β are values satisfying the equations shown next.

$$\widehat{\lambda} = \frac{\sum^{K}{}_{q=1} N_q}{\sum^{K}{}_{q=1}\left(T_q^{\widehat{\beta}} - S_q^{\widehat{\beta}}\right)}$$
$$\widehat{\beta} = \frac{\sum^{K}{}_{q=1} N_q}{\widehat{\lambda}\sum^{K}{}_{q=1}\left[T_q^{\widehat{\beta}}\ln(T_q) - S_q^{\widehat{\beta}}\ln(S_q)\right] - \sum^{K}{}_{q=1}\sum^{N_q}{}_{i=1}\ln(X_{i,q})}$$

where $0 \ln 0$ is defined to be 0. In general, these equations cannot be solved explicitly for $\widehat{\lambda}$ and $\widehat{\beta}$, but must be solved by iterative procedures. Once $\widehat{\lambda}$ and $\widehat{\beta}$ have been estimated, the maximum likelihood estimate of the intensity function is given by:

$$\widehat{u}(t) = \widehat{\lambda}\widehat{\beta}t^{\widehat{\beta}-1}$$

If $S_1 = S_2 = \ldots = S_q = 0$ and $T_1 = T_2 = \ldots = T_q$ $(q = 1, 2, \ldots, K)$ then the maximum likelihood estimates $\widehat{\lambda}$ and $\widehat{\beta}$ are in closed form.

$$\widehat{\lambda} = \frac{\sum^{K}{}_{q=1} N_q}{KT^{\beta}}$$
$$\widehat{\beta} = \frac{\sum^{K}{}_{q=1} N_q}{\sum^{K}{}_{q=1}\sum^{N_q}{}_{i=1}\ln(\frac{T}{X_{iq}})}$$

The following example illustrates these estimation procedures.

Power Law Model Example

For the data in the following table, the starting time for each system is equal to 0 and the ending time for each system is 2,000 hours. Calculate the maximum likelihood estimates $\widehat{\lambda}$ and $\widehat{\beta}$.

Repairable System Failure Data		
System 1 (X_{i1})	**System 2 (X_{i2})**	**System 3 (X_{i3})**
1.2	1.4	0.3
55.6	35.0	32.6
72.7	46.8	33.4
111.9	65.9	241.7
121.9	181.1	396.2
303.6	712.6	444.4

326.9	1005.7	480.8
1568.4	1029.9	588.9
1913.5	1675.7	1043.9
	1787.5	1136.1
	1867.0	1288.1
		1408.1
		1439.4
		1604.8
$N_1 = 9$	$N_2 = 11$	$N_3 = 14$

Solution

Because the starting time for each system is equal to zero and each system has an equivalent ending time, the general equations for $\widehat{\beta}$ and $\widehat{\lambda}$ reduce to the closed form equations. The maximum likelihood estimates of $\hat{\beta}$ and $\hat{\lambda}$ are then calculated as follows:

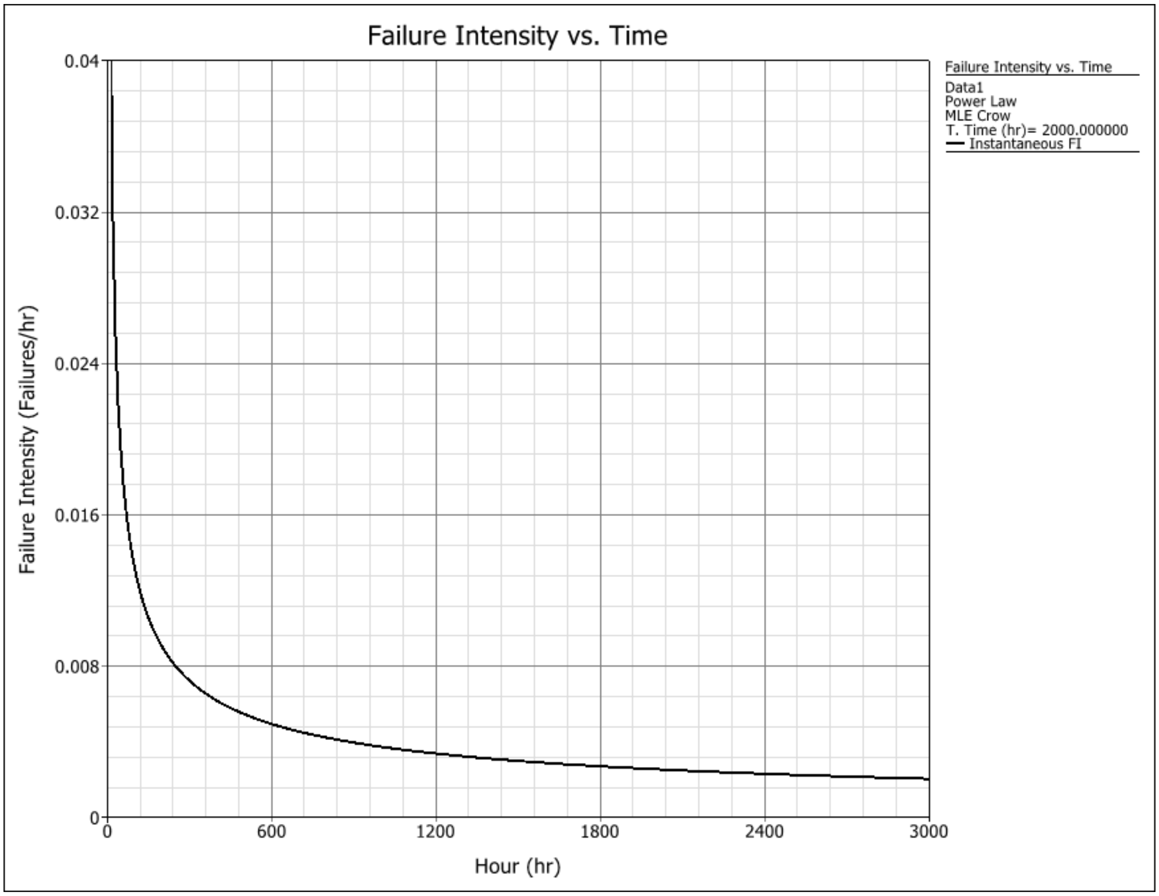
$$\widehat{\beta} = \frac{\sum^{K}{}_{q=1} N_q}{\sum^{K}{}_{q=1} \sum^{N_q}{}_{i=1} \ln(\frac{T}{X_{iq}})} = 0.45300$$

$$\widehat{\lambda} = \frac{\sum^{K}{}_{q=1} N_q}{KT^{\beta}} = 0.36224$$

The system failure intensity function is then estimated by:

$$\widehat{u}(t) = \widehat{\lambda}\widehat{\beta}t^{\widehat{\beta}-1},\ t > 0$$

The next figure is a plot of $\widehat{u}(t)$ over the period (0, 3000). Clearly, the estimated failure intensity function is most representative over the range of the data and any extrapolation should be viewed with the usual caution.



Goodness-of-Fit Tests for Repairable System Analysis

It is generally desirable to test the compatibility of a model and data by a statistical goodness-of-fit test. A parametric Cramer-von Mises goodness-of-fit test is used for the multiple system and repairable system Power Law model, as proposed by Crow in [17]. This goodness-of-fit test is appropriate whenever the start time for each system is 0 and the failure data is complete over the continuous interval $[0, T_q]$ with no gaps in the data. The Chi-Squared test is a goodness-of-fit test that can be applied under more general circumstances. In addition, the Common Beta Hypothesis test also can be used to compare the intensity functions of the individual systems by comparing the β_q values of each system. Lastly, the Laplace Trend test checks for trends within the data. Due to their general application, the Common Beta Hypothesis test and the Laplace Trend test are both presented in Appendix B. The Cramer-von Mises and Chi-Squared goodness-of-fit tests are illustrated next.

Cramer-von Mises Test

To illustrate the application of the Cramer-von Mises statistic for multiple systems data, suppose that K like systems are under study and you wish to test the hypothesis H_1 that their failure times follow a non-homogeneous Poisson process. Suppose information is available for the q^{th} system over the interval $[0, T_q]$, with successive failure times , $(q = 1, 2, \ldots, K)$. The Cramer-von Mises test can be performed with the following steps:

Step 1: If $x_{N_q q} = T_q$ (failure terminated), let $M_q = N_q - 1$, and if $x_{N_q q} < T$ (time terminated), let $M_q = N_q$. Then:

$$M = \sum_{q=1}^{K} M_q$$

Step 2: For each system, divide each successive failure time by the corresponding end time T_q , $i = 1, 2, \ldots, M_q$. Calculate the M values:

$$Y_{iq} = \frac{X_{iq}}{T_q}, i = 1, 2, \ldots, M_q,\ q = 1, 2, \ldots, K$$

Step 3: Next calculate $\bar{\beta}$, the unbiased estimate of β , from:

$$\bar{\beta} = \frac{M - 1}{\sum^{K}{}_{q=1} \sum^{Mq}{}_{i=1} \ln\left(\frac{T_q}{X_{iq}}\right)}$$

Step 4: Treat the Y_{iq} values as one group, and order them from smallest to largest. Name these ordered values $z_1, z_2, \ldots, z_M$, such that $z_1 < \ z_2 < \ldots < z_M$.

Step 5: Calculate the parametric Cramer-von Mises statistic.

$$C_M^2 = \frac{1}{12M} + \sum_{j=1}^{M} (Z_j^{\bar{\beta}} - \frac{2j - 1}{2M})^2$$

Critical values for the Cramer-von Mises test are presented in the Crow-AMSAA (NHPP) page.

Step 6: If the calculated C_M^2 is less than the critical value, then accept the hypothesis that the failure times for the K systems follow the non-homogeneous Poisson process with intensity function $u(t) = \lambda \beta t^{\beta - 1}$.

Chi-Squared Test

The parametric Cramer-von Mises test described above requires that the starting time, S_q , be equal to 0 for each of the K systems. Although not as powerful as the Cramer-von Mises test, the chi-squared test can be applied regardless of the starting times. The expected number of fail-

$\widehat{\lambda} b^{\widehat{\beta}} - \widehat{\lambda} a^{\widehat{\beta}} = \hat{\theta}$, where $\widehat{\lambda}$ and $\widehat{\beta}$ are the maximum likelihood estimates.

The computed χ^2 statistic is:

$$\chi^2 = \sum_{j=1}^{d} \frac{[N(j) - \theta(j)]^2}{\hat{\theta}(j)}$$

where d is the total number of intervals. The random variable χ^2 is approximately chi-square distributed with $df = d - 2$ degrees of freedom. There must be at least three intervals and the length of the intervals do not have to be equal. It is common practice to require that the expected number of failures for each interval, $\theta(j)$, be at least five. If $\chi_0^2 > \chi^2_{\alpha/2, d-2}$ or if $\chi_0^2 < \chi^2_{1-(\alpha/2), d-2}$, reject the null hypothesis.

Cramer-von Mises Example

For the data from power law model example given above, use the Cramer-von Mises test to examine the compatibility of the model at a significance level $\alpha = 0.10$

Solution

Step 1:

$$X_{9,1} = 1913.5 < 2000, \ M_1 = 9$$
$$X_{11,2} = 1867 < 2000, \ M_2 = 11$$
$$X_{14,3} = 1604.8 < 2000, \ M_3 = 14$$
$$M = \sum_{q=1}^{3} M_q = 34$$

Step 2: Calculate Y_{iq}, treat the Y_{iq} values as one group and order them from smallest to largest. Name these ordered values $z_1, z_2, \ldots, z_M$.

Step 3: Calculate:

$$\bar{\beta} = \frac{M-1}{\sum^{K}_{q=1} \sum^{M_q}_{i=1} \ln\left(\frac{T_q}{X_{iq}}\right)} = 0.4397$$

Step 4: Calculate:

$$C_M^2 = \frac{1}{12M} + \sum_{j=1}^{M} \left(Z_j^{\bar{\beta}} - \frac{2j-1}{2M}\right)^2 = 0.0636$$

Step 5: From the table of critical values for the Cramer-von Mises test, find the critical value (CV) for $M=34$ at a significance level $\alpha=0.10\cdot CV=0.172\cdot$

Step 6: Since $C_M^2 < CV$, accept the hypothesis that the failure times for the $K=3$ repairable systems follow the non-homogeneous Poisson process with intensity function $u(t)=\lambda\beta t^{\beta-1}$.

Confidence Bounds for Repairable Systems Analysis

The Weibull++ software provides two methods to estimate the confidence bounds for repairable systems analysis. The Fisher matrix approach is based on the Fisher information matrix and is commonly employed in the reliability field. The Crow bounds were developed by Dr. Larry Crow. See Confidence Bounds for Repairable Systems Analysis for details on how these confidence bounds are calculated.

Confidence Bounds Example

Using the data from the power law model example given above, calculate the mission reliability at $t=2000$ hours and mission time $d=40$ hours along with the confidence bounds at the 90% confidence level.

Solution

The maximum likelihood estimates of $\widehat{\lambda}$ and $\widehat{\beta}$ from the example are:

$$\begin{aligned}\widehat{\beta} =&0.45300\\ \widehat{\lambda} =&0.36224\end{aligned}$$

The mission reliability at $t=2000$ for mission time $d=40$ is:

$$\begin{aligned}\widehat{R}(t) =&e^{-\left[\lambda(t+d)^{\beta}-\lambda t^{\beta}\right]}\\ =&0.90292\end{aligned}$$

At the 90% confidence level and $T=2000$ hours, the Fisher matrix confidence bounds for the mission reliability for mission time $d=40$ are given by:

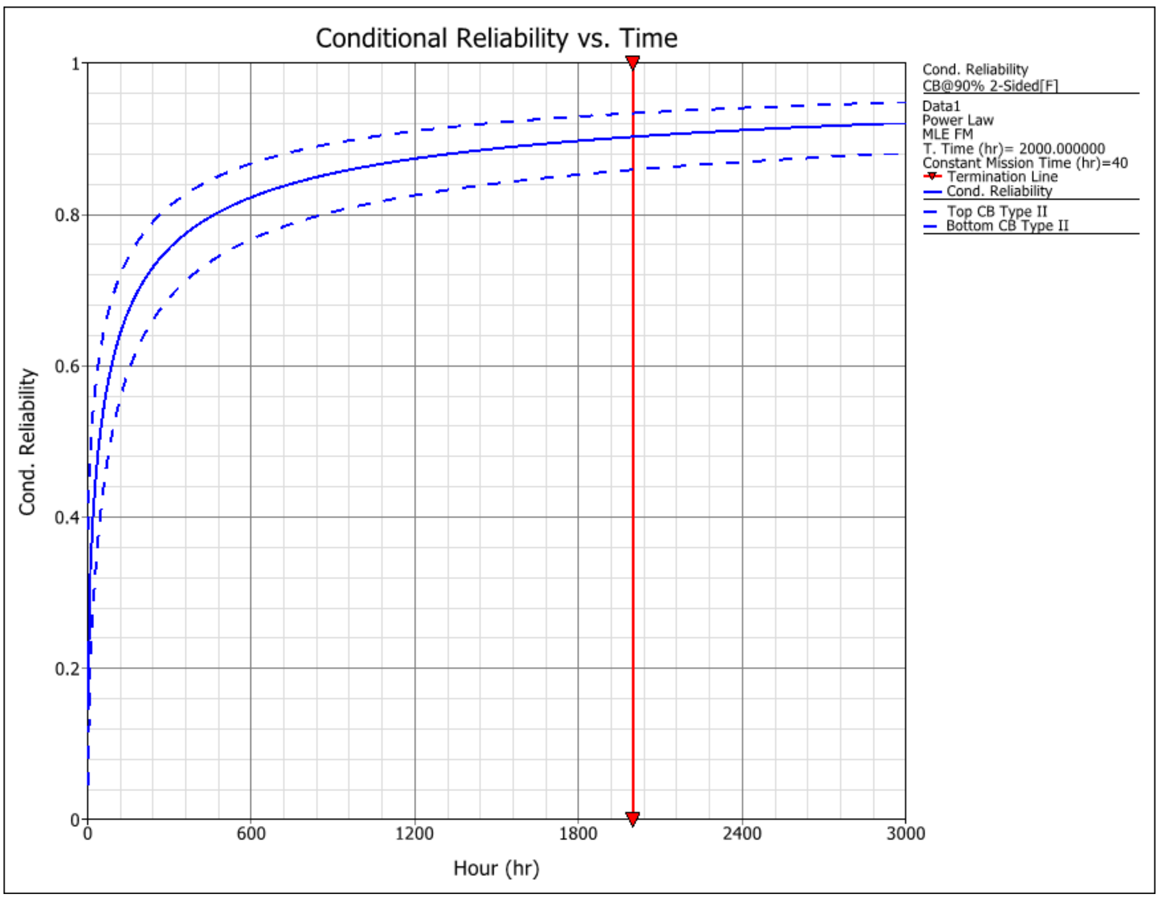
$$CB=\frac{\widehat{R}(t)}{\widehat{R}(t)+(1-\widehat{R}(t))e^{\pm z_{\alpha}\sqrt{Var(\widehat{R}(t))}/\left[\widehat{R}(t)(1-\widehat{R}(t))\right]}}$$

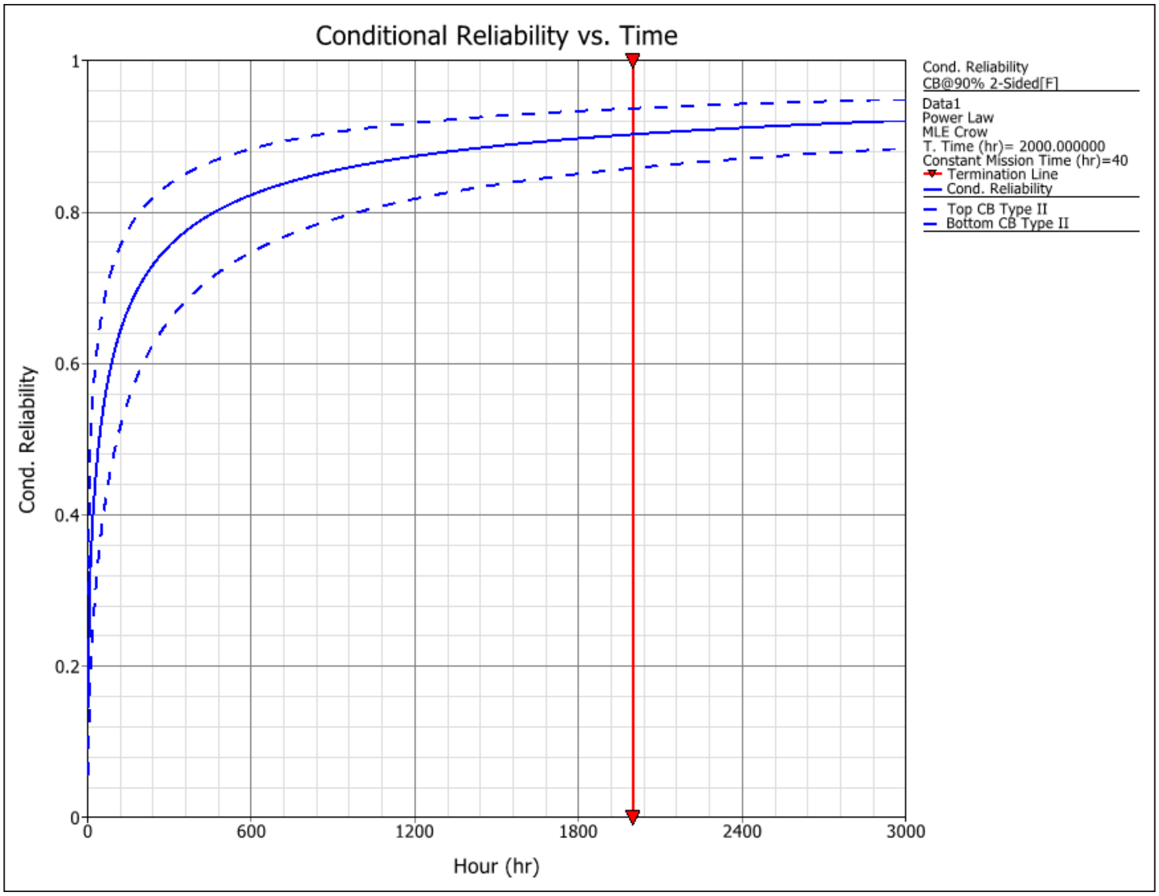
$$\begin{aligned}[\widehat{R}(t)]_L =&0.83711\\ [\widehat{R}(t)]_U =&0.94392\end{aligned}$$

The Crow confidence bounds for the mission reliability are:

$$\begin{aligned}[\widehat{R}(t)]_L &= [\widehat{R}(\tau)]^{\frac{1}{\Pi_1}} \\ &= [0.90292]^{\frac{1}{0.71440}} \\ &= 0.86680 \\ [\widehat{R}(t)]_U &= [\widehat{R}(\tau)]^{\frac{1}{\Pi_2}} \\ &= [0.90292]^{\frac{1}{1.6051}} \\ &= 0.93836\end{aligned}$$

The next two figures show the Fisher matrix and Crow confidence bounds on mission reliability for mission time $d = 40$.





Economical Life Model

One consideration in reducing the cost to maintain repairable systems is to establish an overhaul policy that will minimize the total life cost of the system. However, an overhaul policy makes sense only if $\beta > 1$. It does not make sense to implement an overhaul policy if $\beta < 1$ since wearout is not present. If you assume that there is a point at which it is cheaper to overhaul a system than to continue repairs, what is the overhaul time that will minimize the total life cycle cost while considering repair cost and the cost of overhaul?

Denote C_1 as the average repair cost (unscheduled), C_2 as the replacement or overhaul cost and C_3 as the average cost of scheduled maintenance. Scheduled maintenance is performed for every S miles or time interval. In addition, let N_1 be the number of failures in $[0,t]$, and let N_2 be the number of replacements in $[0,t]$. Suppose that replacement or overhaul occurs at times T, $2T$, and $3T$. The problem is to select the optimum overhaul time $T = T_0$ so as to minimize the long term average system cost (unscheduled maintenance, replacement cost and scheduled maintenance). Since $\beta > 1$, the average system cost is minimized when the system is overhauled (or replaced) at time T_0 such that the instantaneous maintenance cost equals the average system cost. The total system cost between overhaul or replacement is:

$$TSC(T) = C_1 E(N(T)) + C_2 + C_3 \frac{T}{S}$$

So the average system cost is:

$$C(T) = \frac{C_1 E(N(T)) + C_2 + C_3 \frac{T}{S}}{T}$$

The instantaneous maintenance cost at time T is equal to:

$$IMC(T) = C_1 \lambda \beta T^{\beta-1} + \frac{C_3}{S}$$

The following equation holds at optimum overhaul time T_0 :

$$\begin{aligned} C_1 \lambda \beta T_0^{\beta-1} + \frac{C_3}{S} &= \frac{C_1 E(N(T)) + C_2 + C_3 \frac{T}{S}}{T} \\ &= \frac{C_1 \lambda T_0^{\beta} + C_2 + C_3 \frac{T_0}{S}}{T_0} \end{aligned}$$

Therefore:

$$T_0 = \left[\frac{C_2}{\lambda(\beta - 1) C_1} \right]^{1/\beta}$$

But when there is no scheduled maintenance, the equation becomes:

$$C_1 \lambda \beta T_0^{\beta-1} = \frac{C_1 \lambda T_0^{\beta} + C_2}{T_0}$$

and the equation for the optimum overhaul time, T_0 , is the same as in the previous case. Therefore, for periodic maintenance scheduled every S miles, the replacement or overhaul time is the same as for the unscheduled and replacement or overhaul cost model.

More Examples

Automatic Transmission Data Example

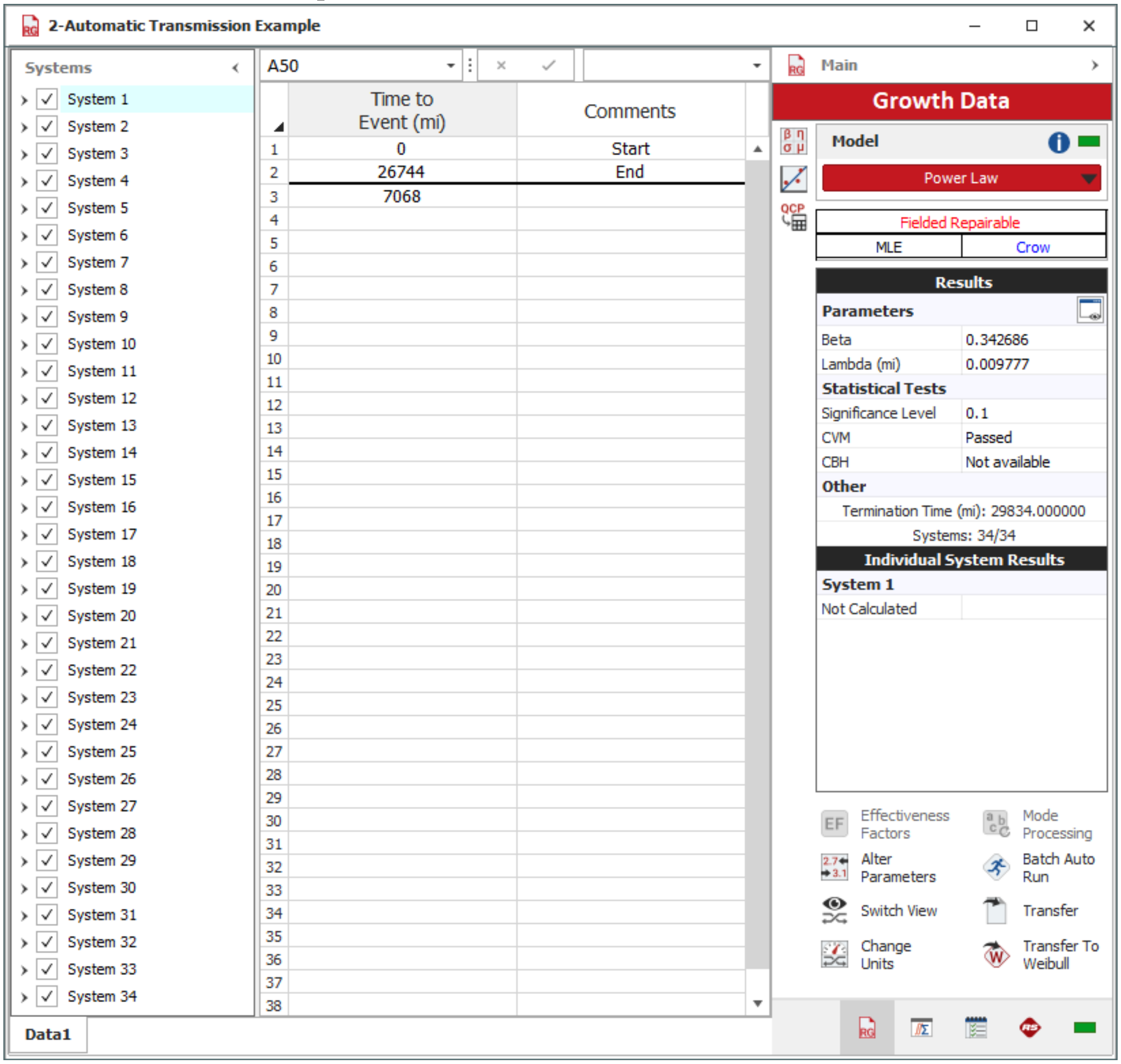
This case study is based on the data given in the article "Graphical Analysis of Repair Data" by Dr. Wayne Nelson [23]. The following table contains repair data on an automatic transmission from a sample of 34 cars. For each car, the data set shows mileage at the time of each transmission repair, along with the latest mileage. The + indicates the latest mileage observed without failure. Car 1, for example, had a repair at 7068 miles and was observed until 26,744 miles. Do the following:

1. Estimate the parameters of the Power Law model.
2. Estimate the number of warranty claims for a 36,000 mile warranty policy for an estimated fleet of 35,000 vehicles.

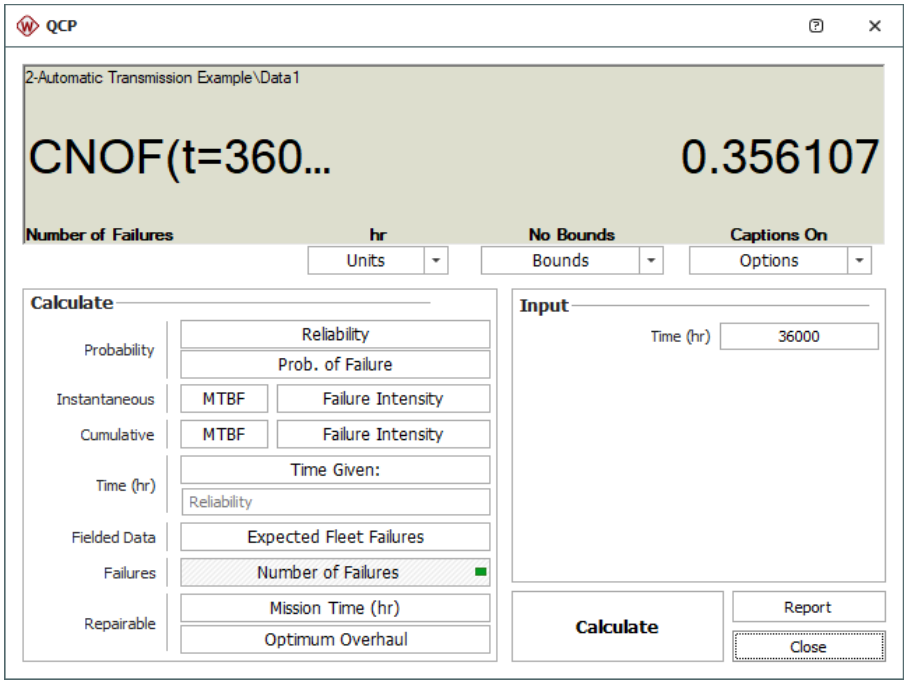
Automatic Transmission Data				
Car	**Mileage**		**Car**	**Mileage**
1	7068, 26744+		18	17955+
2	28, 13809+		19	19507+
3	48, 1440, 29834+		20	24177+
4	530, 25660+		21	22854+
5	21762+		22	17844+
6	14235+		23	22637+
7	1388, 18228+		24	375, 19607+
8	21401+		25	19403+
9	21876+		26	20997+
10	5094, 18228+		27	19175+
11	21691+		28	20425+
12	20890+		29	22149+
13	22486+		30	21144+
14	19321+		31	21237+
15	21585+		32	14281+
16	18676+		33	8250, 21974+
17	23520+		34	19250, 21888+

Solution

1. The estimated Power Law parameters are shown next.



2. The expected number of failures at 36,000 miles can be estimated using the QCP as shown next. The model predicts that 0.3559 failures per system will occur by 36,000 miles. This means that for a fleet of 35,000 vehicles, the expected warranty claims are 0.3559 * 35,000 = 12,456.



Optimum Overhaul Example

Field data have been collected for a system that begins its wearout phase at time zero. The start time for each system is equal to zero and the end time for each system is 10,000 miles. Each system is scheduled to undergo an overhaul after a certain number of miles. It has been determined that the cost of an overhaul is four times more expensive than a repair. The table below presents the data. Do the following:

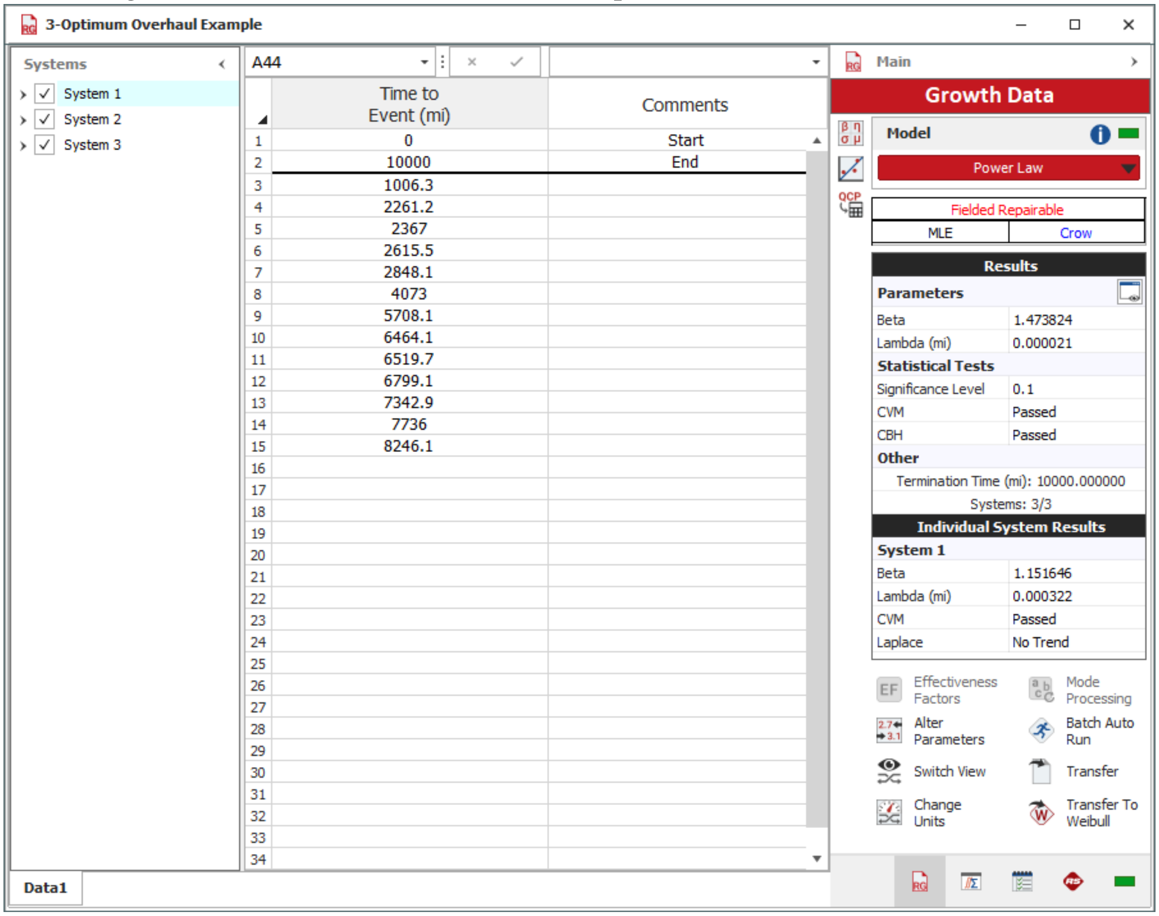
1. Estimate the parameters of the Power Law model.
2. Determine the optimum overhaul interval.
3. If $\beta < 1$, would it be cost-effective to implement an overhaul policy?

Field Data		
System 1	**System 2**	**System 3**
1006.3	722.7	619.1
2261.2	1950.9	1519.1
2367	3259.6	2956.6
2615.5	4733.9	3114.8

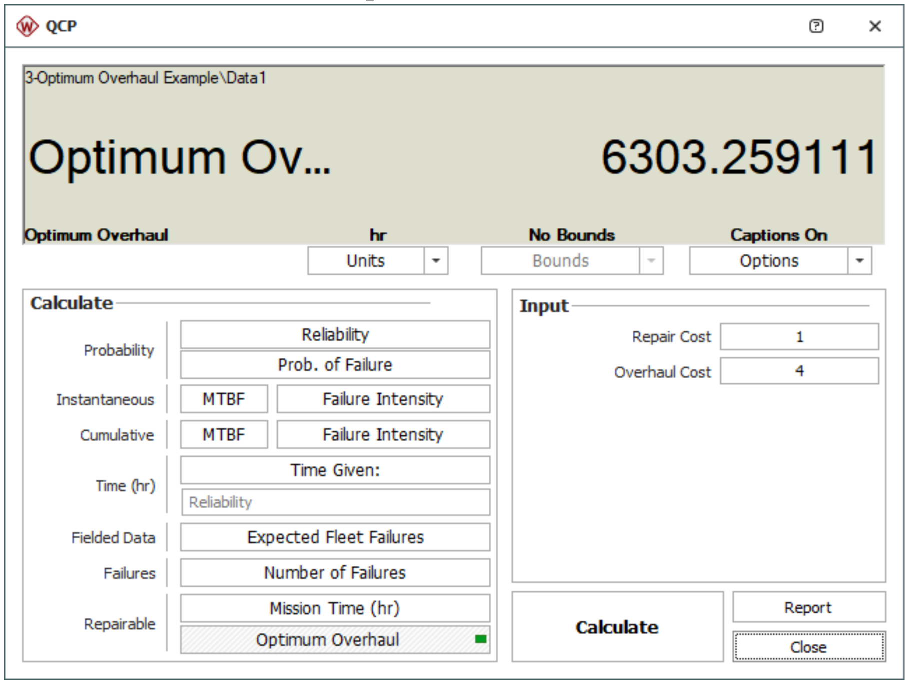
2848.1	5105.1	3657.9
4073	5624.1	4268.9
5708.1	5806.3	6690.2
6464.1	5855.6	6803.1
6519.7	6325.2	7323.9
6799.1	6999.4	7501.4
7342.9	7084.4	7641.2
7736	7105.9	7851.6
8246.1	7290.9	8147.6
	7614.2	8221.9
	8332.1	9560.5
	8368.5	9575.4
	8947.9	
	9012.3	
	9135.9	
	9147.5	
	9601	

Solution

1. The next figure shows the estimated Power Law parameters.



2. The QCP can be used to calculate the optimum overhaul interval, as shown next.



3. Since $\beta > 1$ the systems are wearing out and it would be cost-effective to implement an overhaul policy. An overhaul policy makes sense only if the systems are wearing out.

Crow Extended for Repairable Systems Analysis Example

The failures and fixes of two repairable systems in the field are recorded. Both systems started operating from time 0. System 1 ends at time = 504 and system 2 ends at time = 541. All the BD modes are fixed at the end of the test. A fixed effectiveness factor equal to 0.6 is used. Answer the following questions:

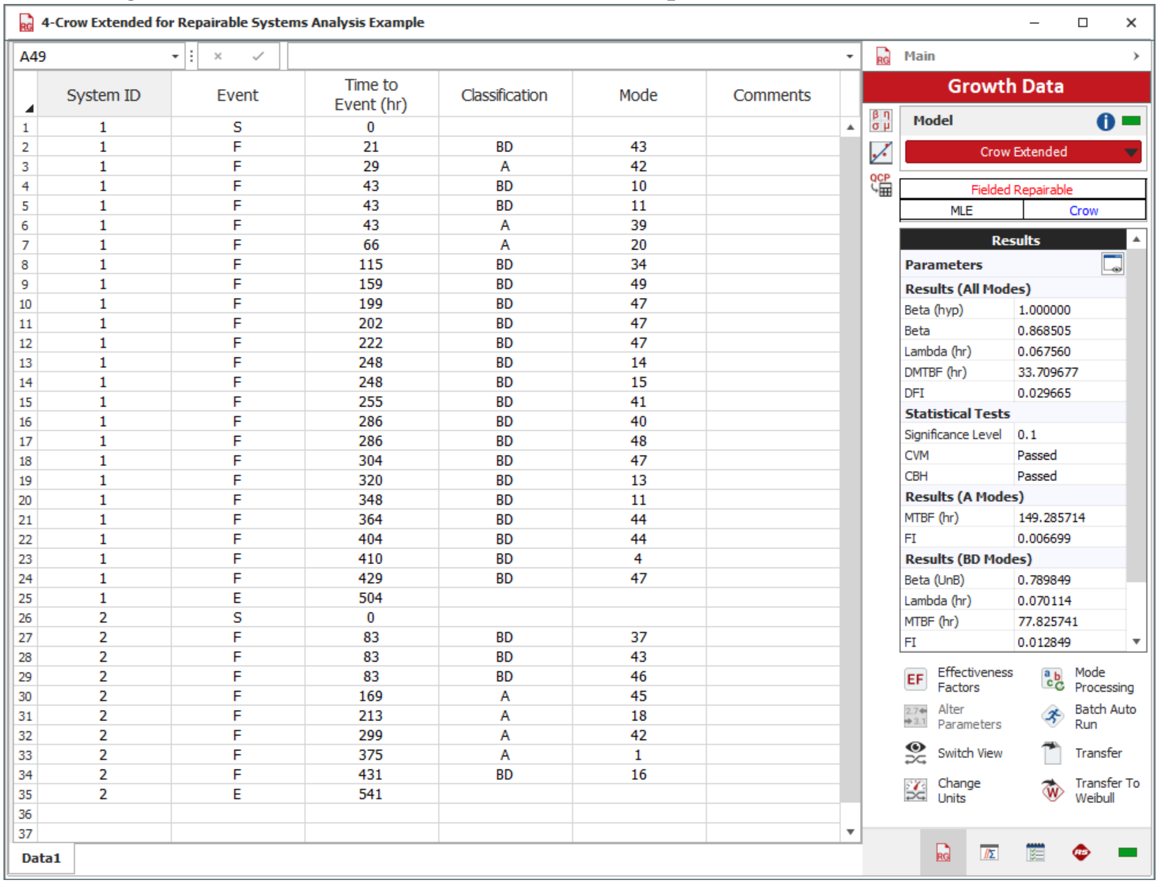
1. Estimate the parameters of the Crow Extended model.

2. Calculate the projected MTBF after the delayed fixes.

3. If no fixes were performed for the future failures, what would be the expected number of failures at time 1,000?

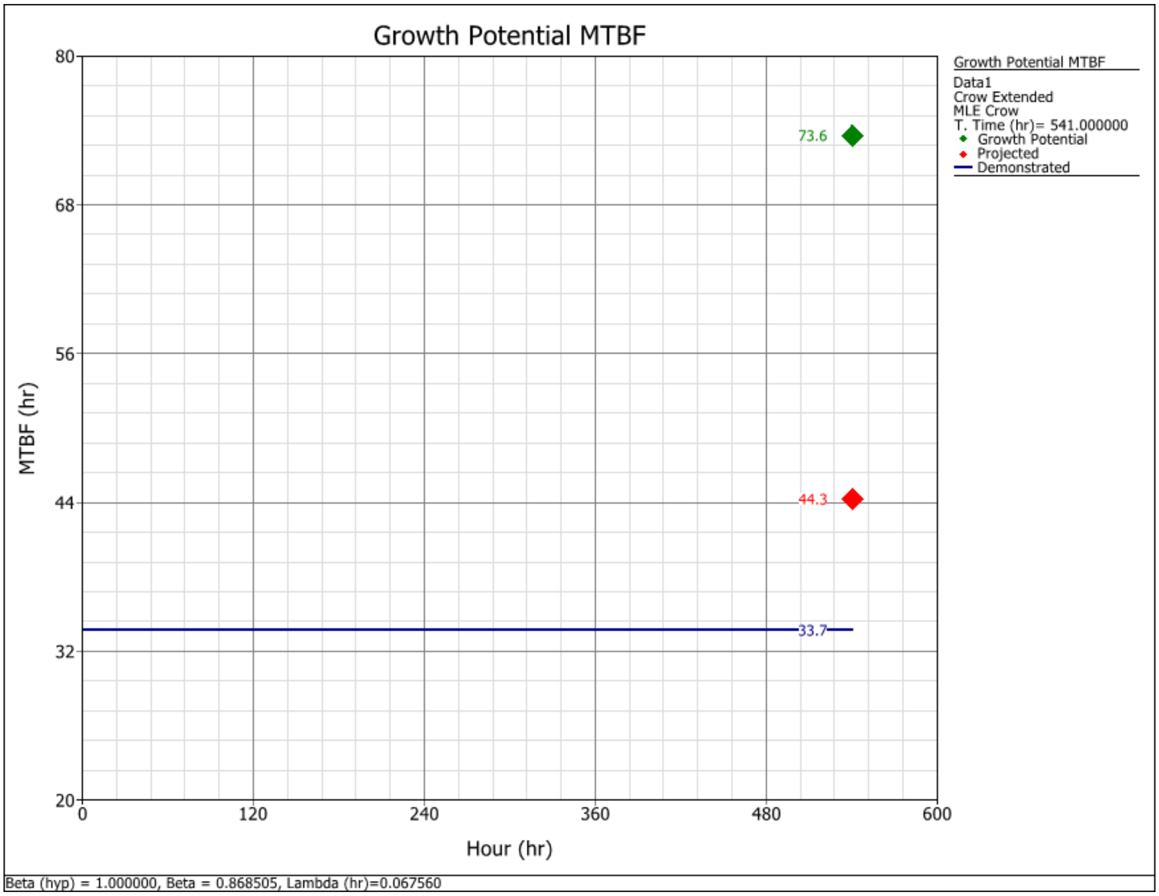
Solution

1. The next figure shows the estimated Crow Extended parameters.

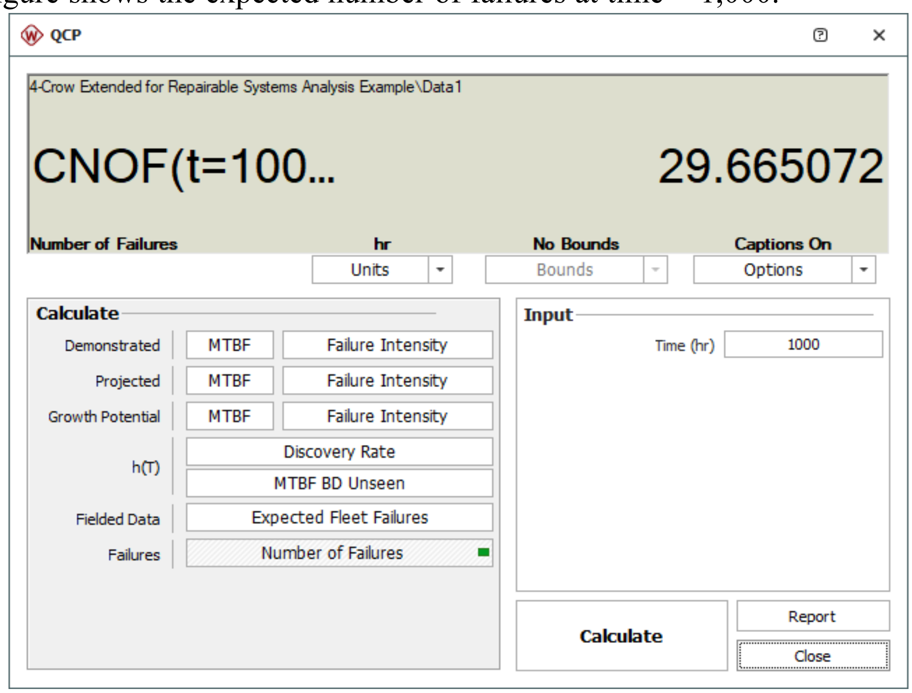


	System ID	Event	Time to Event (hr)	Classification	Mode	Comments
1	1	S	0			
2	1	F	21	BD	43	
3	1	F	29	A	42	
4	1	F	43	BD	10	
5	1	F	43	BD	11	
6	1	F	43	A	39	
7	1	F	66	A	20	
8	1	F	115	BD	34	
9	1	F	159	BD	49	
10	1	F	199	BD	47	
11	1	F	202	BD	47	
12	1	F	222	BD	47	
13	1	F	248	BD	14	
14	1	F	248	BD	15	
15	1	F	255	BD	41	
16	1	F	286	BD	40	
17	1	F	286	BD	48	
18	1	F	304	BD	47	
19	1	F	320	BD	13	
20	1	F	348	BD	11	
21	1	F	364	BD	44	
22	1	F	404	BD	44	
23	1	F	410	BD	4	
24	1	F	429	BD	47	
25	1	E	504			
26	2	S	0			
27	2	F	83	BD	37	
28	2	F	83	BD	43	
29	2	F	83	BD	46	
30	2	F	169	A	45	
31	2	F	213	A	18	
32	2	F	299	A	42	
33	2	F	375	A	1	
34	2	F	431	BD	16	
35	2	E	541			
36						
37						

2. The next figure shows the projected MTBF at time = 541 (i.e., the age of the oldest system).



3. The next figure shows the expected number of failures at time = 1,000.



Operational Testing

Background

Operational testing is basically repairable systems analysis using the Crow Extended model. The general assumptions associated with the Crow Extended model do not change. However, in operational testing, the calculations are conducted with the assumption that $\beta = 1$. Therefore, only delayed fixes, BD modes (test-find-test), are allowed. BC modes and fixes during the test cannot be entered. In this scenario, you want a stable system such that the estimate of β is close to one. The $\beta = 1$ assumption can be verified by checking the confidence bounds on β. If the confidence bounds include one, then you can fail to reject the hypothesis that $\beta = 1$.

Under operational testing:

- The final product has been fielded, why is why this is not developmental testing.
- Only delayed fixes (BD modes) are allowed.
- The configuration is fixed and design changes are kept to a minimum.
- The focus is on age-dependent reliability.
- Testing is generally conducted prior to full production.

Operational testing analysis could also be applied to a system that is already in production and being used by customers in the field. In this case, you will be able to verify the improvement in the system's MTBF based on the specified delayed fixes. Based on this information, along with the cost and time to implement, you can determine if it is cost-effective to apply the fixes to the fielded systems.

Example - Operational Testing

Consider two systems that have been placed into operational testing. The data for each system are given below. Do the following:

1. After estimating the parameters, verify the assumption of $\beta = 1$.
2. Estimate the instantaneous MTBF of the system at the end of the test (demonstrated MTBF).
3. Estimate the MTBF that can be expected after the BD failure modes are addressed (projected MTBF) and the maximum MTBF that can be achieved if all BD modes that exist in the system were discovered and fixed according to the current maintenance strategy (growth potential MTBF).

Operational Testing Data

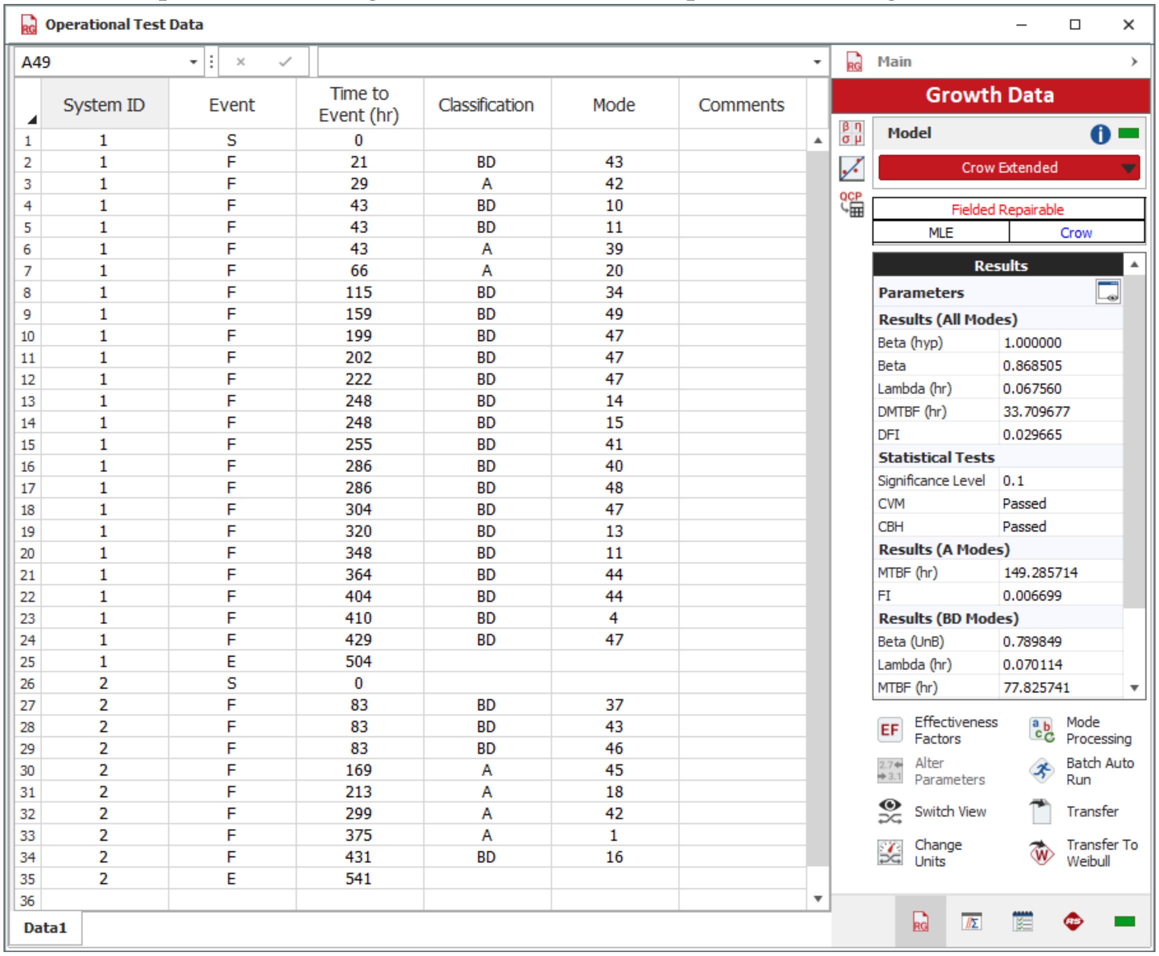
System #	1	2
Start Time (Hr)	0	0
End Time (Hr)	504	541

Failure Times (Hr) and Failure Modes	21 BD43	83 BD37
	29 A42	83 BD43
	43 BD10	83 BD46
	43 BD11	169 A45
	43 A39	213 A18
	66 A20	299 A42
	115 BD34	375 A1
	159 BD49	431 BD16
	199 BD47	
	202 BD47	
	222 BD47	
	248 BD14	
	248 BD15	
	255 BD41	
	286 BD40	
	286 BD48	
	304 BD47	
	320 BD13	
	348 BD11	
	364 BD44	
	404 BD44	
	410 BD4	
	429 BD47	

The BD modes are implemented at the end of the test and assume a fixed effectiveness factor equal to 0.6 (i.e., 40% of the failure intensity will remain after the fixes are implemented).

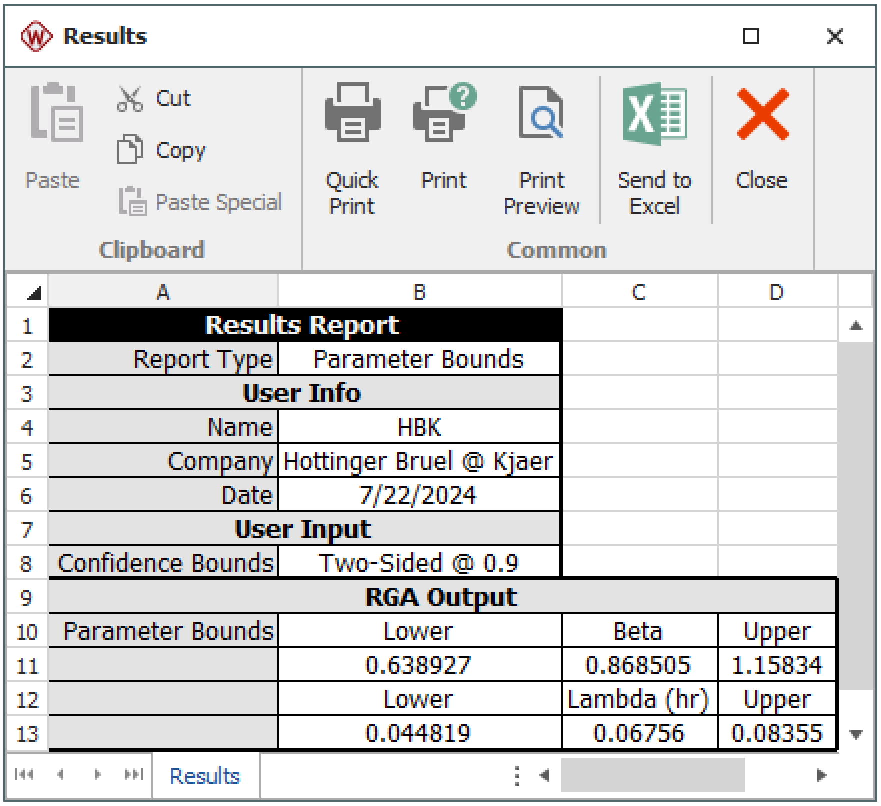
Solution

1. The entered operational testing data and the estimated parameters are given below.



	System ID	Event	Time to Event (hr)	Classification	Mode	Comments
1	1	S	0			
2	1	F	21	BD	43	
3	1	F	29	A	42	
4	1	F	43	BD	10	
5	1	F	43	BD	11	
6	1	F	43	A	39	
7	1	F	66	A	20	
8	1	F	115	BD	34	
9	1	F	159	BD	49	
10	1	F	199	BD	47	
11	1	F	202	BD	47	
12	1	F	222	BD	47	
13	1	F	248	BD	14	
14	1	F	248	BD	15	
15	1	F	255	BD	41	
16	1	F	286	BD	40	
17	1	F	286	BD	48	
18	1	F	304	BD	47	
19	1	F	320	BD	13	
20	1	F	348	BD	11	
21	1	F	364	BD	44	
22	1	F	404	BD	44	
23	1	F	410	BD	4	
24	1	F	429	BD	47	
25	1	E	504			
26	2	S	0			
27	2	F	83	BD	37	
28	2	F	83	BD	43	
29	2	F	83	BD	46	
30	2	F	169	A	45	
31	2	F	213	A	18	
32	2	F	299	A	42	
33	2	F	375	A	1	
34	2	F	431	BD	16	
35	2	E	541			
36						

The assumption of $\beta = 1$ can be verified by looking at the confidence bounds on β via the Quick Calculation Pad (QCP). The 90% 2-sided Crow confidence bounds on β are shown next.

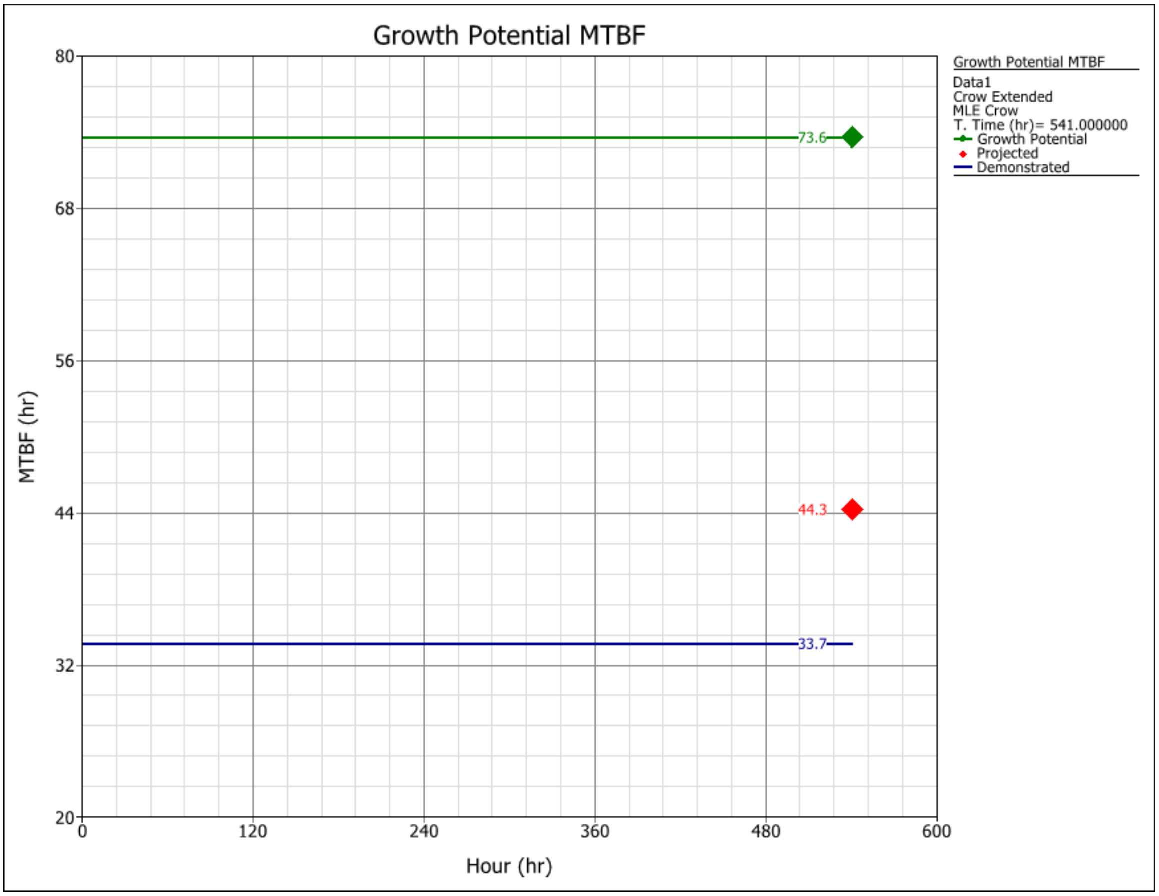


	A	B	C	D
1	Results Report			
2	Report Type	Parameter Bounds		
3	User Info			
4	Name	HBK		
5	Company	Hottinger Bruel @ Kjaer		
6	Date	7/22/2024		
7	User Input			
8	Confidence Bounds	Two-Sided @ 0.9		
9	RGA Output			
10	Parameter Bounds	Lower	Beta	Upper
11		0.638927	0.868505	1.15834
12		Lower	Lambda (hr)	Upper
13		0.044819	0.06756	0.08355

Since the confidence bounds on β include one, then you can fail to reject the hypothesis that $\beta = 1$.

2. From #1, the demonstrated MTBF (DMTBF) equals 33.7097 hours.

3. The projected MTBF and growth potential values can be displayed via the Growth Potential MTBF plot.



The plot shows that the projected MTBF equals 42.3824 hours, and the growth potential MTBF equals 62.9518 hours.

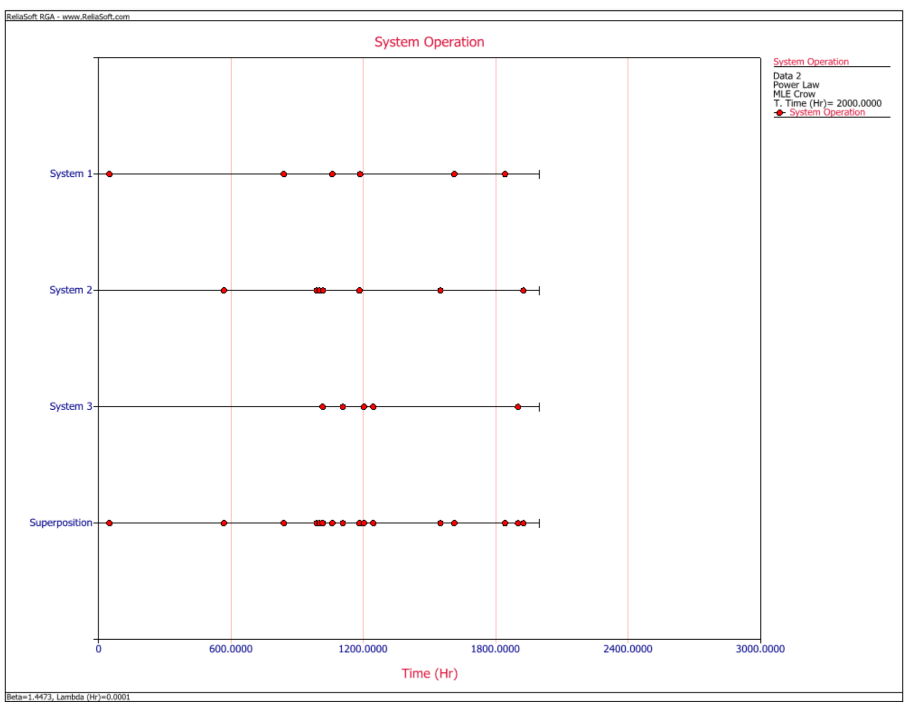
Fleet Data Analysis

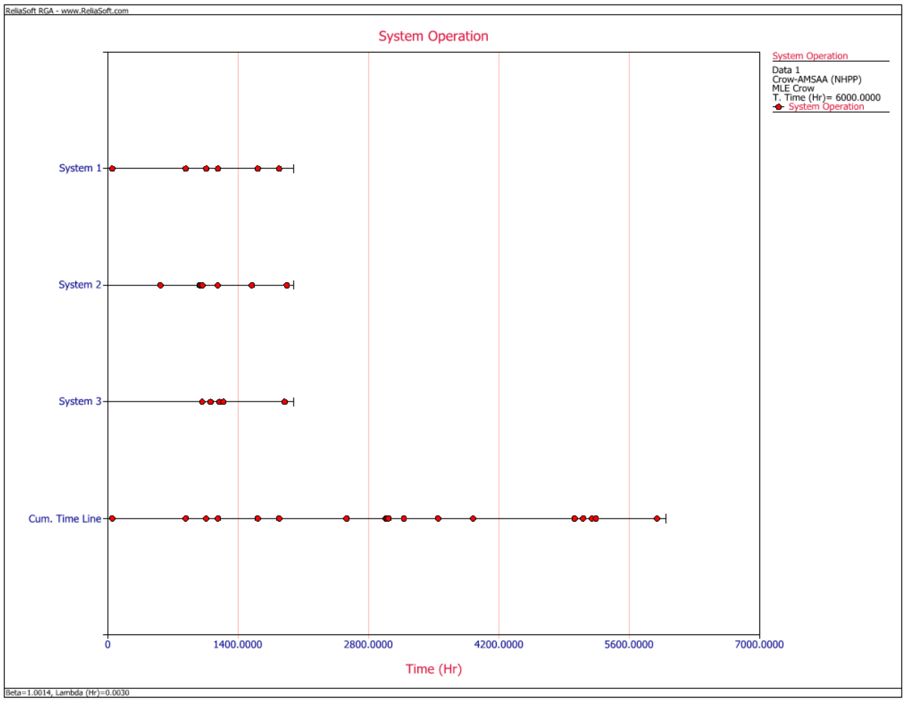
Fleet analysis is similar to the repairable systems analysis described in the previous chapter. The main difference is that a fleet of systems is considered and the models are applied to the fleet failures rather than to the system failures. In other words, repairable system analysis models the number of system failures versus system time, whereas fleet analysis models the number of fleet failures versus fleet time.

The main motivation for fleet analysis is to enable the application of the Crow Extended model for fielded data. In many cases, reliability improvements might be necessary on systems that are already in the field. These types of reliability improvements are essentially delayed fixes (BD modes) as described in the Crow Extended chapter.

Introduction

Recall from the previous chapter that in order to make projections using the Crow Extended model, the β of the combined A and BD modes should be equal to 1. Since the failure intensity in a fielded system might be changing over time (e.g., increasing if the system wears out), this assumption might be violated. In such a scenario, the Crow Extended model cannot be used. However, if a fleet of systems is considered and the number of fleet failures versus fleet time is modeled, the failures might become random. This is because there is a mixture of systems within a fleet, new and old, and when the failures of this mixture of systems are viewed from a cumulative fleet time point of view, they may be random. The next two figures illustrate this concept. The first picture shows the number of failures over system age. It can be clearly seen that as the systems age, the intensity of the failures increases (wearout). The superposition system line, which brings the failures from the different systems under a single timeline, also illustrates this observation. On the other hand, if you take the same four systems and combine their failures from a fleet perspective, and consider fleet failures over cumulative fleet hours, then the failures seem to be random. The second picture illustrates this concept in the System Operation plot when you consider the Cum. Time Line. In this case, the β of the fleet will be equal to 1 and the Crow Extended model can be used for quantifying the effects of future reliability improvements on the fleet.





Methodology

The figures above illustrate that the difference between repairable system data analysis and fleet analysis is the way that the data set is treated. In fleet analysis, the time-to-failure data from each system is stacked to a cumulative timeline. For example, consider the two systems in the following table.

System Data		
System	**Failure Times (hr)**	**End Time (hr)**
1	3, 7	10
2	4, 9, 13	15

Convert to Accumulated Timeline

The data set is first converted to an accumulated timeline, as follows:

- System 1 is considered first. The accumulated timeline is therefore 3 and 7 hours.
- System 1's end time is 10 hours. System 2's first failure is at 4 hours. This failure time is added to System 1's end time to give an accumulated failure time of 14 hours.

- The second failure for System 2 occurred 5 hours after the first failure. This time interval is added to the accumulated timeline to give 19 hours.
- The third failure for System 2 occurred 4 hours after the second failure. The accumulated failure time is 19 + 4 = 23 hours.
- System 2's end time is 15 hours, or 2 hours after the last failure. The total accumulated operating time for the fleet is 25 hours (23 + 2 = 25).

In general, the accumulated operating time Y_j is calculated by:

$$Y_j = X_{i,q} + \sum_{q=1}^{K-1} T_q,\ m = 1, 2, \ldots, N$$

where:

- $X_{i,q}$ is the i^{th} failure of the q^{th} system
- T_q is the end time of the q^{th} system
- K is the total number of systems
- N is the total number of failures from all systems ($N = \sum_{j=1}^{K} N_q$)

As this example demonstrates, the accumulated timeline is determined based on the order of the systems. So if you consider the data in the table by taking System 2 first, the accumulated timeline would be: 4, 9, 13, 18, 22, with an end time of 25. Therefore, the order in which the systems are considered is somewhat important. However, in the next step of the analysis, the data from the accumulated timeline will be grouped into time intervals, effectively eliminating the importance of the order of the systems. Keep in mind that this will NOT always be true. This is true only when the order of the systems was random to begin with. If there is some logic/pattern in the order of the systems, then it will remain even if the cumulative timeline is converted to grouped data. For example, consider a system that wears out with age. This means that more failures will be observed as this system ages and these failures will occur more frequently. Within a fleet of such systems, there will be new and old systems in operation. If the data set collected is considered from the newest to the oldest system, then even if the data points are grouped, the pattern of fewer failures at the beginning and more failures at later time intervals will still be present. If the objective of the analysis is to determine the difference between newer and older systems, then that order for the data will be acceptable. However, if the objective of the analysis is to determine the reliability of the fleet, then the systems should be randomly ordered.

Analyze the Grouped Data

Once the accumulated timeline has been generated, it is then converted into grouped data. To accomplish this, a group interval is required. The group interval length should be chosen so that it is representative of the data. Also note that the intervals do not have to be of equal length. Once the data points have been grouped, the parameters can be obtained using maximum likelihood estimation as described in the Crow-AMSAA (NHPP) chapter. The data from the table above can be grouped into 5 hour intervals. This interval length is sufficiently large to insure that there are failures within each interval. The grouped data set is given in the following table.

Grouped Data	
Failures in Interval	**Interval End Time**
1	5
1	10
1	15
1	20
1	25

The Crow-AMSAA model for grouped failure times is used for the data, and the parameters of the model are solved by satisfying the following maximum likelihood equations (See Crow-AMSAA (NHPP)):

$$\widehat{\lambda} = \frac{n}{T_k^{\widehat{\beta}}}$$

$$\sum_{i=1}^{k} n_i \left[\frac{T_i^{\widehat{\beta}} \ln T_{i-1} - T_{i-1}^{\widehat{\beta}} \ln T_{i-1}}{T_i^{\widehat{\beta}} - T_{i-1}^{\widehat{\beta}}} - \ln T_k \right] = 0$$

Fleet Analysis Example

The following table presents data for a fleet of 27 systems. A cycle is a complete history from overhaul to overhaul. The failure history for the last completed cycle for each system is recorded. This is a random sample of data from the fleet. These systems are in the order in which they were selected. Suppose the intervals to group the current data are 10,000; 20,000; 30,000; 40,000 and the final interval is defined by the termination time. Conduct the fleet analysis.

Sample Fleet Data			
System	Cycle Time T_j	Number of failures N_j	Failure Time X_{ij}
1	1396	1	1396
2	4497	1	4497
3	525	1	525
4	1232	1	1232
5	227	1	227
6	135	1	135
7	19	1	19
8	812	1	812
9	2024	1	2024
10	943	2	316, 943
11	60	1	60
12	4234	2	4233, 4234
13	2527	2	1877, 2527
14	2105	2	2074, 2105
15	5079	1	5079
16	577	2	546, 577
17	4085	2	453, 4085
18	1023	1	1023
19	161	1	161
20	4767	2	36, 4767
21	6228	3	3795, 4375, 6228
22	68	1	68

23	1830	1	1830
24	1241	1	1241
25	2573	2	871, 2573
26	3556	1	3556
27	186	1	186
Total	52110	37	

Solution

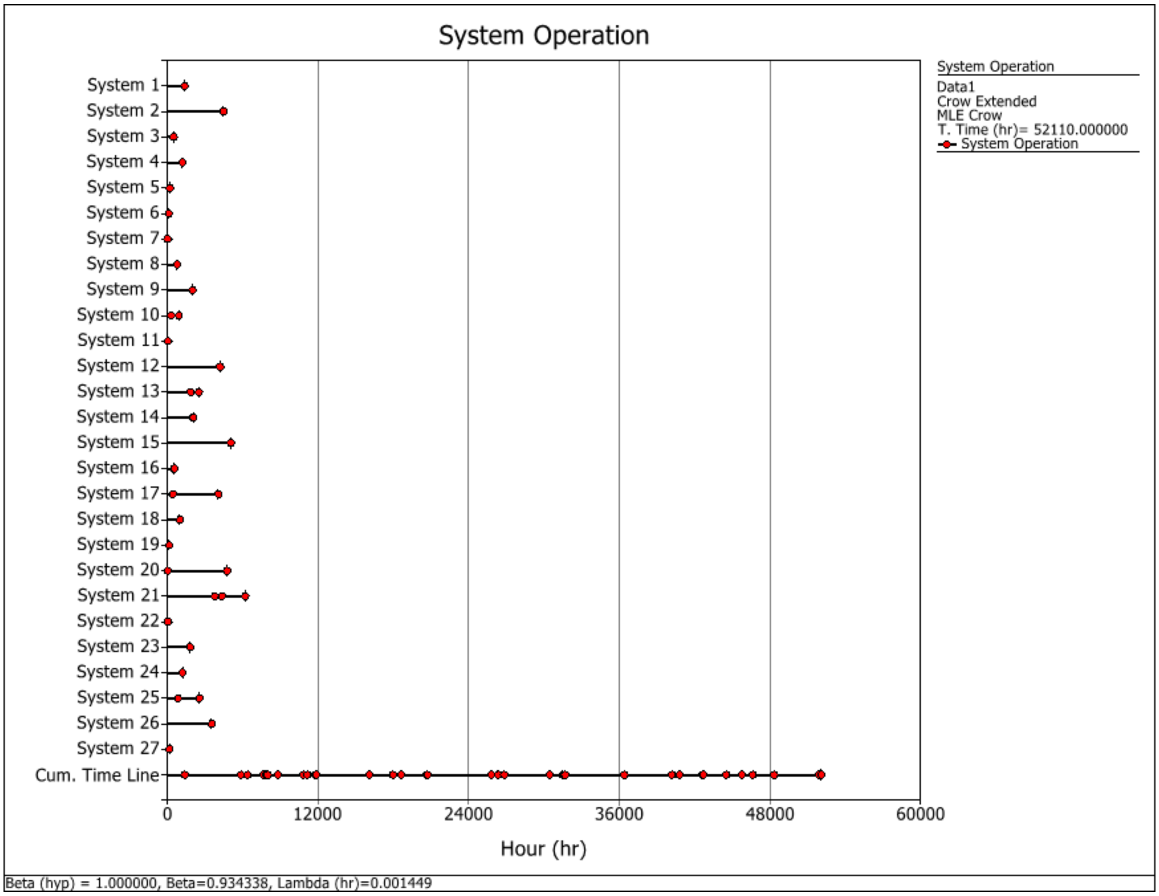
The sample fleet data set can be grouped into 10,000; 20,000; 30,000; 40,000 and 52,110 time intervals. The following table gives the grouped data.

Grouped Data	
Time	**Observed Failures**
10,000	8
20,000	16
30,000	22
40,000	27
52,110	37

Based on the above time intervals, the maximum likelihood estimates of $\widehat{\lambda}$ and $\widehat{\beta}$ for this data set are then given by:

$$\widehat{\lambda} = 0.00147$$
$$\widehat{\beta} = 0.93328$$

The next figure shows the System Operation plot.



Applying the Crow Extended Model to Fleet Data

As it was mentioned previously, the main motivation of the fleet analysis is to apply the Crow Extended model for in-service reliability improvements. The methodology to be used is identical to the application of the Crow Extended model for Grouped Data described in a previous chapter. Consider the fleet data from the example above. In order to apply the Crow Extended model, put $N = 37$ failure times on a cumulative time scale over $(0, T)$, where $T = 52110$. In the example, each T_i corresponds to a failure time X_{ij}. This is often not the situation. However, in all cases the accumulated operating time Y_q at a failure time X_{ir} is:

$$Y_q = X_{i,r} + \sum_{j=1}^{r-1} T_j, \quad q = 1, 2, \ldots, N$$

$$N = \sum_{j=1}^{K} N_j$$

And q indexes the successive order of the failures. Thus, in this example $N = 37,\ Y_1 = 1396,\ Y_2 = 5893,\ Y_3 = 6418, \ldots, Y_{37} = 52110$. See the table below.

Test-Find-Test Fleet Data						
q	Y_q	**Mode**		q	Y_q	**Mode**
1	1396	BD1		20	26361	BD1
2	5893	BD2		21	26392	A
3	6418	A		22	26845	BD8
4	7650	BD3		23	30477	BD1
5	7877	BD4		24	31500	A
6	8012	BD2		25	31661	BD3
7	8031	BD2		26	31697	BD2
8	8843	BD1		27	36428	BD1
9	10867	BD1		28	40223	BD1
10	11183	BD5		29	40803	BD9
11	11810	A		30	42656	BD1
12	11870	BD1		31	42724	BD10
13	16139	BD2		32	44554	BD1
14	16104	BD6		33	45795	BD11
15	18178	BD7		34	46666	BD12
16	18677	BD2		35	48368	BD1
17	20751	BD4		36	51924	BD13
18	20772	BD2		37	52110	BD2
19	25815	BD1				

Each system failure time in the table above corresponds to a problem and a cause (failure mode). The management strategy can be to not fix the failure mode (A mode) or to fix the failure mode with a delayed corrective action (BD mode). There are $N_A = 4$ failures due to A failure modes. There are $N_{BD} = 33$ total failures due to $M = 13$ distinct BD failure modes. Some of the distinct BD modes had repeats of the same problem. For example, mode BD1 had 12 occurrences of the same problem. Therefore, in this example, there are 13 distinct corrective actions corresponding to 13 distinct BD failure modes.

The objective of the Crow Extended model is to estimate the impact of the 13 distinct corrective actions.The analyst will choose an average effectiveness factor (EF) based on the proposed corrective actions and historical experience. Historical industry and government data supports a typical average effectiveness factor $\overline{d}=.70$ for many systems. In this example, an average EF of $\bar{d}=0.4$ was assumed in order to be conservative regarding the impact of the proposed corrective actions. Since there are no BC failure modes (corrective actions applied during the test), the projected failure intensity is:

$$\hat{r}(T)=\left(\frac{N_A}{T}+\sum_{i=1}^{M}(1-d_i)\frac{N_i}{T}\right)+\overline{d}h(T)$$

The first term is estimated by:

$$\widehat{\lambda}_A=\frac{N_A}{T}=0.000077$$

The second term is:

$$\sum_{i=1}^{M}(1-d_i)\frac{N_i}{T}=0.00038$$

This estimates the growth potential failure intensity:

$$\widehat{\gamma}_{GP}(T)=\frac{N_A}{T}+\sum_{i=1}^{M}(1-d_i)\frac{N_i}{T}$$
$$=0.00046$$

To estimate the last term $\overline{d}h(T)$ of the Crow Extended model, partition the data in the table into intervals. This partition consists of D successive intervals. The length of the q^{th} interval is L_q, $q=1,2,\ldots,D$. It is not required that the intervals be of the same length, but there should be several (e.g., at least 5) cycles per interval on average. Also, let $S_1=L_1, S_2=L_1+L_2,\ldots,$ etc. be the accumulated time through the q^{th} interval. For the q^{th} interval, note the number of distinct BD modes, MI_q, appearing for the first time, $q=1,2,\ldots,D$. See the following table.

Grouped Data for Distinct BD Modes			
Interval	**No. of Distinct BD Mode Failures**	**Length**	**Accumulated Time**

1	MI_1	L_1	S_1
2	MI_2	L_2	S_2
.	.	.	.
.	.	.	.
.	.	.	.
D	MI_D	L_D	S_D

The term $\widehat{h}(T)$ is calculated as $\widehat{h}(T)=\widehat{\lambda}\widehat{\beta}T^{\widehat{\beta}-1}$ and the values $\widehat{\lambda}$ and $\widehat{\beta}$ satisfy the maximum likelihood equations for grouped data (given in the Methodology section). This is the grouped data version of the Crow-AMSAA model applied only to the first occurrence of distinct BD modes.

For the data in the first table, the first 4 intervals had a length of 10,000 and the last interval was 12,110. Therefore, $D=5$. This choice gives an average of about 5 overhaul cycles per interval. See the table below.

Grouped Data for Distinct BD Modes from Data in "Applying the Crow Extended Model to Fleet Data"			
Interval	**No. of Distinct BD Mode Failures**	**Length**	**Accumulated Time**
1	4	10000	10000
2	3	10000	20000
3	1	10000	30000
4	0	10000	40000
5	5	12110	52110
Total	13		

Thus:

$$\widehat{\lambda}=0.00330$$
$$\widehat{\beta}=0.76219$$

This gives:

$$\begin{aligned}\widehat{h}(T) =& \widehat{\lambda}\widehat{\beta}T^{\widehat{\beta}-1} \\ =& 0.00019\end{aligned}$$

Consequently, for $\overline{d} = 0.4$ the last term of the Crow Extended model is given by:

$$\overline{d}h(T) = 0.000076$$

The projected failure intensity is:

$$\begin{aligned}\hat{r}(T) =& \frac{N_A}{T} + \sum_{i=1}^{M}(1-d_i)\frac{N_i}{T} + \overline{d}h(T) \\ =& 0.000077 + 0.6 \times (0.00063) + 0.4 \times (0.00019) \\ =& 0.000533\end{aligned}$$

This estimates that the 13 proposed corrective actions will reduce the number of failures per cycle of operation hours from the current $\hat{r}(0) = \frac{N_A+N_{BD}}{T} = 0.00071$ to $\hat{r}(T) = 0.00053.$ The average time between failures is estimated to increase from the current 1408.38 hours to 1876.93 hours.

Confidence Bounds

For fleet data analysis using the Crow-AMSAA model, the confidence bounds are calculated using the same procedure described for the Crow-AMSAA (NHPP) model (See Crow-AMSAA Confidence Bounds). For fleet data analysis using the Crow Extended model, the confidence bounds are calculated using the same procedure described for the Crow Extended model (See Crow Extended Confidence Bounds).

More Examples

Predicting the Number of Failures for Fleet Operation

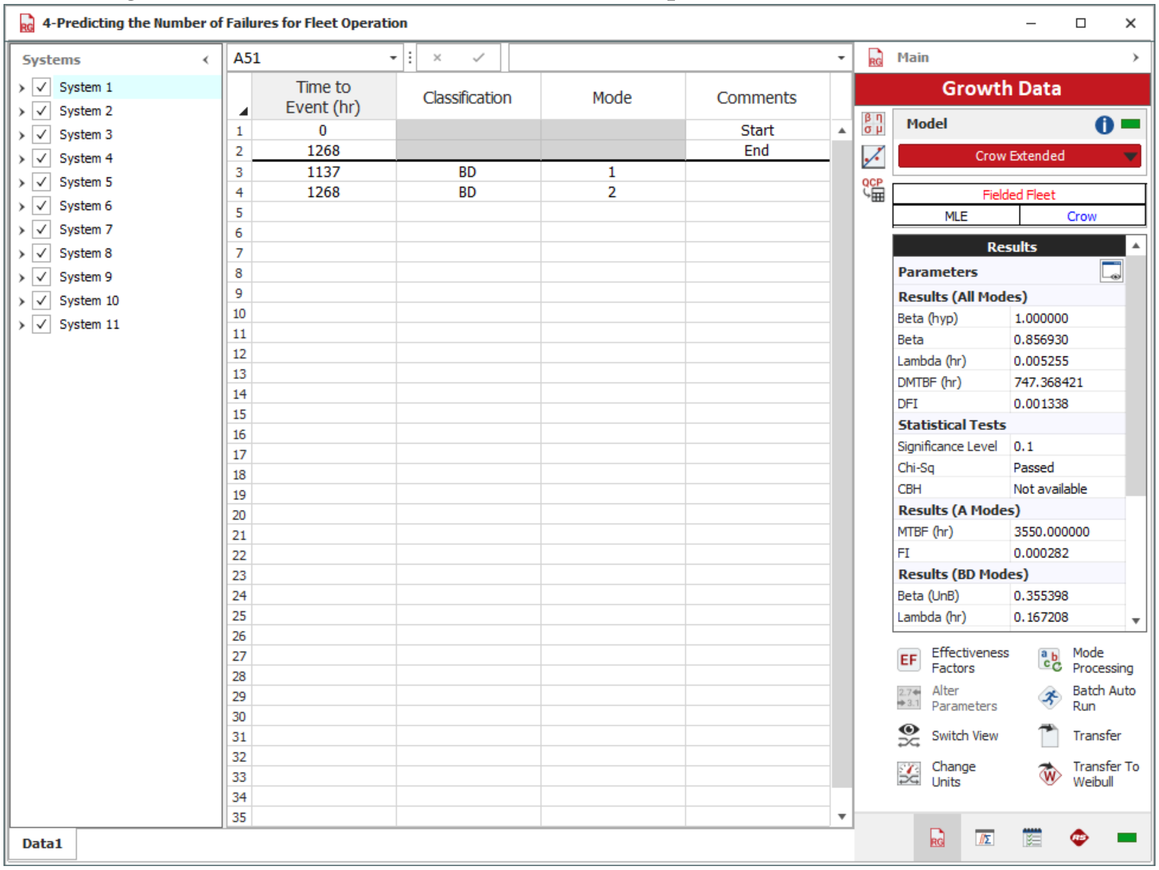
11 systems from the field were chosen for fleet analysis. Each system had at least one failure. All of the systems had a start time equal to zero and the last failure for each system corresponds to the end time. Group the data based on a fixed interval of 3,000 hours, and assume a fixed effectiveness factor equal to 0.4. Do the following:

1. Estimate the parameters of the Crow Extended model.
2. Based on the analysis, does it appear that the systems were randomly ordered?
3. After the implementation of the delayed fixes, how many failures would you expect within the next 4,000 hours of fleet operation.

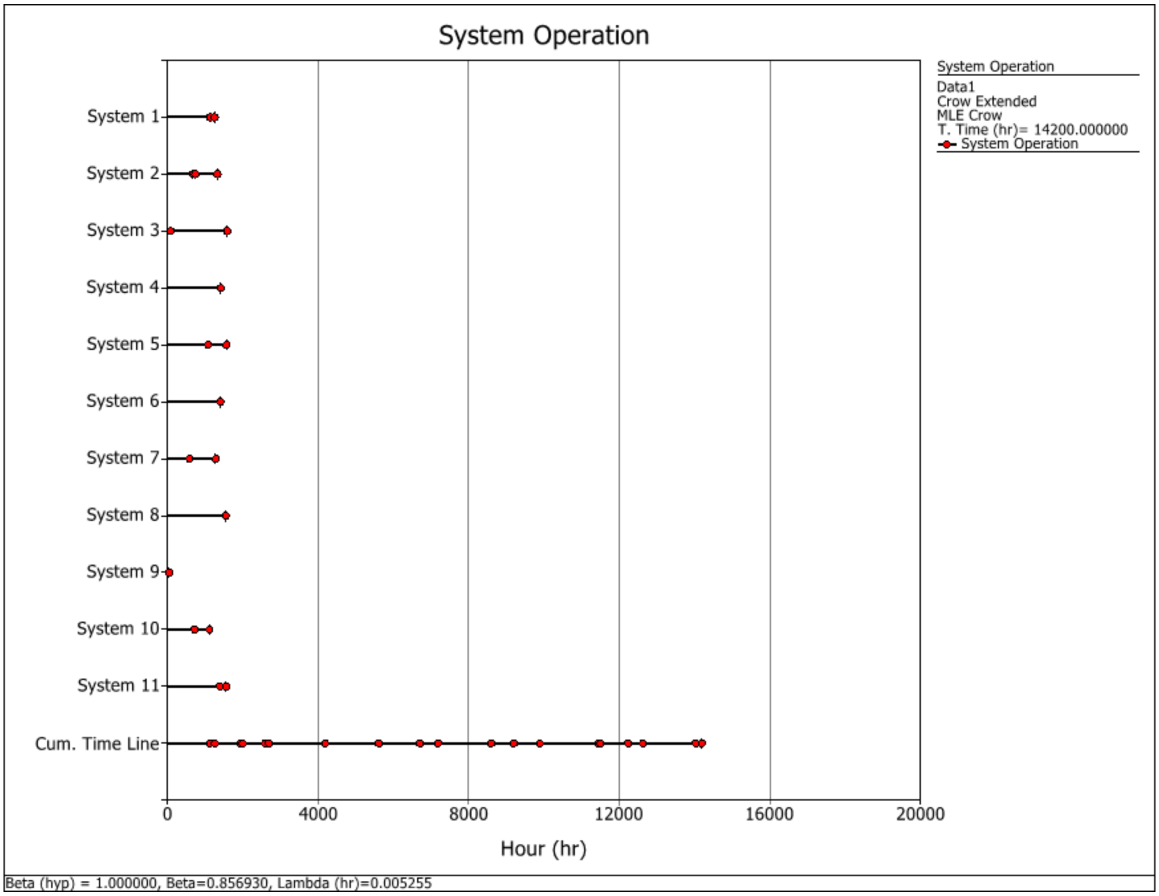
Fleet Data	
System	**Times-to-Failure**
1	1137 BD1, 1268 BD2
2	682 BD3, 744 A, 1336 BD1
3	95 BD1, 1593 BD3
4	1421 A
5	1091 A, 1574 BD2
6	1415 BD4
7	598 BD4, 1290 BD1
8	1556 BD5
9	55 BD4
10	730 BD1, 1124 BD3
11	1400 BD4, 1568 A

Solution

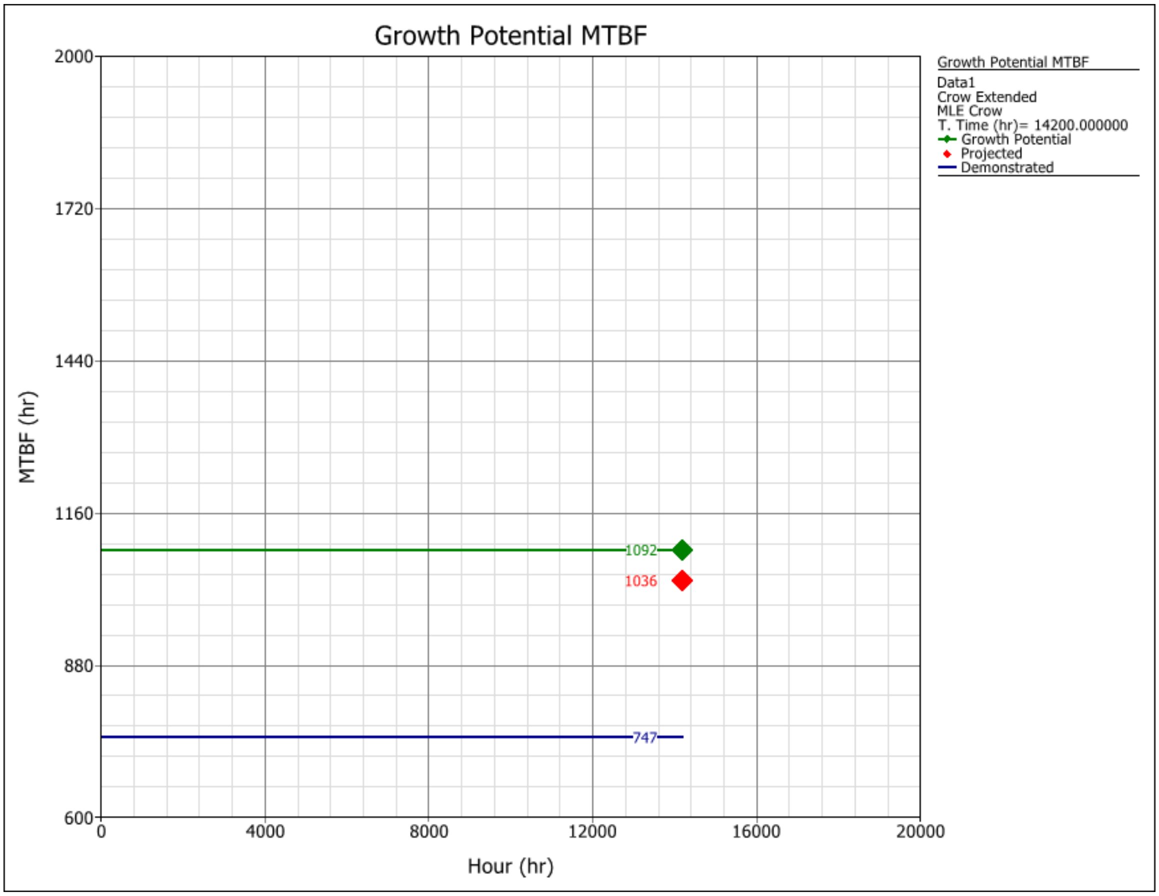
1. The next figure shows the estimated Crow Extended parameters.



2. Upon observing the estimated parameter β, it does appear that the systems were randomly ordered since $\beta = 0.8569$. This value is close to 1. You can also verify that the confidence bounds on β include 1 by going to the QCP and calculating the parameter bounds or by viewing the Beta Bounds plot. However, you can also determine graphically if the systems were randomly ordered by using the System Operation plot as shown below. Looking at the Cum. Time Line, it does not appear that the failures have a trend associated with them. Therefore, the systems can be assumed to be randomly ordered.



3. After implementing the delayed fixes, the system's projected MTBF is equal to 1035.6802 as shown in the plot below.



To estimate the number of failures during the next 4,000 hours, calculate the following:

$$N = \frac{4000}{1035.6802}$$
$$= 3.8622$$

Therefore, it is estimated that $\approx$ 4 failures will be observed during the next 4,000 hours of fleet operation.

Simulation with Reliability Growth Analysis Models

IN THIS CHAPTER

When analyzing developmental systems for reliability growth, and conducting data analysis of fielded repairable systems, it is often useful to experiment with various *what if* scenarios or put together hypothetical analyses before data sets become available in order to plan for the best way to perform the analysis. With that in mind, the Weibull++ software offers applications based on Monte Carlo simulation that can be used in order to:

a. Better understand reliability growth concepts.

b. Experiment with the impact of sample size, test time and growth parameters on analysis results.

c. Construct simulation-based confidence intervals.

d. Better understand concepts behind confidence intervals.

e. Design reliability demonstration tests.

There are two applications of the Monte Carlo simulation in the Weibull++ software. One is called Generate Monte Carlo Data and the other is called SimuMatic.

Generate Monte Carlo Data

Monte Carlo simulation is a computational algorithm in which we randomly generate input variables that follow a specified probability distribution. In the case of reliability growth and repairable system data analysis, we are interested in generating failure times for systems that we assume to have specific characteristics. In our applications we want the inter-arrival times of the failures to follow a non-homogeneous Poisson process with a Weibull failure intensity, as specified in the Crow-AMSAA (NHPP) model.

The first time to failure, t_1, is assumed to follow a Weibull distribution. It is obtained by solving for t_1 :

$$R(t_1) = e^{(-\frac{t_1}{\eta})^\beta} = Uniform(0,1)$$

where:

$$\eta = \left(\frac{1}{\lambda}\right)^{\frac{1}{\beta}}$$

Solving for t_1 yields:

$$t_1 = \eta[-ln(Uniform(0,1))]^{\frac{1}{\beta}}$$

The failure times are then obtained based on the conditional unreliability equation that describes the non-homogeneous Poisson process (NHPP):

$$F(t_i \mid t_{i-1}) = 1 - e^{-\lambda\left[t_i^\beta - t_{i-1}^\beta\right]} = Uniform(0,1)$$

and then solving for t_i yields:

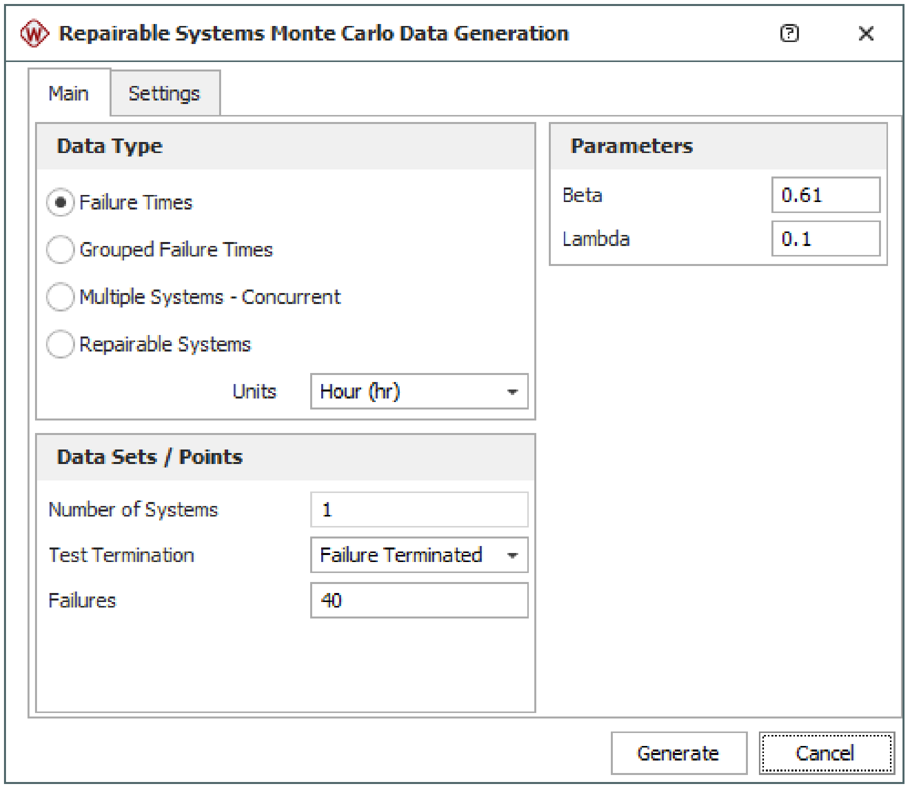
$$t_i = \left[-\frac{ln(1 - Uniform(0,1))}{\lambda} + t_{i-1}^\beta\right]^{\frac{1}{\beta}}$$

To access the data generation utility, choose **Home > Insert > Monte Carlo** and choose the **Repairable Systems Monte Carlo** option. There are different data types that can be generated with the Monte Carlo utility. For all of them, the basic parameters that are always specified are the beta (β) and lambda (λ) parameters of the Crow-AMSAA (NHPP) model. That does not mean that the generated data can be analyzed only with the Crow-AMSAA (NHPP) model. Depending on the data type, the Duane, Crow extended and power law models can also be used. They share the same growth patterns, which are based on the β and λ parameters. In the case of the Duane model, $\beta = 1 - \alpha$, where α is the growth parameter for the Duane model. Below we present the available data types that can generated with the Monte Carlo utility.

- **Failure Times:** The data set is generated assuming a single system. There is a choice between a time terminated test, where the termination time needs to be specified, or a failure terminated test, where the number of failures needs to be specified. The generated failure times data can then be analyzed using the Duane or Crow-AMSAA (NHPP) models, or the Crow extended model if classifications and modes are entered for the failures.

- **Grouped Failure Times:** The data is generated assuming a single system. There is a choice between a time terminated test, where the termination time needs to be specified, or a failure terminated test, where the number of failures needs to be specified. In addition, constant or user-defined intervals need to be specified for the grouping of the data. The generated grouped data can then be analyzed using the Duane or Crow-AMSAA (NHPP) models, or the Crow Extended model if classifications and modes are entered for the failures.
- **Multiple Systems - Concurrent:** In this case, the number of systems needs to be specified. There is a choice between a time terminated test, where the termination time needs to be specified, or a failure terminated test, where the number of failures needs to be specified. The generated folio contains failure times for each of the systems. The data can then be analyzed using the Duane or Crow-AMSAA (NHPP) models, or the Crow Extended model if classifications and modes are entered for the failures.
- **Repairable Systems:** In this case, the number of systems needs to be specified. There is a choice between a time terminated test, where the termination time needs to be specified, or a failure terminated test, where the number of failures needs to be specified. The generated folio contains failure times for each of the systems. The data can then be analyzed using the power law model, or the Crow extended model if classifications and modes are entered for the failures.

The next figure shows the Monte Carlo utility and all the necessary user inputs.



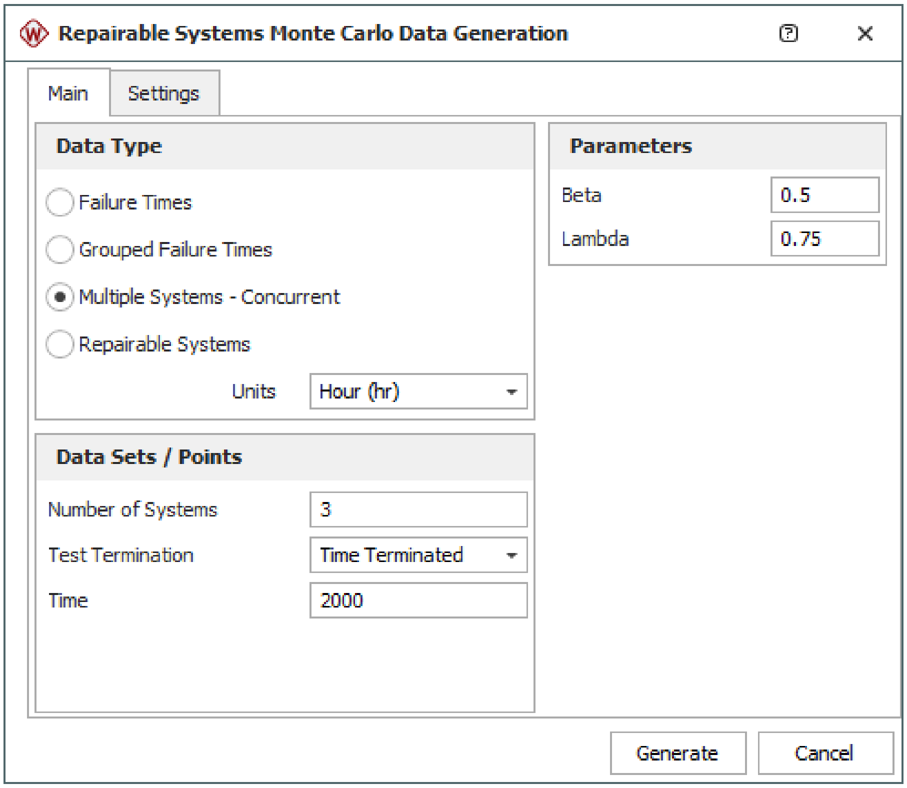
The seed determines the starting point from which the random numbers will be generated. The use of a seed forces the software to use the same sequence of random numbers, resulting in repeatability. In other words, the same failure times can be generated if the same seed, data type, parameters and number of points/systems are used. If no seed is provided, the computer's clock is used to initialize the random number generator and a different set of failure times will be generated at each new request.

Monte Carlo Data Example

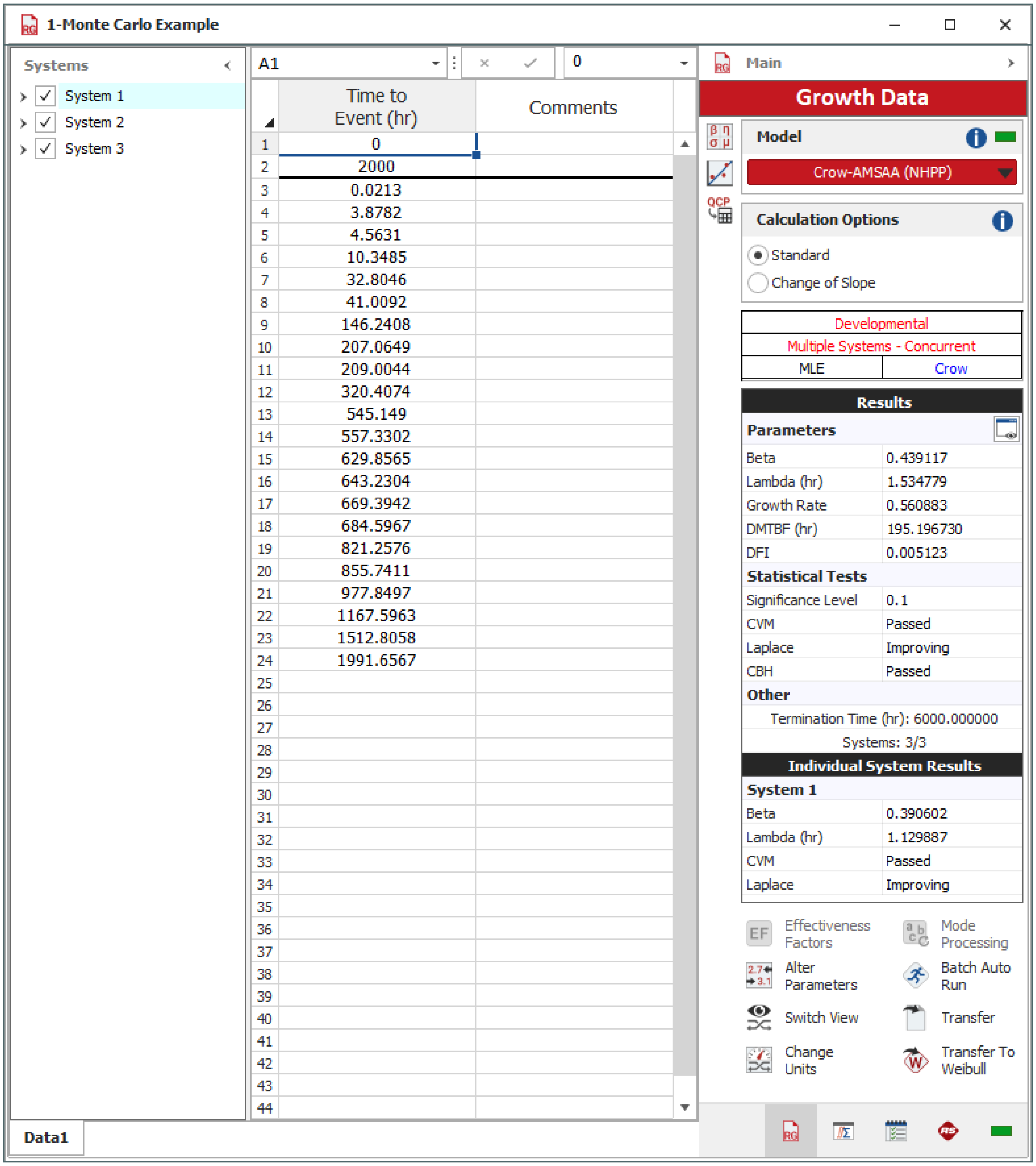
A reliability engineer wants to experiment with different testing scenarios as the reliability growth test of the company's new product is being prepared. From the reliability growth test data of a similar product that was developed previously, the beta and lambda parameters are $\beta = 0.5$ and $\lambda = 0.75.$ Three systems are to be used to generate a representative data set of expected times-to-failure for the upcoming test. The purpose is to explore different test durations in order to demonstrate an MTBF of 200 hours.

Solution

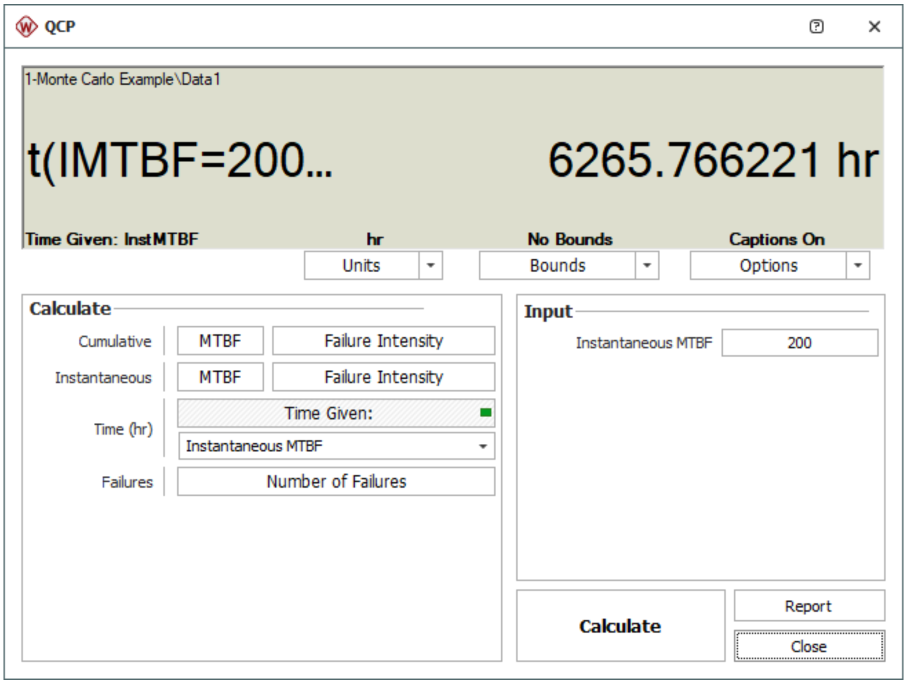
In the Monte Carlo window, the parameters are set to $\beta = 0.5$ and $\lambda = 0.75.$ Since we have three systems, we use the "multiple systems - concurrent" data sheet and then set the number of systems to 3. Initially, the test is set to be time terminated with 2,000 operating hours per system, for a total of 6,000 operating hours. The next figure shows the Monte Carlo window for this example.



The next figure shows the generated failure times data. In this folio, the Advanced Systems View is used, so the data sheet shows the times-to-failure for system 2.



The data can then be analyzed just like a regular folio in the Weibull++ software. In this case, we are interested in analyzing the data with the Crow-AMSAA (NHPP) model to calculate the demonstrated MTBF at the end of the test. In the **Results** area of the folio (shown in the figure above), it can be seen that the demonstrated MTBF at the end of the test is 195.19 hours. Since that does not meet the requirement of demonstrating an MTBF of 200 hours, we can either generate a new Monte Carlo data set with different time termination settings, or access the Quick Calculation Pad (QCP) in this folio to find the time for which the demonstrated (instantaneous) MTBF becomes 200 hours, as shown in the following figure. From the QCP it can be seen that, based on this specific data set, 6265.76 total operating hours are needed to show a demonstrated MTBF of 200 hours.



Note that since the Monte Carlo routine generates random input variables that follow the NHPP based on the specific β and λ values, if the same seed is not used the failure times will be different the next time you run the Monte Carlo routine. Also, because the input variables are pulled from an NHPP with the expected values of β and λ, it should not be expected that the calculated parameters of the generated data set will match exactly the input parameters that were specified. In this example, the input parameters were set as $\beta = 0.5$ and $\lambda = 0.75$, and the data set based on the Monte Carlo generated failure times yielded Crow-AMSAA (NHPP) parameters of $\beta = 0.4391$ and $\lambda = 1.5347$. The next time a data set is generated with a random seed, the calculated parameters will be slightly different, since we are essentially pulling input variables from a predefined distribution. The more simulations that are run, the more the calculated parameters will converge with the expected parameters. In the Weibull++ software, the total number of generated failures with the Monte Carlo utility has to be less than 64,000.

SimuMatic

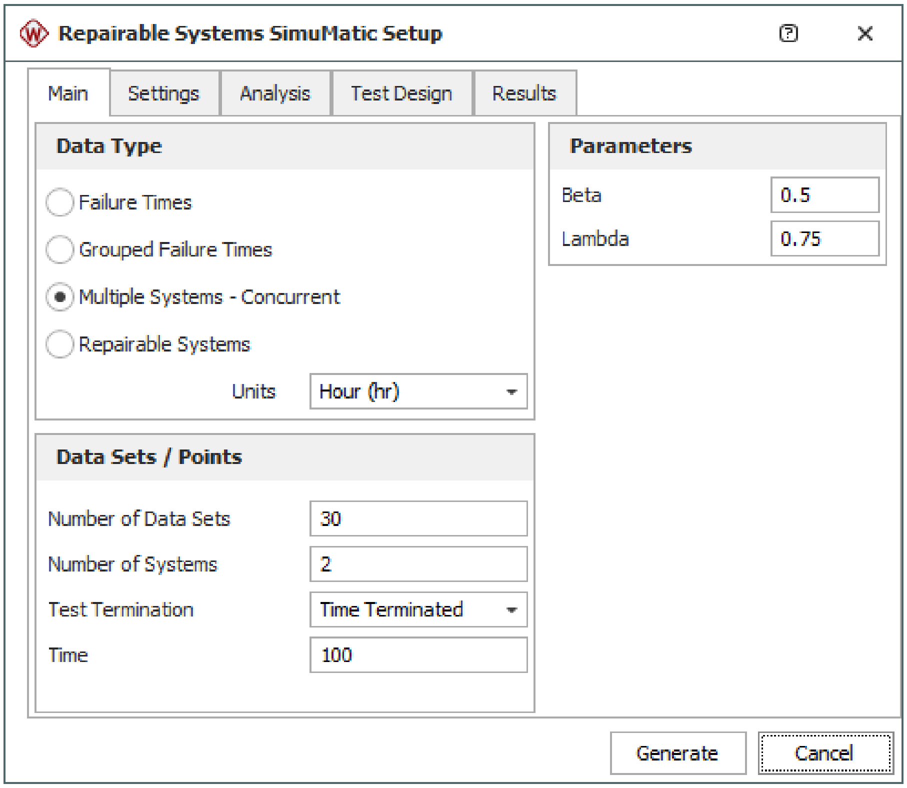
Reliability growth analysis using simulation can be a valuable tool for reliability practitioners. With this approach, reliability growth analyses are performed a large number of times on data sets that have been created using Monte Carlo simulation.

The Weibull++ software's SimuMatic utility generates calculated values of beta and lambda parameters, based on user specified input parameters of beta and lambda. SimuMatic essentially performs a number of Monte Carlo simulations based on user-defined required test time or failure termination settings, and then recalculates the beta and lambda parameters for each of the generated data sets. The number of times that the Monte Carlo data sets are generated and the parameters are re-calculated is also user defined. The final output presents the calculated values of beta and lambda, and allows for various types of analysis.

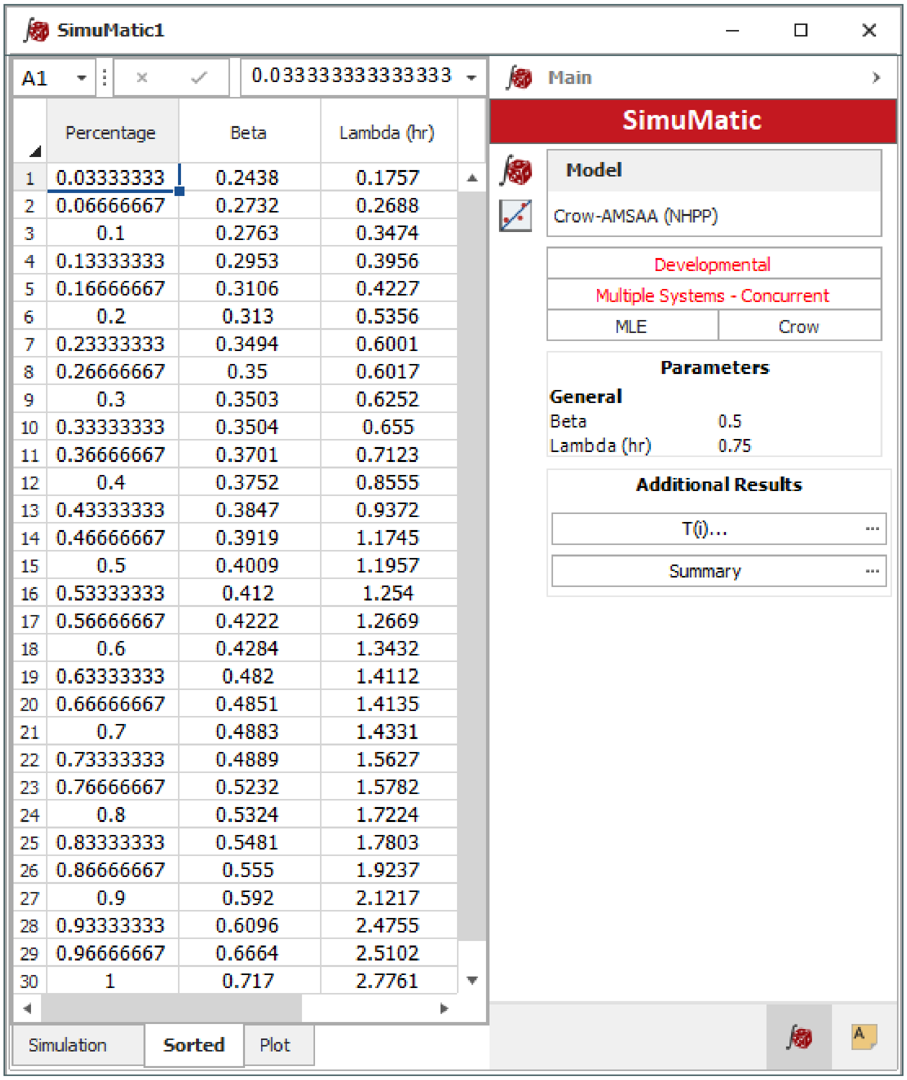
To access the SimuMatic utility, choose **Home > Insert > SimuMatic** and choose the **Repairable Systems SimuMatic** option. For all of the data sets, the basic parameters that are always specified are the beta (β) and lambda (λ) parameters of the Crow-AMSAA (NHPP) model or the power law model.

- **Failure Times**: The data set is generated assuming a single system. There is a choice between a time terminated test, where the termination time needs to be specified, or a failure terminated test, where the number of failures needs to be specified. SimuMatic will return the calculated values of β and λ for a specified number of data sets.
- **Grouped Failure Times**: The data set is generated assuming a single system. There is a choice between a time terminated test, where the termination time needs to be specified, or a failure terminated test, where the number of failures needs to be specified. In addition, constant or user-defined intervals need to be specified for the grouping of the data. SimuMatic will return the calculated values of β and λ for a specified number of data sets.
- **Multiple Systems - Concurrent**: In this case, the number of systems needs to be specified. There is a choice between a time terminated test, where the termination time needs to be specified, or a failure terminated test, where the number of failures needs to be specified. SimuMatic will return the calculated values of β and λ for a specified number of data sets.
- **Repairable Systems:** In this case, the number of systems needs to be specified. There is a choice between a time terminated test, where the termination time needs to be specified, or a failure terminated test, where the number of failures needs to be specified. SimuMatic will return the calculated values of β and λ for a specified number of data sets.

The next figure shows the Main tab of the SimuMatic window where all the necessary user inputs for a multiple systems - concurrent data set have been entered. The Analysis tab allows you to specify the confidence level for simulation-generated confidence bounds, while the Results tab gives you the option to compute for additional results, such as the instantaneous MTBF given a specific test time.

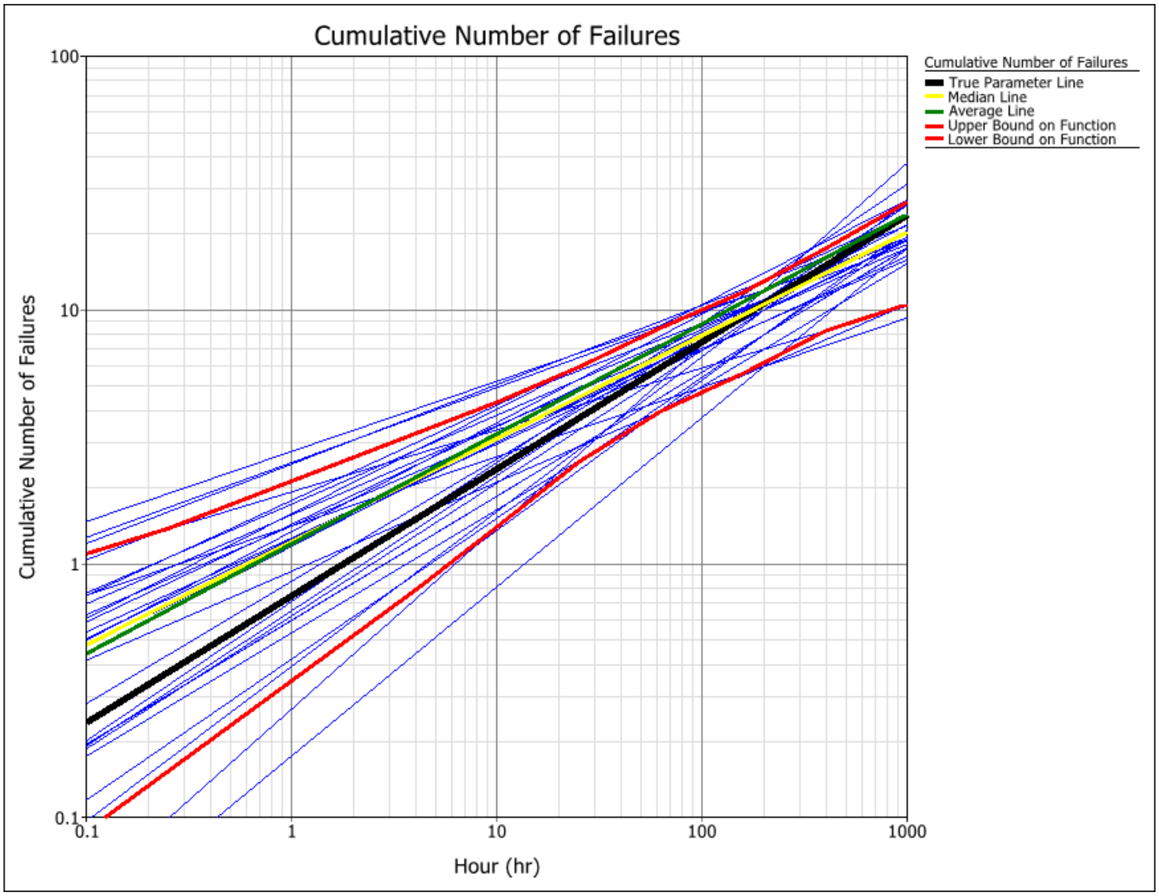


The next figure shows the generated results based on the inputs shown above. The data sheet called "Sorted" allows us to extract conclusions about the simulation-generated confidence bounds because the lambda and beta parameters and any other additional output are sorted by percentage.



	Percentage	Beta	Lambda (hr)
1	0.03333333	0.2438	0.1757
2	0.06666667	0.2732	0.2688
3	0.1	0.2763	0.3474
4	0.13333333	0.2953	0.3956
5	0.16666667	0.3106	0.4227
6	0.2	0.313	0.5356
7	0.23333333	0.3494	0.6001
8	0.26666667	0.35	0.6017
9	0.3	0.3503	0.6252
10	0.33333333	0.3504	0.655
11	0.36666667	0.3701	0.7123
12	0.4	0.3752	0.8555
13	0.43333333	0.3847	0.9372
14	0.46666667	0.3919	1.1745
15	0.5	0.4009	1.1957
16	0.53333333	0.412	1.254
17	0.56666667	0.4222	1.2669
18	0.6	0.4284	1.3432
19	0.63333333	0.482	1.4112
20	0.66666667	0.4851	1.4135
21	0.7	0.4883	1.4331
22	0.73333333	0.4889	1.5627
23	0.76666667	0.5232	1.5782
24	0.8	0.5324	1.7224
25	0.83333333	0.5481	1.7803
26	0.86666667	0.555	1.9237
27	0.9	0.592	2.1217
28	0.93333333	0.6096	2.4755
29	0.96666667	0.6664	2.5102
30	1	0.717	2.7761

The following plot shows the simulation-confidence bounds for the cumulative number of failures based on the input parameters specified.



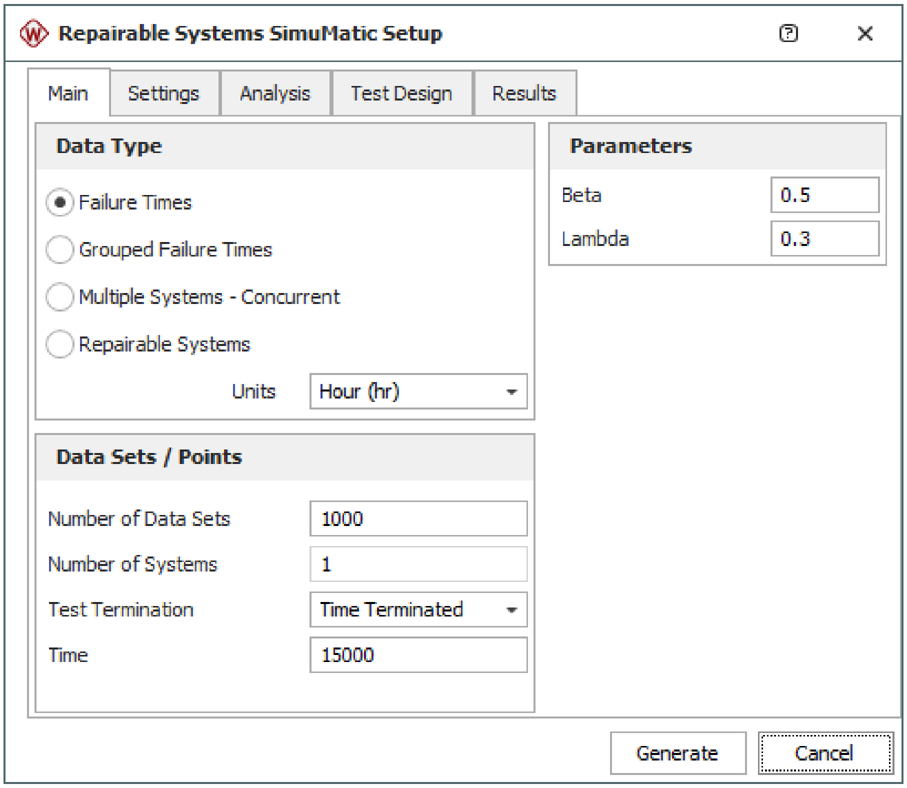
SimuMatic Example

A manufacturer wants to design a reliability growth test for a redesigned product, in order to achieve an MTBF of 1,000 hours. Simulation is chosen to estimate the 1-sided 90% confidence bound on the required time to achieve the goal MTBF of 1,000 hours and the 1-sided 90% lower confidence bound on the MTBF at the end of the test time. The total test time is expected to be 15,000 hours. Based on historical data for the previous version, the expected beta and lambda parameters of the test are 0.5 and 0.3, respectively. Do the following:

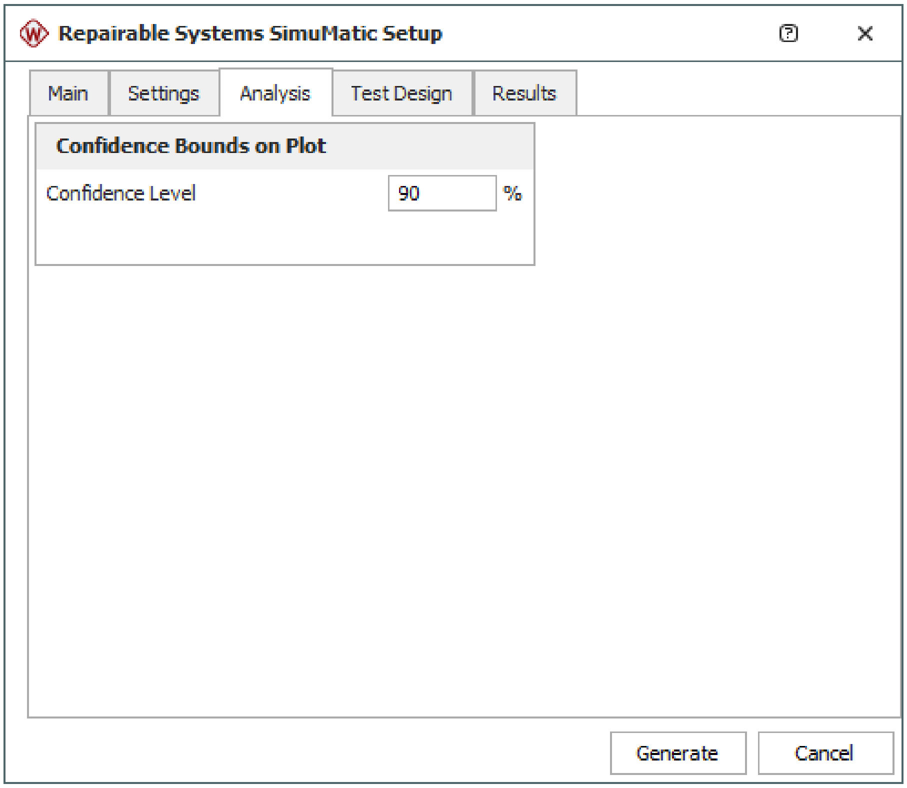
1. Generate 1,000 data sets using SimuMatic along with the required output.
2. Plot the instantaneous MTBF vs. time with the 90% confidence bounds.
3. Estimate the 1-sided 90% lower confidence bound on time for an MTBF of 1,000 hours.
4. Estimate the 1-sided 90% lower confidence bound on the instantaneous MTBF at the end of the test.

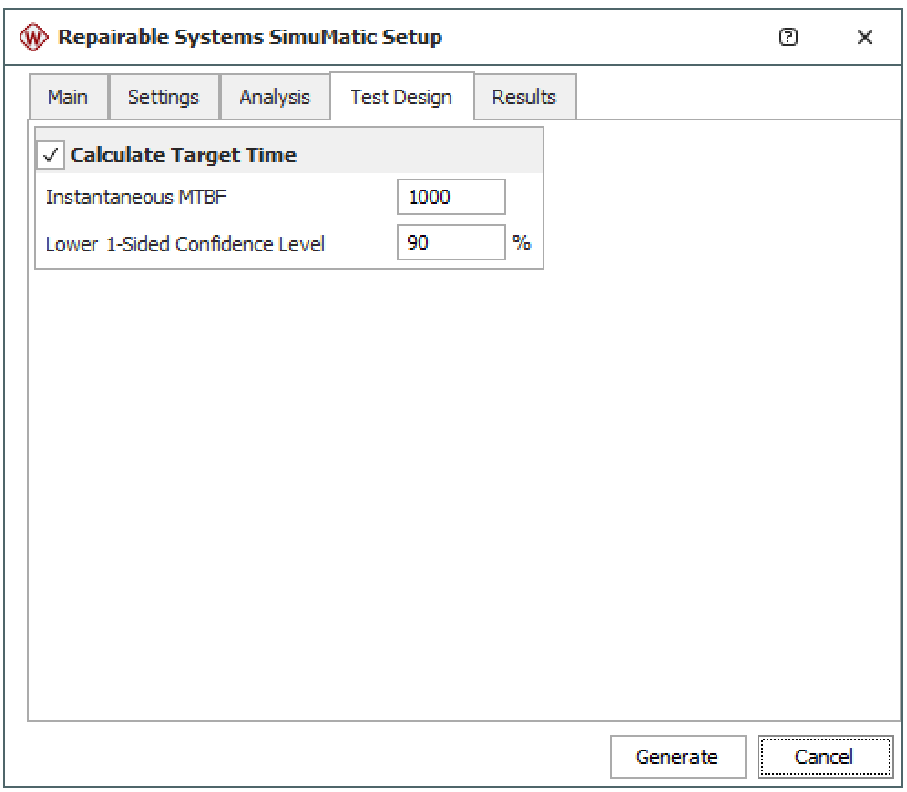
Solution

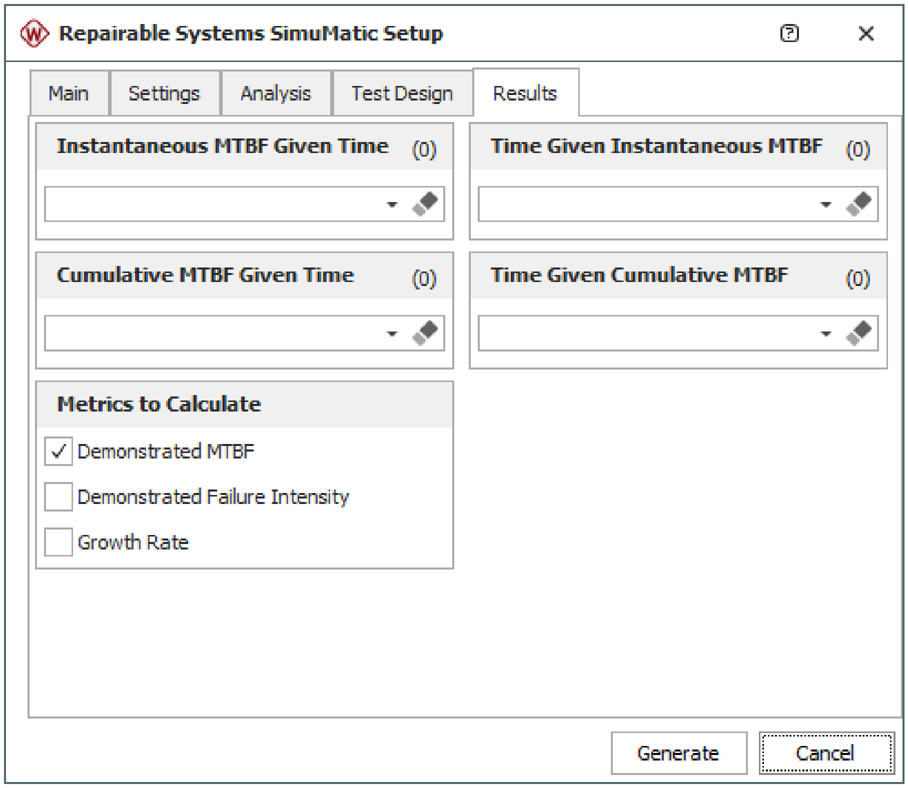
1. The next figure shows the SimuMatic window with all the appropriate inputs for creating the data sets.



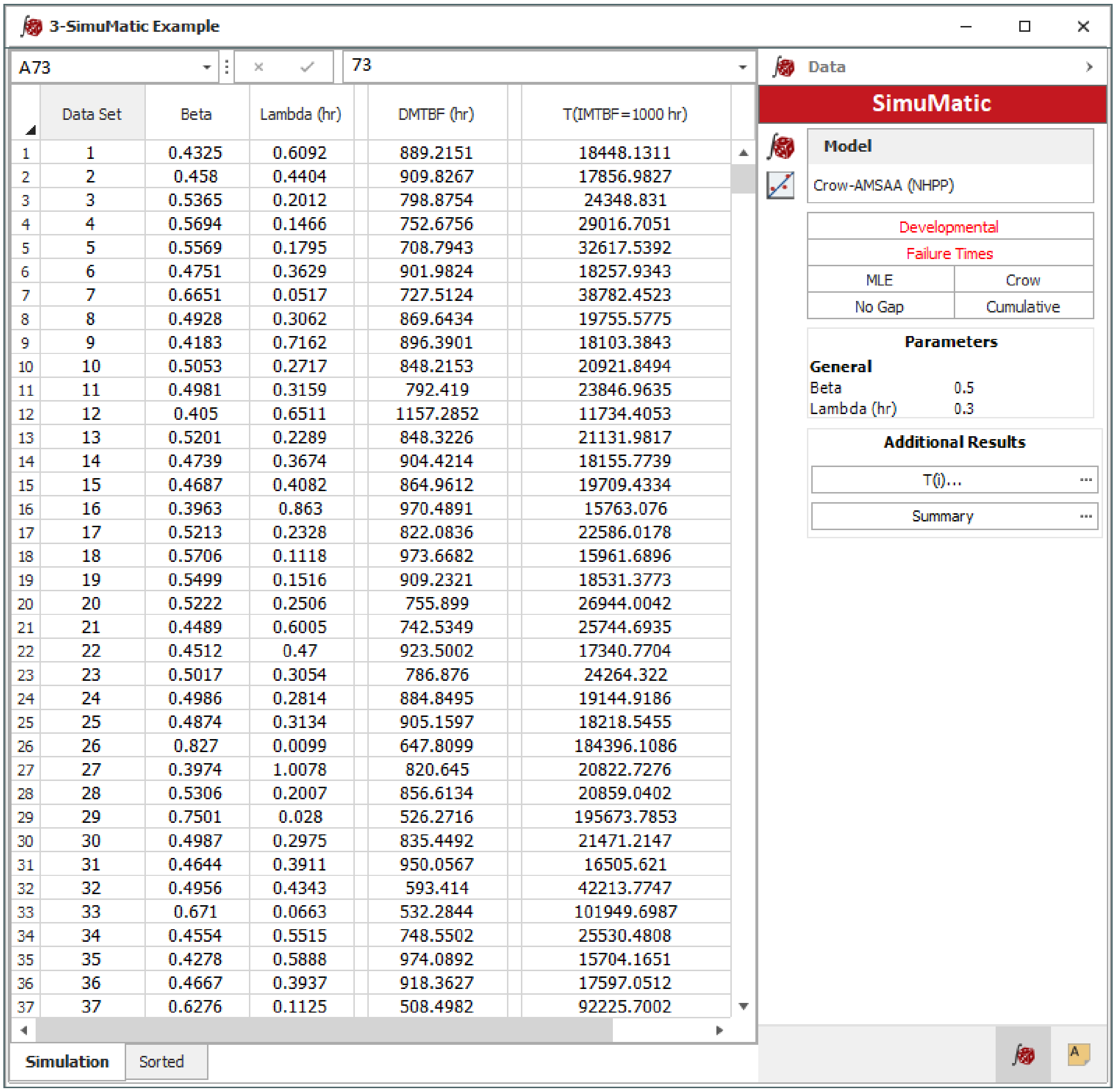
The next three figures show the settings in the Analysis, Test Design and Results tab of the SimuMatic window in order to obtain the desired outputs.


Repairable Systems SimuMatic Setup
Main
Settings
Analysis
Test Design
Results
Confidence Bounds on Plot
Confidence Level
90
%
Generate
Cancel


Repairable Systems SimuMatic Setup
Main
Settings
Analysis
Test Design
Results
Calculate Target Time
Instantaneous MTBF
1000
Lower 1-Sided Confidence Level
90
%
Generate
Cancel

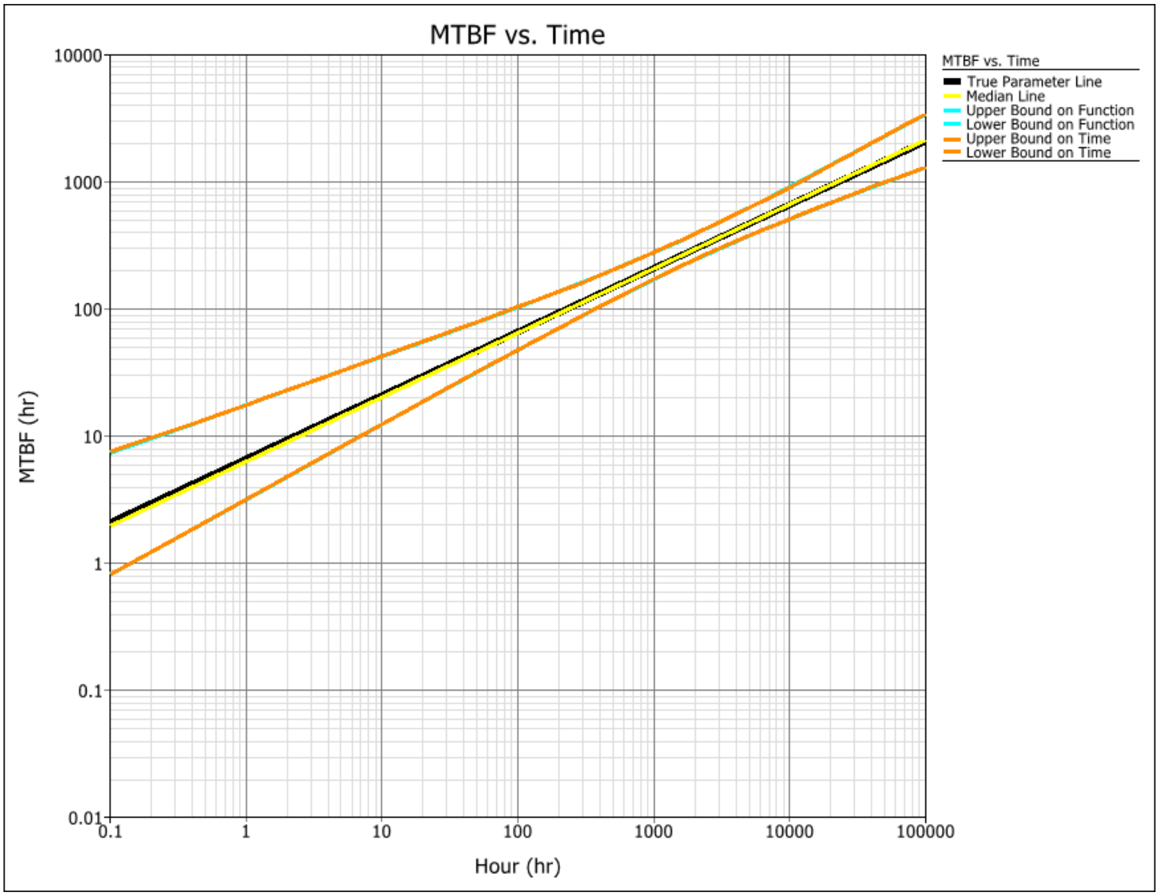


The following figure displays the results of the simulation. The columns labeled "Beta" and "Lambda" contain the different parameters obtained by calculating each data set generated via simulation for the 1,000 data sets. The "DMTBF" column contains the instantaneous MTBF at 15,000 hours (the end of test time), given the parameters obtained by calculating each data set generated via simulation. The "T(IMTBF=1000 Hr)" column contains the time required for the MTBF to reach 1,000 hours, given the parameters obtained from the simulation.

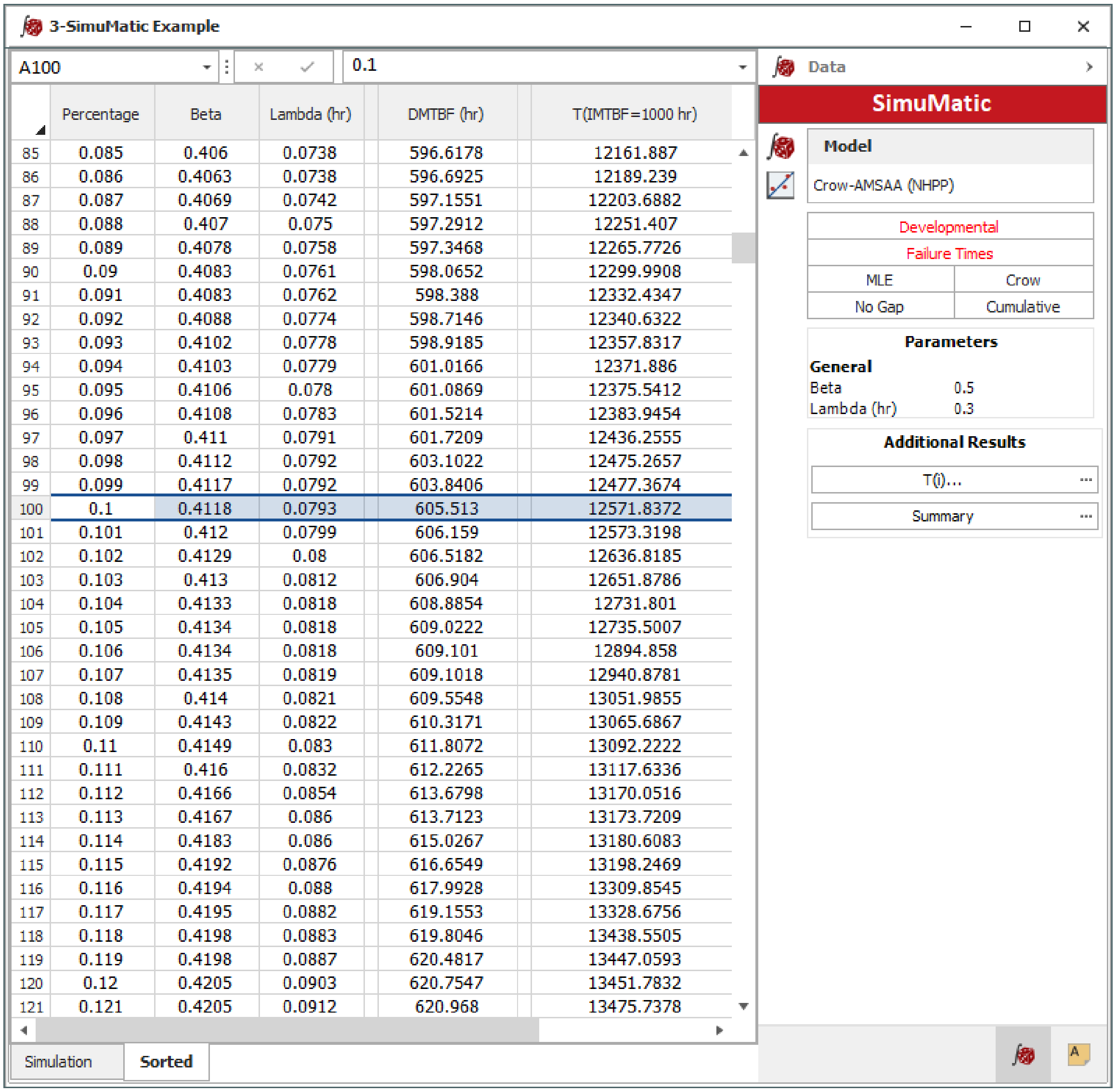


	Data Set	Beta	Lambda (hr)	DMTBF (hr)	T(IMTBF=1000 hr)
1	1	0.4325	0.6092	889.2151	18448.1311
2	2	0.458	0.4404	909.8267	17856.9827
3	3	0.5365	0.2012	798.8754	24348.831
4	4	0.5694	0.1466	752.6756	29016.7051
5	5	0.5569	0.1795	708.7943	32617.5392
6	6	0.4751	0.3629	901.9824	18257.9343
7	7	0.6651	0.0517	727.5124	38782.4523
8	8	0.4928	0.3062	869.6434	19755.5775
9	9	0.4183	0.7162	896.3901	18103.3843
10	10	0.5053	0.2717	848.2153	20921.8494
11	11	0.4981	0.3159	792.419	23846.9635
12	12	0.405	0.6511	1157.2852	11734.4053
13	13	0.5201	0.2289	848.3226	21131.9817
14	14	0.4739	0.3674	904.4214	18155.7739
15	15	0.4687	0.4082	864.9612	19709.4334
16	16	0.3963	0.863	970.4891	15763.076
17	17	0.5213	0.2328	822.0836	22586.0178
18	18	0.5706	0.1118	973.6682	15961.6896
19	19	0.5499	0.1516	909.2321	18531.3773
20	20	0.5222	0.2506	755.899	26944.0042
21	21	0.4489	0.6005	742.5349	25744.6935
22	22	0.4512	0.47	923.5002	17340.7704
23	23	0.5017	0.3054	786.876	24264.322
24	24	0.4986	0.2814	884.8495	19144.9186
25	25	0.4874	0.3134	905.1597	18218.5455
26	26	0.827	0.0099	647.8099	184396.1086
27	27	0.3974	1.0078	820.645	20822.7276
28	28	0.5306	0.2007	856.6134	20859.0402
29	29	0.7501	0.028	526.2716	195673.7853
30	30	0.4987	0.2975	835.4492	21471.2147
31	31	0.4644	0.3911	950.0567	16505.621
32	32	0.4956	0.4343	593.414	42213.7747
33	33	0.671	0.0663	532.2844	101949.6987
34	34	0.4554	0.5515	748.5502	25530.4808
35	35	0.4278	0.5888	974.0892	15704.1651
36	36	0.4667	0.3937	918.3627	17597.0512
37	37	0.6276	0.1125	508.4982	92225.7002

2. The next figure shows the plot of the instantaneous MTBF with the 90% confidence bounds.

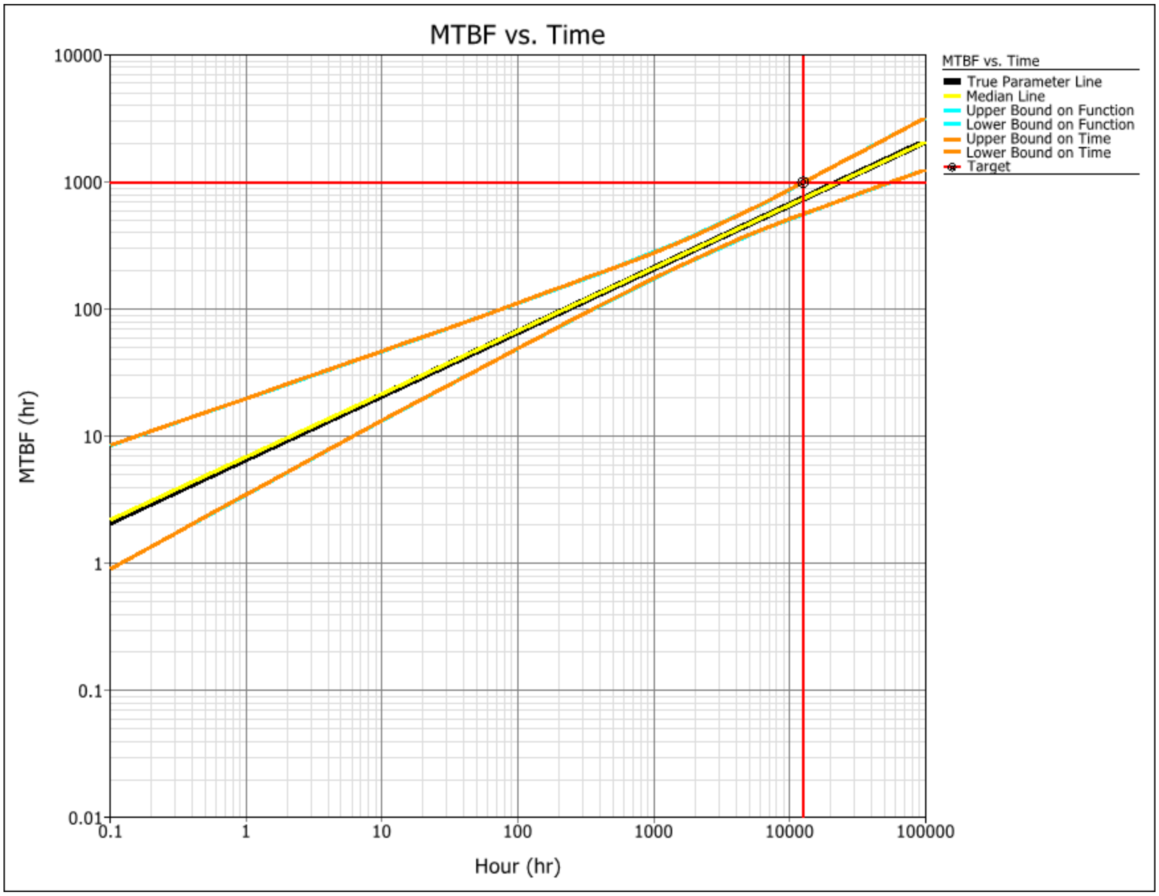


3. The 1-sided 90% lower confidence bound on time, assuming MTBF = 1,000 hours, can be obtained from the results of the simulation. In the "Sorted" data sheet, this is the target DMTBF value that corresponds to 10.00%, as shown in the next figure. Therefore the 1-sided 90% lower confidence bound on time is 12,571.83 hours.



	Percentage	Beta	Lambda (hr)	DMTBF (hr)	T(IMTBF=1000 hr)
85	0.085	0.406	0.0738	596.6178	12161.887
86	0.086	0.4063	0.0738	596.6925	12189.239
87	0.087	0.4069	0.0742	597.1551	12203.6882
88	0.088	0.407	0.075	597.2912	12251.407
89	0.089	0.4078	0.0758	597.3468	12265.7726
90	0.09	0.4083	0.0761	598.0652	12299.9908
91	0.091	0.4083	0.0762	598.388	12332.4347
92	0.092	0.4088	0.0774	598.7146	12340.6322
93	0.093	0.4102	0.0778	598.9185	12357.8317
94	0.094	0.4103	0.0779	601.0166	12371.886
95	0.095	0.4106	0.078	601.0869	12375.5412
96	0.096	0.4108	0.0783	601.5214	12383.9454
97	0.097	0.411	0.0791	601.7209	12436.2555
98	0.098	0.4112	0.0792	603.1022	12475.2657
99	0.099	0.4117	0.0792	603.8406	12477.3674
100	0.1	0.4118	0.0793	605.513	12571.8372
101	0.101	0.412	0.0799	606.159	12573.3198
102	0.102	0.4129	0.08	606.5182	12636.8185
103	0.103	0.413	0.0812	606.904	12651.8786
104	0.104	0.4133	0.0818	608.8854	12731.801
105	0.105	0.4134	0.0818	609.0222	12735.5007
106	0.106	0.4134	0.0818	609.101	12894.858
107	0.107	0.4135	0.0819	609.1018	12940.8781
108	0.108	0.414	0.0821	609.5548	13051.9855
109	0.109	0.4143	0.0822	610.3171	13065.6867
110	0.11	0.4149	0.083	611.8072	13092.2222
111	0.111	0.416	0.0832	612.2265	13117.6336
112	0.112	0.4166	0.0854	613.6798	13170.0516
113	0.113	0.4167	0.086	613.7123	13173.7209
114	0.114	0.4183	0.086	615.0267	13180.6083
115	0.115	0.4192	0.0876	616.6549	13198.2469
116	0.116	0.4194	0.088	617.9928	13309.8545
117	0.117	0.4195	0.0882	619.1553	13328.6756
118	0.118	0.4198	0.0883	619.8046	13438.5505
119	0.119	0.4198	0.0887	620.4817	13447.0593
120	0.12	0.4205	0.0903	620.7547	13451.7832
121	0.121	0.4205	0.0912	620.968	13475.7378

4. The next figure shows the 1-sided 90% lower confidence bound on time in the instantaneous MTBF plot. This is indicated by the target lines on the plot.



5. The 1-sided 90% lower confidence bound on the instantaneous MTBF at the end of the test is again obtained from the "Sorted" data sheet by looking at the value in the "IMTBF (15,000)" column that corresponds to 10.00%. As seen in the simulation results shown above, the 1-sided 90% lower confidence bound on the instantaneous MTBF at the end of the test is 605.93 hours.

Reliability Demonstration Test Design for Repairable Systems

IN THIS CHAPTER

The Weibull++ software provides a utility that allows you to design reliability demonstration tests for repairable systems. For example, you may want to design a test to demonstrate a specific cumulative MTBF for a specific operating period at a specific confidence level for a repairable system. The same applies for demonstrating an instantaneous MTBF or cumulative and instantaneous failure intensity at a given time t.

Underlying Theory

The failure process in a repairable system is considered to be a non-homogeneous Poisson process (NHPP) with power law failure intensity. The instantaneous failure intensity at time t is:

$$\lambda_i\left(t\right)=\lambda\beta t^{\beta-1}=\lambda_c\left(t\right)\beta$$

So, the cumulative failure intensity at time t is:

$$\lambda_c\left(t\right)=\lambda t^{\beta-1}=\frac{\lambda_i\left(t\right)}{\beta}$$

The instantaneous MTBF is:

$$\begin{aligned} MTBF_i(t) &= \frac{1}{\lambda_{i(t)}} \\ &= \frac{1}{\lambda\beta} t^{1-\beta} \\ &= \frac{MTBF_c(t)}{\beta} \end{aligned}$$

The cumulative MTBF at time t is:

$$MTBF_c(t) = \frac{1}{\lambda_c(t)} = \frac{1}{\lambda} t^{1-\beta} = MTBF_i(t)\,\beta$$

The relation between the confidence level, required test time, number of systems under test and allowed total number of failures in the test is:

$$1 - CL = \sum_{i=0}^{r} \frac{\left(m\lambda T^{\beta}\right)^{i} \exp(-m\lambda T^{\beta})}{i!}$$

where:

- T is the total test time for each system.
- m is the number of systems under test.
- r is the number of allowed failures in the test.
- CL is the confidence level.

Given any three of the parameters, the equation above can be solved for the fourth unknown parameter. Note that when $\beta = 1$, the number of failures is a homogeneous Poisson process, and the time between failures is given by the exponential distribution.

Example: Solve for Time

The objective is to design a test to demonstrate that the number of failures per system in five years is less than or equal to 10. In other words, demonstrate that the cumulative MTBF for a repairable system is less than or equal to 0.5 during a five year operating period, with 80% confidence level. Assume that $\beta = 1$, the number of systems for the test is $m = 6$ and that the number of allowed failures in the test is $r = 2$.

Solution

Since the given requirement is the number of failures, we transfer the requirement to the cumulative MTBF or cumulative failure intensity.

$$MTBF_c = \frac{5}{10} = 0.5year$$

then:

$$\lambda_c = \frac{1}{MTBF_c} = 2\ failures/\ \ year$$

We can then solve for λ :

$$\lambda_c\left(t\right) = \lambda t^{\beta - 1}$$

For the five year period:

$$\lambda_c\left(5\right) = \lambda \cdot 5^{\beta - 1}$$

Using the values of λ and β, we have:

$$2 = \lambda \cdot 5^{1-1}$$

Then solving for λ yields:

$$\lambda = 2$$

We can then solve for the required test time, T, for each system:

$$1 - CL = \sum_{i=0}^{r} \frac{\left(m\lambda T^{\beta}\right)^{i} \exp(-m\lambda T^{\beta})}{i!}$$

or:

$$1 - 0.8 = \sum_{i=0}^{2} \frac{\left(6 \cdot 0.894 \cdot T^{1}\right)^{i} \exp(-6 \cdot 2 \cdot T^{1})}{i!}$$

or:

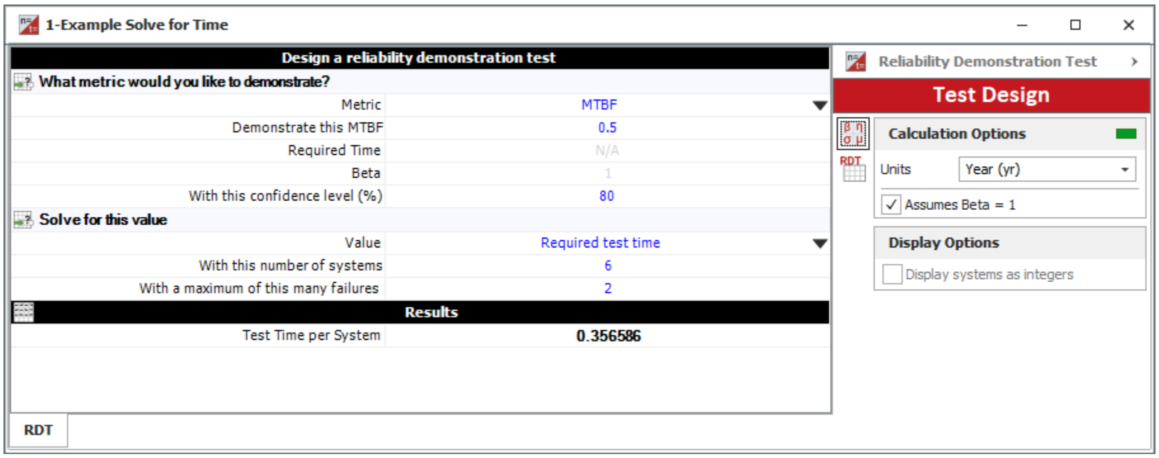
$$\begin{aligned} 0.2 =& exp(-6 \cdot 2 \cdot T^{1}) \\ &+ \frac{(6 \cdot 2 \cdot T^{1})^{1} exp(-6 \cdot 2 \cdot T^{1})}{1!} \\ &+ \frac{6 \cdot 2 \cdot T^{1})^{2} exp(-6 \cdot 2 \cdot T^{1})}{2!} \end{aligned}$$

Solving the above equation numerically yields:

$$T = 0.36$$

In other words, for this example we have to test for 0.36 years to demonstrate that the number of failures per system in five years is less than or equal to 10.

The same result can be obtained in the Weibull++ software, by using the Design of Reliability Tests (DRT) tool. The next figure shows the calculated required test time per system of 0.3566 on the results of the example.

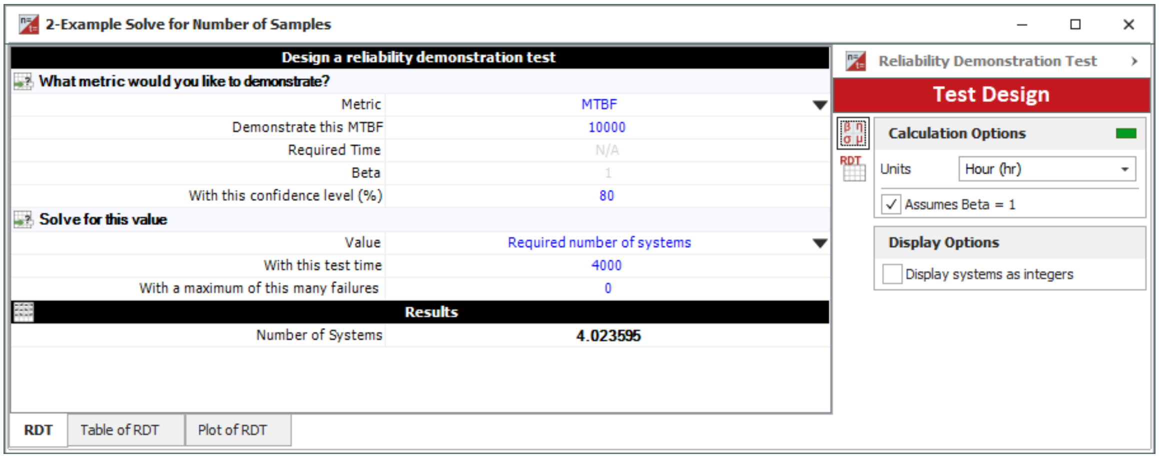


Example: Solve for Number of Samples

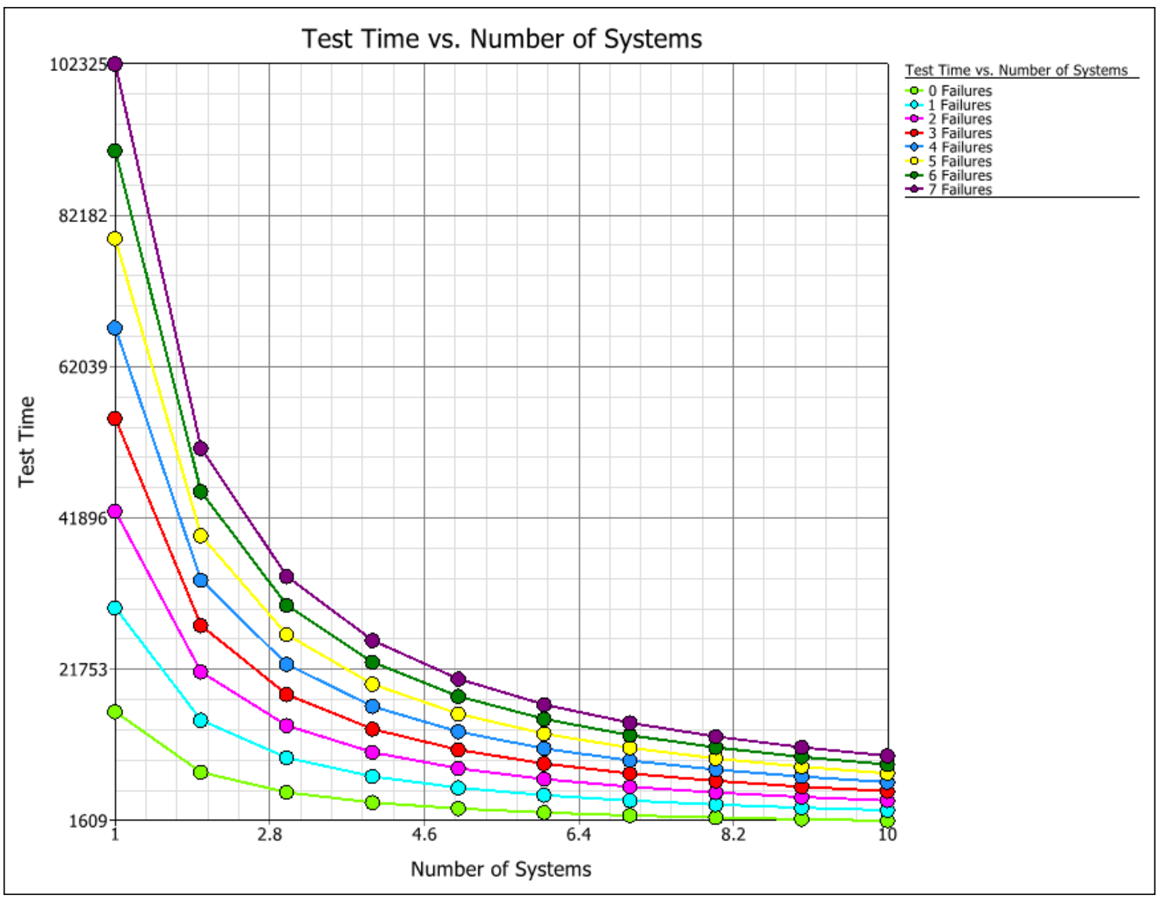
At the end of a reliability growth testing program, a manufacturer wants to demonstrate that a new product has achieved an MTBF of 10,000 hours with 80% confidence. The available time for the demonstration test is 4,000 hours for each test unit. Assuming zero failures are allowed, what is the required number of units to be tested in order to demonstrate the desired MTBF?

Solution

We can obtain the required number of units by using the Design of Reliability Tests (DRT) tool in the Weibull++ software. Since this is a demonstration test then $\beta = 1$, and no growth will be achieved. The results are shown next. It can be seen that in order to demonstrate a 10,000 hours MTBF with 80% confidence, 5 test units will be required.



The next figure shows the different combinations of test time per unit, and the number of units in the test for different numbers of allowable failures. It helps to visually examine other possible test scenarios.



Appendices

IN THIS CHAPTER

Failure Discounting

During a reliability growth test, once a failure has been analyzed and corrective actions for that specific failure mode have been implemented, the probability of its recurrence is diminished, as given in Lloyd [4]. Then for the success/failure data that follow, the value of the failure for which corrective actions have already been implemented should be subtracted from the total number of failures. However, certain questions arise, such as to what extent should the failure value be diminished or discounted, and how should the failure value be defined? One answer would be to use engineering judgment (e.g., a panel of specialists would agree that the probability of failure has been reduced by 50% or 90% and therefore, that failure should be given a value of 0.5 or 0.9). The obvious disadvantage of this approach is its arbitrariness and the difficulty of reaching an agreement. Therefore, a statistical basis needs to be selected, one that is repeatable and less arbitrary. Failure discounting is applied when using the Lloyd-Lipow, logistic, and the standard and modified Gompertz models.

The value of the failure, f, is chosen to be the upper confidence limit on the probability of failure based on the number of successful tests following implementation of the corrective action. The failure value is given by the following equation:

$$f=1-(1-CL)^{\frac{1}{S_n}}$$

where:

- CL is the confidence level.
- S_n is the number of successful tests after the first success following the corrective action.

For example, after one successful test following a corrective action, $S_n=1$, the failure is given a value of 0.9 based on a 90% confidence level. After two successful tests, $S_n=2$, the failure is given a value of 0.684, and so on. The procedure for applying this method is illustrated in the next example.

Example

Assume that during the 22 launches given in the first table below, the first failure was caused by Mode 1, the second and fourth failures were caused by Mode 2, the third and fifth failures were caused by Mode 3, the sixth failure was caused by Mode 4 and the seventh failure was caused by Mode 5.

1. Find the standard Gompertz reliability growth curve using the results of the first 15 launches.
2. Find the predicted reliability after launch 22.
3. Calculate the reliability after launch 22 based on the full data set from the second table, and compare it with the estimate obtained for question 2.

Launch Sequence with Failure Modes and Failure Values							
Launch Number	**Result/Mode**	**Failure 1**	**Failure 2**	**Failure 3**	**Failure 4**	**Failure 5**	**Sum of Failures**
1	F1	1.000					1.000
2	F2	1.000	1.000				2.000
3	F3	0.900	1.000	1.000			2.900
4	S	0.684	0.900	1.000			2.584
5	F2	0.536	1.000	0.900			2.436
6	F3	0.438	1.000	1.000			2.438
7	S	0.369	0.900	1.000			2.269
8	S	0.319	0.684	0.900			1.902
9	S	0.280	0.536	0.684			1.500
10	S	0.250	0.438	0.536			1.224
11	S	0.226	0.369	0.438			1.032
12	S	0.206	0.319	0.369			0.894

13	S	0.189	0.280	0.319			0.788
14	S	0.175	0.250	0.280			0.705
15	S	0.162	0.226	0.250			0.638
16	S	0.152	0.206	0.226			0.584
17	F4	0.142	0.189	0.206	1.000		1.537
18	S	0.134	0.175	0.189	1.000		1.498
19	F5	0.127	0.162	0.175	0.900	1.000	2.364
20	S	0.120	0.152	0.162	0.684	1.000	2.118
21	S	0.114	0.142	0.152	0.536	0.900	1.844
22	S	0.109	0.134	0.142	0.438	0.684	1.507

Comparison of the Predicted Reliability with the Actual Data

Launch Number	Calculated Reliability (%)	ln(R)	Gompertz Reliability (%)
1	0.000		
2	0.000		
3	3.333	1.204	
4	35.406	3.567	28.617
5	51.283	3.937	45.883
6	59.372	4.084	60.841
7	67.585	4.213	72.017
8	76.219	4.334	79.654
9	83.334	4.423	84.600
		$S_1 = 24.558$	
10	87.764	4.475	87.701

11	90.614	4.507	89.609
12	92.555	4.528	90.769
13	93.939	4.543	91.469
14	94.964	4.553	91.891
15	95.746	4.562	92.143
		S_2 = 27.167	
16	96.356	4.568	92.295
17	90.960	4.510	92.385
18	91.681	4.518	92.439
19	87.560	4.472	92.472
20	89.411	4.493	92.491
21	91.219	4.513	92.503
		S_3 = 27.076	
22	93.152	4.534	92.510

Solution

1. In the table above, the failures are represented by columns "Failure 1", "Failure 2", etc. The "Result/Mode" column shows whether each launch is a failure (indicated by the failure modes F1, F2, etc.) or a success (S). The values of failure are based on $CL = 0.90$ and are calculated from:

$$f = 1 - (1 - CL)^{\frac{1}{S_n}}$$

These values are summed and the reliability is calculated from:

$$R = \left[1 - \left(\frac{\sum_{i=1}^{N} f_i}{n}\right)\right] \cdot 100\ \%$$

where N is the number of failures and n is the number of events, tests, runs or launches.

- Failure 1 is Mode 1; it occurs at launch 1 and it does not recur throughout the process. So at launch 3, $S_n = 1$, and so on.

- Failure 2 is Mode 2; it occurs at launch 2 and it recurs at launch 5. Therefore, $S_n = 1$ at launch 4 and at launch 7, and so on.
- Failure 3 is Mode 3; it occurs at launch 3 and it recurs at launch 6. Therefore, $S_n = 1$ at launch 5 and at launch 8, and so on.
- Failure 6 is Mode 4; it occurs at launch 17 and it does not recur throughout the process. So at launch 19, $S_n = 1$, and so on.
- Failure 7 is Mode 5; it occurs at launch 19 and it does not recur throughout the process. So at launch 21, $S_n = 1$, and so on.

For launch 3 and failure 1, $S_n = 1$.

$$f_{1/3} = 1 - (1 - 0.90)^{1/1} = 0.900$$

For launch 4 and failure 1, $S_n = 2$.

$$f_{1/4} = 1 - (1 - 0.90)^{1/2} = 0.684$$

And so on. Calculate the initial values of the Gompertz parameters using the second table above. Based on the equations from the Gompertz Models chapter, the initial values are:

$$\begin{aligned} c &= \left(\frac{S_2 - S_3}{S_2 - S_1}\right)^{\frac{1}{n \cdot I}} \\ &= \left[\frac{27.167 - 27.076}{27.167 - 24.558}\right]^{\frac{1}{6}} \\ &= 0.572 \end{aligned}$$

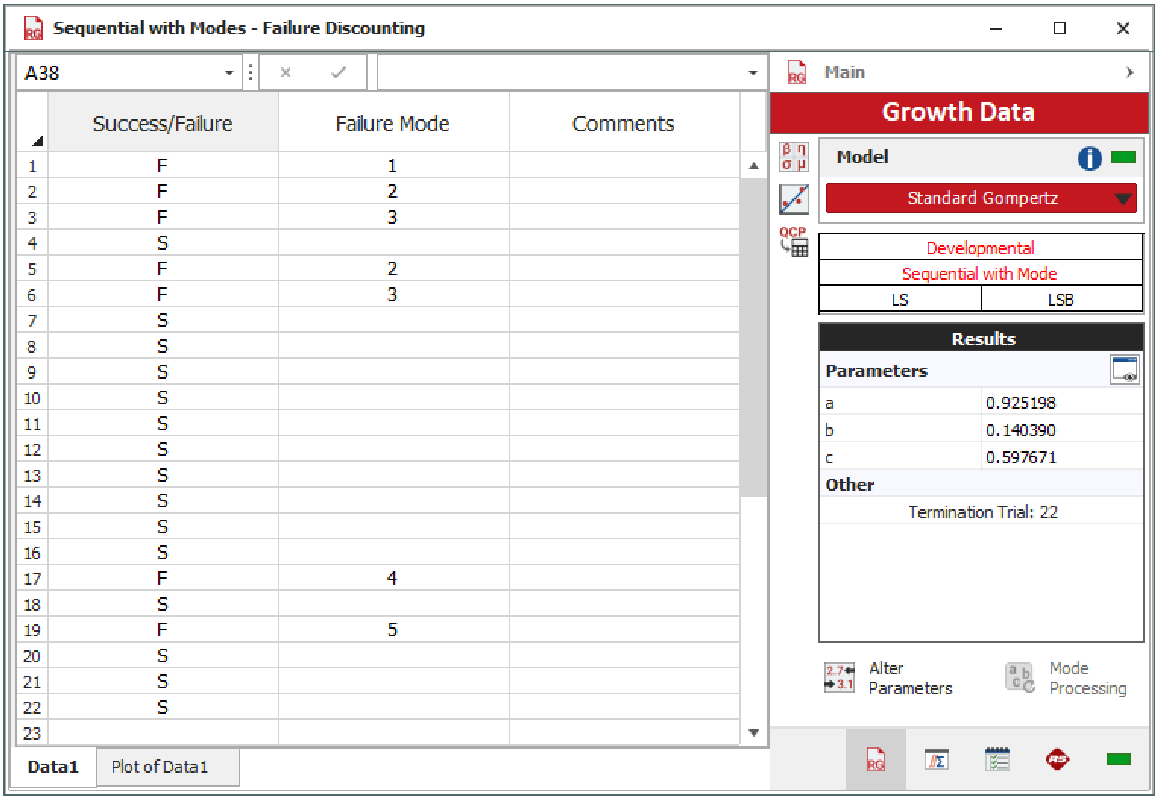
$$\begin{aligned} a &= e^{\left[\frac{1}{n}\left(S_1 + \frac{S_2 - S_1}{1 - e^{n \cdot I}}\right)\right]} \\ &= e^{\left[\frac{1}{6}\left(24.558 + \frac{27.167 - 24.558}{1 - 0.572^6}\right)\right]} \\ &= 94.024\% \end{aligned}$$

$$\begin{aligned} b &= e^{\left[\frac{(S_2 - S_1)(c - 1)}{(1 - c^n)^2}\right]} \\ &= e^{\left[\frac{(27.167 - 24.558)(0.572 - 1)}{(1 - 0.572^6)^2}\right]} \\ &= 0.301 \end{aligned}$$

Now, since the initial values have been determined, the Gauss-Newton method can be used. Substituting $Y_i = R_i, g_1^{(0)} = 94.024, g_2^{(0)} = 0.301, g_3^{(0)} = 0.572$. The iterations are continued to solve for the parameters. Using the Weibull++ software, the estimators of the parameters for the given example are:

$$\hat{a} = 0.9252$$
$$\hat{b} = 0.1404$$
$$\hat{c} = 0.5977$$

The next figure shows the entered data and the estimated parameters.



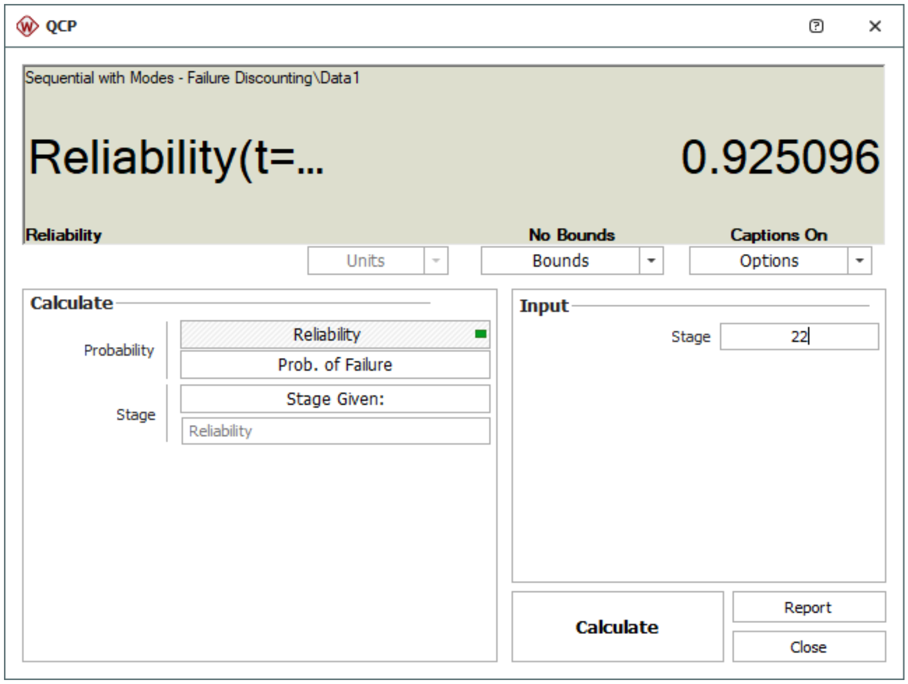
The Gompertz reliability growth curve may now be written as follows where L_G is the number of launches, with the first successful launch being counted as $L_G = 1$. Therefore, L_G is equal to 19, since reliability growth starts with launch 4.

$$R = 0.9299(0.0943)^{0.7170^{L_G}}$$

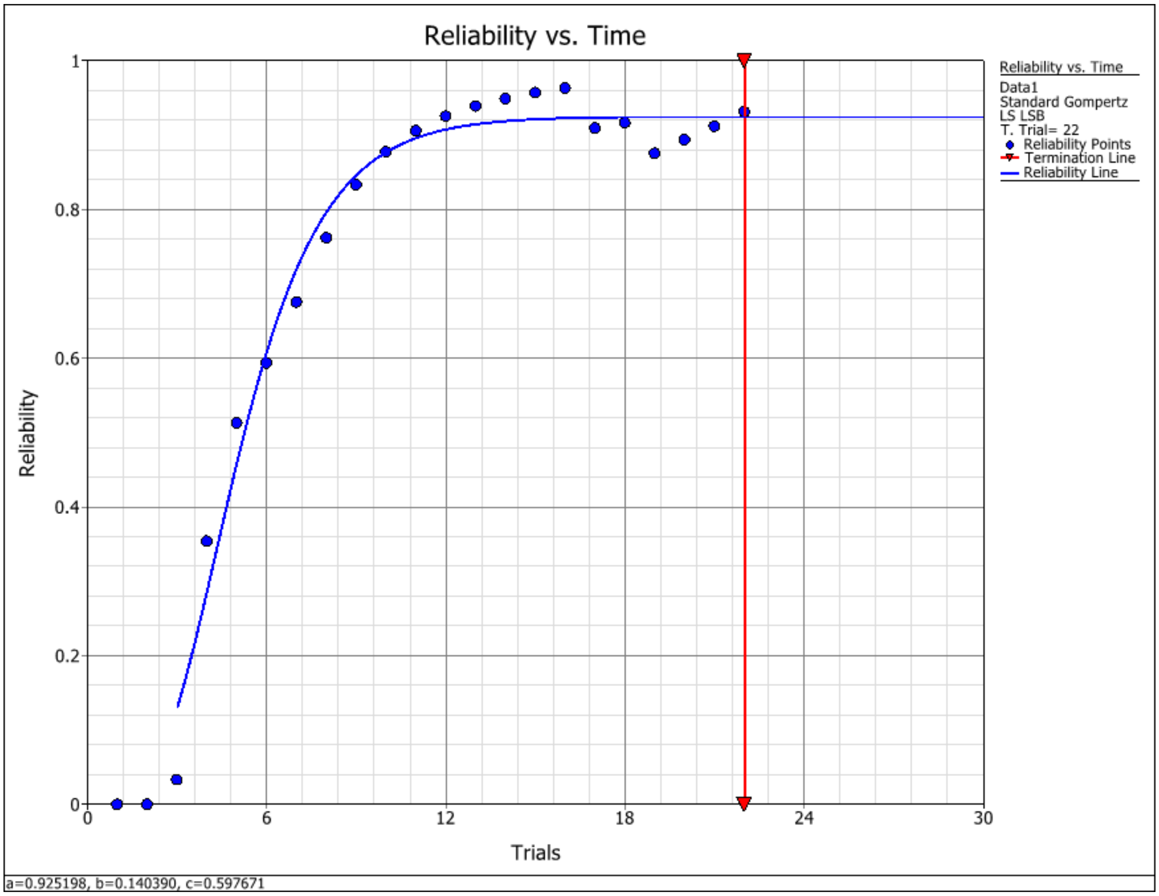
2. The predicted reliability after launch 22 is therefore:

$$R = 0.9252(0.1404)^{0.5977^{19}}$$
$$= 0.9251$$

The predicted reliability after launch 22 is calculated using the Quick Calculation Pad (QCP), as shown next.



3. In the second table, the predicted reliability values are compared with the reliabilities that are calculated from the raw data using failure discounting. It can be seen in the table, and in the following figure, that the Gompertz curve appears to provide a good fit to the actual data.



Hypothesis Tests

The Weibull++ software provides two types of hypothesis tests: common beta hypothesis (CBH) and Laplace trend. Both tests are applicable to the following data types:

- Times-to-failure data
  - Multiple Systems - Concurrent Operating Times
  - Multiple Systems with Dates
  - Multiple Systems with Event Codes
- Fielded data
  - Repairable Systems
  - Fleet

Common Beta Hypothesis Test

The common beta hypothesis (CBH) tests the hypothesis that all systems in the data set have similar values of beta. As shown by Crow [17], suppose that K number of systems are under test. Each system has an intensity function given by:

$$u_q(t) = \lambda_q \beta_q t^{\beta_q - 1}$$

where $q = 1, \ldots, K$. You can compare the intensity functions of each of the systems by comparing the β_q of each system. When conducting an analysis of data consisting of multiple systems, you expect that each of the systems performed in a similar manner. In particular, you would expect the interarrival rate of the failures across the systems to be fairly consistent. Therefore, the CBH test evaluates the hypothesis, H_o , such that $\beta_1 = \beta_2 = \ldots = \beta_K$. Let $\tilde{\beta}_q$ denote the conditional maximum likelihood estimate of β_q , which is given by:

$$\tilde{\beta}_q = \frac{\sum^{K}{}_{q=1} M_q}{\sum^{K}{}_{q=1} \sum^{M_q}{}_{i=1} \ln\left(\frac{T_q}{X_{iq}}\right)}$$

where:

- $K = 1$.
- $M_q = N_q$ if data on the q^{th} system is time terminated or $M_q = (N_q - 1)$ if data on the q^{th} system is failure terminated (N_q is the number of failures on the q^{th} system).
- X_{iq} is the i^{th} time-to-failure on the q^{th} system.

Then for each system, assume that:

$$\chi_q^2 = \frac{2M_q \beta_q}{\tilde{\beta}_q}$$

are conditionally distributed as independent chi-squared random variables with $2M_q$ degrees of freedom. When $K = 2$, you can test the null hypothesis, H_o , using the following statistic:

$$F = \frac{\frac{\chi_1^2}{2M_1}}{\frac{\chi_2^2}{2M_2}}$$

If H_o is true, then F equals $\frac{\tilde{\beta}_2}{\tilde{\beta}_1}$ and conditionally has an F-distribution with $(2M_1, 2M_2)$ degrees of freedom. The critical value, F , can then be determined by referring to the chi-squared tables. Now, if $K \geq 2$, then the likelihood ratio procedure can be used to test the hypothesis $\beta_1 = \beta_2 = \ldots = \beta_K$, as discussed in Crow [17]. Consider the following statistic:

$$L=\sum_{q=1}^{K} M_q \ln(\tilde{\beta}_q) - M\ln(\beta^*)$$

where:

- $M=\sum_{q=1}^{K} M_q$
- $\beta^* = \frac{M}{\sum_{q=1}^{K} \frac{M_q}{\tilde{\beta}_q}}$

Also, let:

$$a = 1+\frac{1}{6(K-1)}\left[\sum_{q=1}^{K}\frac{1}{M_q}-\frac{1}{M}\right]$$

Calculate the statistic D, such that:

$$D=\frac{2L}{a}$$

The statistic D is approximately distributed as a chi-squared random variable with $(K-1)$ degrees of freedom. Then after calculating D, refer to the chi-squared tables with $(K-1)$ degrees of freedom to determine the critical points. H_o is true if the statistic D falls between the critical points.

Common Beta Hypothesis Example

Consider the data in the following table.

Repairable System Data			
	System 1	System 2	System 3
Start	0	0	0
End	2000	2000	2000
Failures	1.2	1.4	0.3
	55.6	35	32.6
	72.7	46.8	33.4

	111.9	65.9	241.7
	121.9	181.1	396.2
	303.6	712.6	444.4
	326.9	1005.7	480.8
	1568.4	1029.9	588.9
	1913.5	1675.7	1043.9
		1787.5	1136.1
		1867	1288.1
			1408.1
			1439.4
			1604.8

Given that the intensity function for the q^{th} system is $u_q(t) = \lambda_q \beta_q t^{\beta_q - 1}$, test the hypothesis that $\beta_1 = \beta_2$ while assuming a significance level equal to 0.05. Calculate the maximum likelihood estimates of $\tilde{\beta}_1$ and $\tilde{\beta}_2$. Therefore:

$$\tilde{\beta}_1 = 0.3753$$
$$\tilde{\beta}_2 = 0.4657$$

Then $\frac{\tilde{\beta}_2}{\tilde{\beta}_1} = 1.2408$. Calculate the statistic F with a significance level of 0.05.

$$F = 2.0980$$

Since $1.2408 < 2.0980$ we fail to reject the null hypothesis that $\beta_1 = \beta_2$ at the 5% significance level.

Now suppose that we test the hypothesis that $\beta_1 = \beta_2 = \beta_3$. Calculate the statistic D.

$$D = 0.5260$$

Using the chi-square tables with $K - 1 = 2$ degrees of freedom, the critical values at the 2.5 and 97.5 percentiles are 0.1026 and 5.9915, respectively. Since $0.1026 < D < 5.9915$, we fail to reject the null hypothesis that $\beta_1 = \beta_2 = \beta_3$ at the 5% significance level.

Laplace Trend Test

The Laplace trend test evaluates the hypothesis that a trend does not exist within the data. The Laplace trend test can determine whether the system is deteriorating, improving, or if there is no trend at all. Calculate the test statistic, U, using the following equation:

$$U=\frac{\frac{\sum_{i=1}^{N} X_i}{N}-\frac{T}{2}}{T\sqrt{\frac{1}{12N}}}$$

where:

- T = total operating time (termination time)
- X_i = age of the system at the i^{th} successive failure
- N = total number of failures

The test statistic U is approximately a standard normal random variable. The critical value is read from the standard normal tables with a given significance level, α.

Laplace Trend Test Example

Consider once again the data given in the table above. Check for a trend within System 1 assuming a significance level of 0.10. Calculate the test statistic U for System 1.

$$U=-2.6121$$

From the standard normal tables with a significance level of 0.10, the critical value is equal to 1.645. If $-1.645<U<1.645$ then we would fail to reject the hypothesis of no trend. However, since $U<-1.645$ then an improving trend exists within System 1. If $U>1.645$ then a deteriorating trend would exist.

Crow-AMSAA Confidence Bounds

In this appendix, we will present the two methods used in the Weibull++ software to estimate the confidence bounds for the Crow-AMSAA (NHPP) model when applied to developmental testing data. The Fisher matrix approach is based on the Fisher information matrix and is commonly employed in the reliability field. The Crow bounds were developed by Dr. Larry Crow.

Note regarding the Crow Bounds calculations: The equations that involve the use of the chi-squared distribution assume left-tail probability.

Individual (Non-Grouped) Data

This section presents the confidence bounds for the Crow-AMSAA model under developmental testing when the failure times are known. The confidence bounds for when the failure times are not known are presented in the Grouped Data section.

Beta

FISHER MATRIX BOUNDS

The parameter β must be positive, thus $\ln\beta$ is treated as being normally distributed as well.

$$\frac{\ln\hat{\beta}-\ln\beta}{\sqrt{Var(\ln\hat{\beta})}} \sim N(0,1)$$

The approximate confidence bounds are given as:

$$CB_{\beta}=\hat{\beta}e^{\pm z_{\alpha}\sqrt{Var(\hat{\beta})}/\hat{\beta}}$$

α in z_{α} is different ($\alpha/2$, α) according to a 2-sided confidence interval or a 1-sided confidence interval, and variances can be calculated using the Fisher matrix.

$$\begin{bmatrix}-\frac{\partial^2\Lambda}{\partial\lambda^2} & -\frac{\partial^2\Lambda}{\partial\lambda\partial\beta}\\ -\frac{\partial^2\Lambda}{\partial\lambda\partial\beta} & -\frac{\partial^2\Lambda}{\partial\beta^2}\end{bmatrix}^{-1}_{\beta=\hat{\beta},\lambda=\hat{\lambda}}=\begin{bmatrix}Var(\hat{\lambda}) & Cov(\hat{\beta},\hat{\lambda})\\ Cov(\hat{\beta},\hat{\lambda}) & Var(\hat{\beta})\end{bmatrix}$$

Λ is the natural log-likelihood function:

$$\Lambda=N\ln\lambda+N\ln\beta-\lambda T^{\beta}+(\beta-1)\sum_{i=1}^{N}\ln T_i$$

And:

$$\frac{\partial^2\Lambda}{\partial\lambda^2}=-\frac{N}{\lambda^2}$$

$$\frac{\partial^2\Lambda}{\partial\beta^2}=-\frac{N}{\beta^2}-\lambda T^{\beta}(\ln T)^2$$

$$\frac{\partial^2\Lambda}{\partial\lambda\partial\beta}=-T^{\beta}\ln T$$

CROW BOUNDS

Failure Terminated

For the 2-sided $(1-\alpha)$ 100% confidence interval on β, calculate:

$$D_L = \frac{N \cdot \chi^2_{\frac{\alpha}{2},2(N-1)}}{2(N-1)(N-2)}$$
$$D_U = \frac{N \cdot \chi^2_{1-\frac{\alpha}{2},2(N-1)}}{2(N-1)(N-2)}$$

Thus, the confidence bounds on β are:

$$\beta_L = D_L \cdot \hat{\beta}$$
$$\beta_U = D_U \cdot \hat{\beta}$$

Time Terminated

For the 2-sided $(1-\alpha)$ 100% confidence interval on β, calculate:

$$D_L = \frac{\chi^2_{\frac{\alpha}{2},2N}}{2(N-1)}$$
$$D_U = \frac{\chi^2_{1-\frac{\alpha}{2},2N}}{2(N-1)}$$

The confidence bounds on β are:

$$\beta_L = D_L \cdot \hat{\beta}$$
$$\beta_U = D_U \cdot \hat{\beta}$$

Growth Rate

Since the growth rate, α, is equal to $1-\beta$, the confidence bounds for both the Fisher matrix and Crow methods are:

$$\alpha_L = 1 - \beta_U$$
$$\alpha_U = 1 - \beta_L$$

β_L and β_U are obtained using the methods described above in the confidence bounds on Beta.

Lambda

FISHER MATRIX BOUNDS

The parameter λ must be positive; thus, $\ln \lambda$ is treated as being normally distributed as well. These bounds are based on:

$$\frac{\ln \hat{\lambda} - \ln \lambda}{\sqrt{Var(\ln \hat{\lambda})}} \sim N(0,1)$$

The approximate confidence bounds on λ are given as:

$$CB_{\lambda} = \hat{\lambda} e^{\pm z_{\alpha} \sqrt{Var(\hat{\lambda})}/\hat{\lambda}}$$

where:

$$\hat{\lambda} = \frac{n}{T^{*\hat{\beta}}}$$

The variance calculation is the same as given above in the confidence bounds on Beta.

CROW BOUNDS

Failure Terminated

For the 2-sided $(1-\alpha)$ 100% confidence interval, the confidence bounds on λ are:

$$\lambda_L = \frac{\chi^2_{\frac{\alpha}{2},2N}}{2T^{\hat{\beta}}}$$
$$\lambda_U = \frac{\chi^2_{1-\frac{\alpha}{2},2N}}{2T^{\hat{\beta}}}$$

where:

- N = total number of failures.
- T = termination time.

Time Terminated

For the 2-sided $(1-\alpha)$ 100% confidence interval, the confidence bounds on λ are:

$$\lambda_L = \frac{\chi^2_{\frac{\alpha}{2},2N}}{2T^{\hat{\beta}}}$$
$$\lambda_U = \frac{\chi^2_{1-\frac{\alpha}{2},2N+2}}{2T^{\hat{\beta}}}$$

where:

- N = total number of failures.
- T = termination time.

Cumulative Number of Failures

FISHER MATRIX BOUNDS

The cumulative number of failures, $N(t)$, must be positive, thus $\ln N(t)$ is treated as being normally distributed.

$$\frac{\ln \hat{N}(t) - \ln N(t)}{\sqrt{Var(\ln \hat{N}(t))}} \sim N(0,1)$$

$$N(t) = \hat{N}(t) e^{\pm z_\alpha \sqrt{Var(\hat{N}(t))}/\hat{N}(t)}$$

where:

$$\hat{N}(t) = \hat{\lambda} t^{\hat{\beta}}$$

$$Var(\hat{N}(t)) = \left(\frac{\partial \hat{N}(t)}{\partial \beta}\right)^2 Var(\hat{\beta}) + \left(\frac{\partial \hat{N}(t)}{\partial \lambda}\right)^2 Var(\hat{\lambda}) + 2\left(\frac{\partial \hat{N}(t)}{\partial \beta}\right)\left(\frac{\partial \hat{N}(t)}{\partial \lambda}\right) cov(\hat{\beta}, \hat{\lambda})$$

The variance calculation is the same as given above in the confidence bounds on Beta. And:

$$\frac{\partial \hat{N}(t)}{\partial \beta} = \hat{\lambda} t^{\hat{\beta}} \ln t$$
$$\frac{\partial \hat{N}(t)}{\partial \lambda} = t^{\hat{\beta}}$$

CROW BOUNDS

The Crow cumulative number of failure confidence bounds are:

$$N(t)_L = \frac{t}{\hat{\beta}} IFI(t)_L$$
$$N(t)_U = \frac{t}{\hat{\beta}} IFI(t)_U$$

where $IFI(t)_L$ and $IFI(t)_U$ are calculated using the process for calculating the confidence bounds on instantaneous failure intensity.

Cumulative Failure Intensity

FISHER MATRIX BOUNDS

The cumulative failure intensity, $\lambda_c(t)$, must be positive, thus $\ln \lambda_c(t)$ is treated as being normally distributed.

$$\frac{\ln \hat{\lambda}_c(t) - \ln \lambda_c(t)}{\sqrt{Var(\ln \hat{\lambda}_c(t))}} \sim N(0,1)$$

The approximate confidence bounds on the cumulative failure intensity are then estimated from:

$$CB = \hat{\lambda}_c(t) e^{\pm z_\alpha \sqrt{Var(\hat{\lambda}_c(t))}/\hat{\lambda}_c(t)}$$

where:

$$\hat{\lambda}_c(t) = \hat{\lambda} t^{\hat{\beta}-1}$$

and:

$$Var(\hat{\lambda}_c(t)) = \left(\frac{\partial \lambda_c(t)}{\partial \beta}\right)^2 Var(\hat{\beta}) + \left(\frac{\partial \lambda_c(t)}{\partial \lambda}\right)^2 Var(\hat{\lambda}) + 2\left(\frac{\partial \lambda_c(t)}{\partial \beta}\right)\left(\frac{\partial \lambda_c(t)}{\partial \lambda}\right) cov(\hat{\beta},\ \hat{\lambda})$$

The variance calculation is the same as given above in the confidence bounds on Beta. And:

$$\frac{\partial \lambda_c(t)}{\partial \beta} = \hat{\lambda} t^{\hat{\beta}-1} \ln t$$
$$\frac{\partial \lambda_c(t)}{\partial \lambda} = t^{\hat{\beta}-1}$$

CROW BOUNDS

The Crow bounds on the cumulative failure intensity (CFI) are given below. Let:

$$N = \hat{\lambda} t^{\hat{\beta}}$$

Failure Terminated

$$CFI_L = \frac{\chi^2_{\frac{\alpha}{2}, 2N}}{2 \cdot t}$$

$$CFI_U = \frac{\chi^2_{1-\frac{\alpha}{2}, 2N}}{2 \cdot t}$$

Time Terminated

$$CFI_L = \frac{\chi^2_{\frac{\alpha}{2}, 2N}}{2 \cdot t}$$

$$CFI_U = \frac{\chi^2_{1-\frac{\alpha}{2}, 2N+2}}{2 \cdot t}$$

Cumulative MTBF

FISHER MATRIX BOUNDS

The cumulative MTBF, $m_c(t)$, must be positive, thus $\ln m_c(t)$ is treated as being normally distributed as well.

$$\frac{\ln \hat{m}_c(t) - \ln m_c(t)}{\sqrt{Var(\ln \hat{m}_c(t))}} \sim N(0,1)$$

The approximate confidence bounds on the cumulative MTBF are then estimated from:

$$CB = \hat{m}_c(t) e^{\pm z_\alpha \sqrt{Var(\hat{m}_c(t))}/\hat{m}_c(t)}$$

where:

$$\hat{m}_c(t) = \frac{1}{\hat{\lambda}} t^{1-\hat{\beta}}$$

$$\begin{aligned} Var(\hat{m}_c(t)) = & \left(\frac{\partial m_c(t)}{\partial \beta}\right)^2 Var(\hat{\beta}) + \left(\frac{\partial m_c(t)}{\partial \lambda}\right)^2 Var(\hat{\lambda}) \\ & + 2\left(\frac{\partial m_c(t)}{\partial \beta}\right)\left(\frac{\partial m_c(t)}{\partial \lambda}\right) cov(\hat{\beta}, \hat{\lambda}) \end{aligned}$$

The variance calculation is the same as given above in the confidence bounds on Beta. And:

$$\frac{\partial m_c(t)}{\partial \beta} = -\frac{1}{\hat{\lambda}} t^{1-\hat{\beta}} \ln t$$
$$\frac{\partial m_c(t)}{\partial \lambda} = -\frac{1}{\hat{\lambda}^2} t^{1-\hat{\beta}}$$

CROW BOUNDS

The 2-sided confidence bounds on the cumulative MTBF $(CMTBF)$ are given by:

$$CMTBF_L = \frac{1}{CFI_U}$$
$$CMTBF_U = \frac{1}{CFI_L}$$

where CFI_L and CFI_U are calculated using the process for calculating the confidence bounds on cumulative failure intensity.

Instantaneous MTBF

FISHER MATRIX BOUNDS

The instantaneous MTBF, $m_i(t)$, must be positive, thus $\ln m_i(t)$ is treated as being normally distributed as well.

$$\frac{\ln \hat{m}_i(t) - \ln m_i(t)}{\sqrt{Var(\ln \hat{m}_i(t))}} \sim N(0,1)$$

The approximate confidence bounds on the instantaneous MTBF are then estimated from:

$$CB = \hat{m}_i(t) e^{\pm z_\alpha \sqrt{Var(\hat{m}_i(t))}/\hat{m}_i(t)}$$

where:

$$\hat{m}_i(t) = \frac{1}{\lambda \beta t^{\beta-1}}$$

$$Var(\hat{m}_i(t)) = \left(\frac{\partial m_i(t)}{\partial \beta}\right)^2 Var(\hat{\beta}) + \left(\frac{\partial m_i(t)}{\partial \lambda}\right)^2 Var(\hat{\lambda}) + 2\left(\frac{\partial m_i(t)}{\partial \beta}\right)\left(\frac{\partial m_i(t)}{\partial \lambda}\right) cov(\hat{\beta}, \hat{\lambda}).$$

The variance calculation is the same as given above in the confidence bounds on Beta. And:

$$\frac{\partial m_i(t)}{\partial \beta} = -\frac{1}{\hat{\lambda}\hat{\beta}^2}t^{1-\hat{\beta}} - \frac{1}{\hat{\lambda}\hat{\beta}}t^{1-\hat{\beta}}\ln t$$
$$\frac{\partial m_i(t)}{\partial \lambda} = -\frac{1}{\hat{\lambda}^2\hat{\beta}}t^{1-\hat{\beta}}$$

CROW BOUNDS

Failure Terminated

For failure terminated data and the 2-sided confidence bounds on instantaneous MTBF $(IMTBF)$, consider the following equation:

$$G(\mu|n) = \int_0^{\infty} \frac{e^{-x}x^{n-2}}{(n-2)!}\sum_{i=0}^{n-1}\frac{1}{i!}\left(\frac{\mu}{x}\right)^i \exp\left(-\frac{\mu}{x}\right)dx$$

Find the values p_1 and p_2 by finding the solution $G\left(\frac{n^2}{c}\middle| n\right) = \frac{\alpha}{2}$ and $G\left(\frac{n^2}{c}\middle| n\right) = 1-\frac{\alpha}{2}$ for the lower and upper bounds, respectively.

If using the biased parameters, $\hat{\beta}$ and $\hat{\lambda}$, then the upper and lower confidence bounds are:

$$IMTBF_L = IMTBF \cdot p_1$$
$$IMTBF_U = IMTBF \cdot p_2$$

where $IMTBF = \frac{1}{\hat{\lambda}\hat{\beta}t^{\hat{\beta}-1}}$.

If using the unbiased parameters, $\bar{\beta}$ and $\bar{\lambda}$, then the upper and lower confidence bounds are:

$$IMTBF_L = IMTBF \cdot \left(\frac{N-2}{N}\right) \cdot p_1$$
$$IMTBF_U = IMTBF \cdot \left(\frac{N-2}{N}\right) \cdot p_2$$

where $IMTBF = \frac{1}{\bar{\lambda}\bar{\beta}t^{\bar{\beta}-1}}$.

Time Terminated

Consider the following equation where $I_1(.)$ is the modified Bessel function of order one:

$$H(x|k) = \sum_{j=1}^{k}\frac{x^{2j-1}}{2^{2j-1}(j-1)!j!I_1(x)}$$

Find the values Π_1 and Π_2 by finding the solution x to $H(x|k)=\frac{\alpha}{2}$ and $H(x|k)=1-\frac{\alpha}{2}$ in the cases corresponding to the lower and upper bounds, respectively. Calculate $\Pi=\frac{4n^2}{x^2}$ for each case.

If using the biased parameters, $\hat{\beta}$ and $\hat{\lambda}$, then the upper and lower confidence bounds are:

$$IMTBF_L = IMTBF\cdot\Pi_1$$
$$IMTBF_U = IMTBF\cdot\Pi_2$$

where $IMTBF=\frac{1}{\hat{\lambda}\hat{\beta}t^{\hat{\beta}-1}}$.

If using the unbiased parameters, $\bar{\beta}$ and $\bar{\lambda}$, then the upper and lower confidence bounds are:

$$IMTBF_L = IMTBF\cdot\left(\frac{N-1}{N}\right)\cdot\Pi_1$$
$$IMTBF_U = IMTBF\cdot\left(\frac{N-1}{N}\right)\cdot\Pi_2$$

where $IMTBF=\frac{1}{\bar{\lambda}\bar{\beta}t^{\bar{\beta}-1}}$.

Instantaneous Failure Intensity

FISHER MATRIX BOUNDS

The instantaneous failure intensity, $\lambda_i(t)$, must be positive, thus $\ln\lambda_i(t)$ is treated as being normally distributed.

$$\frac{\ln\hat{\lambda}_i(t)-\ln\lambda_i(t)}{\sqrt{Var(\ln\hat{\lambda}_i(t))}} \sim N(0,1)$$

The approximate confidence bounds on the instantaneous failure intensity are then estimated from:

$$CB=\hat{\lambda}_i(t)e^{\pm z_\alpha\sqrt{Var(\hat{\lambda}_i(t))}/\hat{\lambda}_i(t)}$$

where

$$\lambda_i(t)=\lambda\beta t^{\beta-1}$$

$$Var(\hat{\lambda}_i(t)) = \left(\frac{\partial \lambda_i(t)}{\partial \beta}\right)^2 Var(\hat{\beta}) + \left(\frac{\partial \lambda_i(t)}{\partial \lambda}\right)^2 Var(\hat{\lambda}) + 2\left(\frac{\partial \lambda_i(t)}{\partial \beta}\right)\left(\frac{\partial \lambda_i(t)}{\partial \lambda}\right) cov(\hat{\beta},\ \hat{\lambda})$$

The variance calculation is the same as given above in the confidence bounds on Beta. And:

$$\frac{\partial \lambda_i(t)}{\partial \beta} = \hat{\lambda} t^{\hat{\beta}-1} + \hat{\lambda}\hat{\beta} t^{\hat{\beta}-1} \ln t$$
$$\frac{\partial \lambda_i(t)}{\partial \lambda} = \hat{\beta} t^{\hat{\beta}-1}$$

CROW BOUNDS

The 2-sided confidence bounds on the instantaneous failure intensity (IFI) are given by:

$$IFI_L = \frac{1}{IMTBF_U}$$
$$IFI_U = \frac{1}{IMTBF_L}$$

where $IMTBF_L$ and $IMTBF_U$ are calculated using the process presented for the confidence bounds on the instantaneous MTBF.

Time Given Cumulative Failure Intensity

FISHER MATRIX BOUNDS

The time, T, must be positive, thus $\ln T$ is treated as being normally distributed.

$$\frac{\ln \hat{T} - \ln T}{\sqrt{Var(\ln \hat{T})}} \sim N(0,1)$$

Confidence bounds on the time are given by:

$$CB = \hat{T} e^{\pm z_\alpha \sqrt{Var(\hat{T})}/\hat{T}}$$

where:

$$Var(\hat{T}) = \left(\frac{\partial T}{\partial \beta}\right)^2 Var(\hat{\beta}) + \left(\frac{\partial T}{\partial \lambda}\right)^2 Var(\hat{\lambda}) + 2\left(\frac{\partial T}{\partial \beta}\right)\left(\frac{\partial T}{\partial \lambda}\right) cov(\hat{\beta},\ \hat{\lambda})$$

The variance calculation is the same as given above in the confidence bounds on Beta. And:

$$\frac{\partial T}{\partial \beta}=\frac{-\left(\frac{\lambda_c(T)}{\lambda}\right)^{1/(\beta-1)}\ln\left(\frac{\lambda_c(T)}{\lambda}\right)}{(1-\beta)^2}$$

$$\frac{\partial T}{\partial \lambda}=\left(\frac{\lambda_c(T)}{\lambda}\right)^{1/(\beta-1)}\frac{1}{\lambda(1-\beta)}$$

CROW BOUNDS

The 2-sided confidence bounds on time given cumulative failure intensity (CFI) are given by:

$$\hat{t}=\left(\frac{CFI}{\hat{\lambda}}\right)^{\frac{1}{\hat{\beta}-1}}$$

Then estimate the number of failures, N, such that:

$$N=\hat{\lambda}\hat{t}^{\hat{\beta}}$$

The lower and upper confidence bounds on time are then estimated using:

$$t_L=\frac{\chi^2_{\frac{\alpha}{2},2N}}{2\cdot CFI}$$

$$t_U=\frac{\chi^2_{1-\frac{\alpha}{2},2N+2}}{2\cdot CFI}$$

Time Given Cumulative MTBF

FISHER MATRIX BOUNDS

The time, T, must be positive, thus $\ln T$ is treated as being normally distributed.

$$\frac{\ln\hat{T}-\ln T}{\sqrt{Var(\ln\hat{T})}} \sim N(0,1)$$

Confidence bounds on the time are given by:

$$CB=\hat{T}e^{\pm z_\alpha\sqrt{Var(\hat{T})}/\hat{T}}$$

where:

$$Var(\hat{T})=\left(\frac{\partial T}{\partial \beta}\right)^2 Var(\hat{\beta})+\left(\frac{\partial T}{\partial \lambda}\right)^2 Var(\hat{\lambda})+2\left(\frac{\partial T}{\partial \beta}\right)\left(\frac{\partial T}{\partial \lambda}\right)cov(\hat{\beta},\ \hat{\lambda})$$

The variance calculation is the same as given above in the confidence bounds on Beta. And:

$$\hat{T} = (\lambda \cdot m_c)^{1/(1-\beta)}$$

$$\frac{\partial T}{\partial \beta} = \frac{(\lambda \cdot m_c)^{1/(1-\beta)} \ln(\lambda \cdot m_c)}{(1-\beta)^2}$$

$$\frac{\partial T}{\partial \lambda} = \frac{(\lambda \cdot m_c)^{1/(1-\beta)}}{\lambda(1-\beta)}$$

CROW BOUNDS

The 2-sided confidence bounds on time given cumulative MTBF $(CMTBF)$ are estimated using the process for calculating the confidence bounds on time given cumulative failure intensity (CFI) where $CFI = \frac{1}{CMTBF}$.

Time Given Instantaneous MTBF

FISHER MATRIX BOUNDS

The time, T, must be positive, thus $\ln T$ is treated as being normally distributed.

$$\frac{\ln \hat{T} - \ln T}{\sqrt{Var(\ln \hat{T})}} \sim N(0,1)$$

Confidence bounds on the time are given by:

$$CB = \hat{T} e^{\pm z_\alpha \sqrt{Var(\hat{T})}/\hat{T}}$$

where:

$$Var(\hat{T}) = \left(\frac{\partial T}{\partial \beta}\right)^2 Var(\hat{\beta}) + \left(\frac{\partial T}{\partial \lambda}\right)^2 Var(\hat{\lambda}) + 2\left(\frac{\partial T}{\partial \beta}\right)\left(\frac{\partial T}{\partial \lambda}\right) cov(\hat{\beta}, \ \hat{\lambda})$$

The variance calculation is the same as given above in the confidence bounds on Beta. And:

$$\hat{T} = (\lambda \beta \cdot MTBF_i)^{1/(1-\beta)}$$

$$\frac{\partial T}{\partial \beta} = (\lambda\beta \cdot MTBF_i)^{1/(1-\beta)} \left[\frac{1}{(1-\beta)^2} \ln(\lambda\beta \cdot MTBF_i) + \frac{1}{\beta(1-\beta)} \right]$$

$$\frac{\partial T}{\partial \lambda} = \frac{(\lambda\beta \cdot MTBF_i)^{1/(1-\beta)}}{\lambda(1-\beta)}$$

CROW BOUNDS

Failure Terminated

If the unbiased value $\bar{\beta}$ is used then:

$$IMTBF = IMTBF \cdot \frac{N-2}{N}$$

where:

- $IMTBF$ = instantaneous MTBF.
- N = total number of failures.

Calculate the constants p_1 and p_2 using procedures described for the confidence bounds on instantaneous MTBF. The lower and upper confidence bounds on time are then given by:

$$\hat{t}_L = \left(\frac{\lambda\beta \cdot IMTBF}{p_1} \right)^{\frac{1}{1-\beta}}$$

$$\hat{t}_U = \left(\frac{\lambda\beta \cdot IMTBF}{p_2} \right)^{\frac{1}{1-\beta}}$$

Time Terminated

If the unbiased value $\bar{\beta}$ is used then:

$$IMTBF = IMTBF \cdot \frac{N-1}{N}$$

where:

- $IMTBF$ = instantaneous MTBF.
- N = total number of failures.

Calculate the constants Π_1 and Π_2 using procedures described for the confidence bounds on instantaneous MTBF. The lower and upper confidence bounds on time are then given by:

$$\hat{t}_L = \left(\frac{\lambda\beta \cdot IMTBF}{\Pi_1}\right)^{\frac{1}{1-\beta}}$$

$$\hat{t}_U = \left(\frac{\lambda\beta \cdot IMTBF}{\Pi_2}\right)^{\frac{1}{1-\beta}}$$

Time Given Instantaneous Failure Intensity

FISHER MATRIX BOUNDS

The time, T, must be positive, thus $\ln T$ is treated as being normally distributed.

$$\frac{\ln \hat{T} - \ln T}{\sqrt{Var(\ln \hat{T})}} \sim N(0,1)$$

Confidence bounds on the time are given by:

$$CB = \hat{T} e^{\pm z_{\alpha} \sqrt{Var(\hat{T})}/\hat{T}}$$

where:

$$Var(\hat{T}) = \left(\frac{\partial T}{\partial \beta}\right)^2 Var(\hat{\beta}) + \left(\frac{\partial T}{\partial \lambda}\right)^2 Var(\hat{\lambda}) + 2\left(\frac{\partial T}{\partial \beta}\right)\left(\frac{\partial T}{\partial \lambda}\right) cov(\hat{\beta},\ \hat{\lambda})$$

The variance calculation is the same as given above in the confidence bounds on Beta. And:

$$\hat{T} = \left(\frac{\lambda_i(T)}{\lambda\beta}\right)^{1/(\beta-1)}$$

$$\frac{\partial T}{\partial \beta} = \left(\frac{\lambda_i(T)}{\lambda\beta}\right)^{1/(\beta-1)} \left[-\frac{\ln(\frac{\lambda_i(T)}{\lambda\beta})}{(\beta-1)^2} + \frac{1}{\beta(1-\beta)}\right]$$

$$\frac{\partial T}{\partial \lambda} = \left(\frac{\lambda_i(T)}{\lambda\beta}\right)^{1/(\beta-1)} \frac{1}{\lambda(1-\beta)}$$

CROW BOUNDS

The 2-sided confidence bounds on time given instantaneous failure intensity (IFI) are estimated using the process for calculating the confidence bounds on time given instantaneous MTBF where $IMTBF = \dfrac{1}{IFI}$.

Grouped Data

This section presents the confidence bounds for the Crow-AMSAA model when using Grouped data.

Beta (Grouped)

FISHER MATRIX BOUNDS

The parameter β must be positive, thus $\ln\beta$ is treated as being normally distributed as well.

$$\frac{\ln\hat{\beta}-\ln\beta}{\sqrt{Var(\ln\hat{\beta})}} \sim N(0,1)$$

The approximate confidence bounds are given as:

$$CB_{\beta}=\hat{\beta}e^{\pm z_{\alpha}\sqrt{Var(\hat{\beta})}/\hat{\beta}}$$

$\hat{\beta}$ can be obtained by $\sum_{i=1}^{K} n_i\left(\frac{T_i^{\hat{\beta}}\ln T_i-T_{i-1}^{\hat{\beta}}\ln T_{i-1}}{T_i^{\hat{\beta}}-T_{i-1}^{\hat{\beta}}}-\ln T_k\right)=0$.

All variance can be calculated using the Fisher matrix:

$$\begin{bmatrix} -\frac{\partial^2\Lambda}{\partial\lambda^2} & -\frac{\partial^2\Lambda}{\partial\lambda\partial\beta} \\ -\frac{\partial^2\Lambda}{\partial\lambda\partial\beta} & -\frac{\partial^2\Lambda}{\partial\beta^2} \end{bmatrix}^{-1}_{\beta=\hat{\beta},\lambda=\hat{\lambda}} = \begin{bmatrix} Var(\hat{\lambda}) & Cov(\hat{\beta},\hat{\lambda}) \\ Cov(\hat{\beta},\hat{\lambda}) & Var(\hat{\beta}) \end{bmatrix}$$

Λ is the natural log-likelihood function where $\ln^2 T=(\ln T)^2$ and:

$$\Lambda=\sum_{i=1}^{k}\left[n_i\ln(\lambda T_i^{\beta}-\lambda T_{i-1}^{\beta})-(\lambda T_i^{\beta}-\lambda T_{i-1}^{\beta})-\ln n_i!\right]$$

$$\frac{\partial^2\Lambda}{\partial\lambda^2}=-\frac{n}{\lambda^2}$$

$$\frac{\partial^2\Lambda}{\partial\beta^2}=\sum_{i=1}^{k}\left[n_i\left(\frac{(T_i^{\hat{\beta}}\ln^2 T_i-T_{i-1}^{\hat{\beta}}\ln^2 T_{i-1})(T_i^{\hat{\beta}}-T_{i-1}^{\hat{\beta}})-\left(T_i^{\hat{\beta}}\ln T_i-T_{i-1}^{\hat{\beta}}\ln T_{i-1}\right)^2}{\left(T_i^{\hat{\beta}}-T_{i-1}^{\hat{\beta}}\right)^2}\right) - \left(\lambda T_i^{\hat{\beta}}\ln^2 T_i-\lambda T_{i-1}^{\hat{\beta}}\ln^2 T_{i-1}\right)\right]$$

$$\frac{\partial^2\Lambda}{\partial\lambda\partial\beta}=-T_K^{\beta}\ln T_k$$

CROW BOUNDS

The 2-sided confidence bounds on $\hat{\beta}$ are given by first calculating:

$$P(i) = \frac{T_i}{T_K};\ i = 1, 2, \ldots, K$$

where:

- T_i = interval end time for the i^{th} interval.
- K = number of intervals.
- T_K = end time for the last interval.

Next:

$$A = \sum_{i=1}^{K} \frac{\left[P(i)^{\hat{\beta}} \ln P(i)^{\hat{\beta}} - P(i-1)^{\hat{\beta}} \ln P(i-1)^{\hat{\beta}}\right]^2}{\left[P(i)^{\hat{\beta}} - P(i-1)^{\hat{\beta}}\right]}$$

And:

$$c = \frac{1}{\sqrt{A}}$$

Then:

$$S = \frac{\left(z_{1-\frac{\alpha}{2}}\right) \cdot c}{\sqrt{N}}$$

where:

- $z_{1-\frac{\alpha}{2}}$ = inverse standard normal.
- N = number of failures.

The 2-sided confidence bounds on β are then $\hat{\beta}\,(1 \pm S)$.

Growth Rate (Grouped)

Since the growth rate, α, is equal to $1-\beta$, the confidence bounds for both the Fisher matrix and Crow methods are:

$$\alpha_L = 1 - \beta_U$$
$$\alpha_U = 1 - \beta_L$$

β_L and β_U are obtained using the methods described above in the confidence bounds on Beta.

Lambda (Grouped)

FISHER MATRIX BOUNDS

The parameter λ must be positive, thus $\ln\lambda$ is treated as being normally distributed as well. These bounds are based on:

$$\frac{\ln\hat{\lambda}-\ln\lambda}{\sqrt{Var(\ln\hat{\lambda})}} \sim N(0,1)$$

The approximate confidence bounds on λ are given as:

$$CB_\lambda=\hat{\lambda}e^{\pm z_\alpha\sqrt{Var(\hat{\lambda})}/\hat{\lambda}}$$

where:

$$\hat{\lambda}=\frac{n}{T_k^{\hat{\beta}}}$$

The variance calculation is the same as given above in the confidence bounds on Beta.

CROW BOUNDS

Failure Terminated

For failure terminated data, the 2-sided $(1-\alpha)$ 100% confidence interval, the confidence bounds on λ are:

$$\lambda_L=\frac{\chi^2_{\frac{\alpha}{2},2N}}{2\cdot T_k^{\beta}}$$

$$\lambda_U=\frac{\chi^2_{1-\frac{\alpha}{2},2N}}{2\cdot T_k^{\beta}}$$

where:

- N = total number of failures.
- T_K = end time of last interval.

Time Terminated

For time terminated data, the 2-sided $(1-\alpha)$ 100% confidence interval, the confidence bounds on λ are:

$$\lambda_L = \frac{\chi^2_{\frac{\alpha}{2},2N}}{2\cdot T_k^{\beta}}$$
$$\lambda_U = \frac{\chi^2_{1-\frac{\alpha}{2},2N+2}}{2\cdot T_k^{\beta}}$$

where:

- N = total number of failures.
- T_K = end time of last interval.

Cumulative Number of Failures (Grouped)

FISHER MATRIX BOUNDS

The cumulative number of failures, $N(t)$, must be positive, thus $\ln N(t)$ is treated as being normally distributed.

$$\frac{\ln \hat{N}(t) - \ln N(t)}{\sqrt{Var(\ln \hat{N}(t))}} \sim N(0,1)$$

$$N(t) = \hat{N}(t)e^{\pm z_\alpha \sqrt{Var(\hat{N}(t))}/\hat{N}(t)}$$

where:

$$\hat{N}(t) = \hat{\lambda}t^{\hat{\beta}}$$

$$Var(\hat{N}(t)) = \left(\frac{\partial \hat{N}(t)}{\partial \beta}\right)^2 Var(\hat{\beta}) + \left(\frac{\partial \hat{N}(t)}{\partial \lambda}\right)^2 Var(\hat{\lambda}) + 2\left(\frac{\partial \hat{N}(t)}{\partial \beta}\right)\left(\frac{\partial \hat{N}(t)}{\partial \lambda}\right) cov(\hat{\beta},\hat{\lambda})$$

The variance calculation is the same as given above in the confidence bounds on Beta. And:

$$\frac{\partial \hat{N}(t)}{\partial \beta} = \hat{\lambda}t^{\hat{\beta}} \ln t$$
$$\frac{\partial \hat{N}(t)}{\partial \lambda} = t^{\hat{\beta}}$$

CROW BOUNDS

The 2-sided confidence bounds on the cumulative number of failures are given by:

$$N(t)_L = \frac{t}{\hat{\beta}} IFI_L$$

$$N(t)_U = \frac{t}{\hat{\beta}} IFI_U$$

where IFI_L and IFI_U are calculated based on the procedures for the confidence bounds on the instantaneous failure intensity.

Cumulative Failure Intensity (Grouped)

FISHER MATRIX BOUNDS

The cumulative failure intensity, $\lambda_c(t)$, must be positive, thus $\ln \lambda_c(t)$ is treated as being normally distributed.

$$\frac{\ln \hat{\lambda}_c(t) - \ln \lambda_c(t)}{\sqrt{Var(\ln \hat{\lambda}_c(t))}} \sim N(0,1)$$

The approximate confidence bounds on the cumulative failure intensity are then estimated from:

$$CB = \hat{\lambda}_c(t) e^{\pm z_\alpha \sqrt{Var(\hat{\lambda}_c(t))}/\hat{\lambda}_c(t)}$$

where:

$$\hat{\lambda}_c(t) = \hat{\lambda} t^{\hat{\beta}-1}$$

and:

$$Var(\hat{\lambda}_c(t)) = \left(\frac{\partial \lambda_c(t)}{\partial \beta}\right)^2 Var(\hat{\beta}) + \left(\frac{\partial \lambda_c(t)}{\partial \lambda}\right)^2 Var(\hat{\lambda}) + 2\left(\frac{\partial \lambda_c(t)}{\partial \beta}\right)\left(\frac{\partial \lambda_c(t)}{\partial \lambda}\right) cov(\hat{\beta},\ \hat{\lambda})$$

The variance calculation is the same as given above in the confidence bounds on Beta. And:

$$\frac{\partial \lambda_c(t)}{\partial \beta} = \hat{\lambda} t^{\hat{\beta}-1} \ln t$$

$$\frac{\partial \lambda_c(t)}{\partial \lambda} = t^{\hat{\beta}-1}$$

CROW BOUNDS

The 2-sided confidence bounds on the cumulative failure intensity (CFI) are given below. Let:

$$N = \hat{\lambda} t^{\hat{\beta}}$$

Then:

$$CFI_L = \frac{\chi^2_{\frac{\alpha}{2},2N}}{2 \cdot t}$$

$$CFI_U = \frac{\chi^2_{1-\frac{\alpha}{2},2N+2}}{2 \cdot t}$$

Cumulative MTBF (Grouped)

FISHER MATRIX BOUNDS

The cumulative MTBF, $m_c(t)$, must be positive, thus $\ln m_c(t)$ is treated as being normally distributed as well.

$$\frac{\ln \hat{m}_c(t) - \ln m_c(t)}{\sqrt{Var(\ln \hat{m}_c(t))}} \sim N(0,1)$$

The approximate confidence bounds on the cumulative MTBF are then estimated from:

$$CB = \hat{m}_c(t) e^{\pm z_\alpha \sqrt{Var(\hat{m}_c(t))}/\hat{m}_c(t)}$$

where:

$$\hat{m}_c(t) = \frac{1}{\hat{\lambda}} t^{1-\hat{\beta}}$$

$$Var(\hat{m}_c(t)) = \left(\frac{\partial m_c(t)}{\partial \beta}\right)^2 Var(\hat{\beta}) + \left(\frac{\partial m_c(t)}{\partial \lambda}\right)^2 Var(\hat{\lambda}) + 2\left(\frac{\partial m_c(t)}{\partial \beta}\right)\left(\frac{\partial m_c(t)}{\partial \lambda}\right) cov(\hat{\beta}, \hat{\lambda})$$

The variance calculation is the same as given above in the confidence bounds on Beta. And:

$$\frac{\partial m_c(t)}{\partial \beta} = -\frac{1}{\hat{\lambda}} t^{1-\hat{\beta}} \ln t$$

$$\frac{\partial m_c(t)}{\partial \lambda} = -\frac{1}{\hat{\lambda}^2} t^{1-\hat{\beta}}$$

CROW BOUNDS

The 2-sided confidence bounds on cumulative MTBF $(CMTBF)$ are given by:

$$CMTBF_L = \frac{1}{CFI_U}$$

$$CMTBF_U = \frac{1}{CFI_L}$$

where CFI_L and CFI_U are calculating using the process for calculating the confidence bounds on the cumulative failure intensity.

Instantaneous MTBF (Grouped)

FISHER MATRIX BOUNDS

The instantaneous MTBF, $m_i(t)$, must be positive, thus $\ln m_i(t)$ is approximately treated as being normally distributed as well.

$$\frac{\ln \hat{m}_i(t) - \ln m_i(t)}{\sqrt{Var(\ln \hat{m}_i(t))}} \sim N(0,1)$$

The approximate confidence bounds on the instantaneous MTBF are then estimated from:

$$CB = \hat{m}_i(t) e^{\pm z_\alpha \sqrt{Var(\hat{m}_i(t))}/\hat{m}_i(t)}$$

where:

$$\hat{m}_i(t) = \frac{1}{\lambda \beta t^{\beta-1}}$$

$$Var(\hat{m}_i(t)) = \left(\frac{\partial m_i(t)}{\partial \beta}\right)^2 Var(\hat{\beta}) + \left(\frac{\partial m_i(t)}{\partial \lambda}\right)^2 Var(\hat{\lambda}) + 2\left(\frac{\partial m_i(t)}{\partial \beta}\right)\left(\frac{\partial m_i(t)}{\partial \lambda}\right) cov(\hat{\beta},\ \hat{\lambda})$$

The variance calculation is the same as given above in the confidence bounds on Beta. And:

$$\frac{\partial m_i(t)}{\partial \beta} = -\frac{1}{\hat{\lambda}\hat{\beta}^2} t^{1-\hat{\beta}} - \frac{1}{\hat{\lambda}\hat{\beta}} t^{1-\hat{\beta}} \ln t$$

$$\frac{\partial m_i(t)}{\partial \lambda} = -\frac{1}{\hat{\lambda}^2 \hat{\beta}} t^{1-\hat{\beta}}$$

CROW BOUNDS

The 2-sided confidence bounds on instantaneous MTBF ($IMTBF$) are given by first calculating:

$$P(i) = \frac{T_i}{T_K};\ i = 1, 2, \ldots, K$$

where:

- T_i = interval end time for the i^{th} interval.
- K = number of intervals.
- T_K = end time for the last interval.

Calculate:

$$A = \sum_{i=1}^{K} \frac{\left[P(i)^{\hat{\beta}} \ln P(i)^{\hat{\beta}} - P(i-1)^{\hat{\beta}} \ln P(i-1)^{\hat{\beta}}\right]^2}{\left[P(i)^{\hat{\beta}} - P(i-1)^{\hat{\beta}}\right]}$$

Next:

$$D = \sqrt{\frac{1}{A} + 1}$$

And:

$$W = \frac{\left(z_{1-\frac{\alpha}{2}}\right) \cdot D}{\sqrt{N}}$$

where:

- $z_{1-\frac{\alpha}{2}}$ = inverse standard normal.
- N = number of failures.

The 2-sided confidence bounds on instantaneous MTBF are then $IMTBF(1 \pm W)$.

Instantaneous Failure Intensity (Grouped)

FISHER MATRIX BOUNDS

The instantaneous failure intensity, $\lambda_i(t)$, must be positive, thus $\ln\lambda_i(t)$ is treated as being normally distributed.

$$\frac{\ln\hat{\lambda}_i(t)-\ln\lambda_i(t)}{\sqrt{Var(\ln\hat{\lambda}_i(t))}}\sim N(0,1)$$

The approximate confidence bounds on the instantaneous failure intensity are then estimated from:

$$CB=\hat{\lambda}_i(t)e^{\pm z_\alpha\sqrt{Var(\hat{\lambda}_i(t))}/\hat{\lambda}_i(t)}$$

where $\lambda_i(t)=\lambda\beta t^{\beta-1}$ and:

$$\begin{aligned}Var(\hat{\lambda}_i(t))=&\left(\frac{\partial\lambda_i(t)}{\partial\beta}\right)^2 Var(\hat{\beta})+\left(\frac{\partial\lambda_i(t)}{\partial\lambda}\right)^2 Var(\hat{\lambda})\\&+2\left(\frac{\partial\lambda_i(t)}{\partial\beta}\right)\left(\frac{\partial\lambda_i(t)}{\partial\lambda}\right)cov(\hat{\beta},\ \hat{\lambda})\end{aligned}$$

The variance calculation is the same as given above in the confidence bounds on Beta. And:

$$\begin{aligned}\frac{\partial\lambda_i(t)}{\partial\beta}=&\hat{\lambda}t^{\hat{\beta}-1}+\hat{\lambda}\hat{\beta}t^{\hat{\beta}-1}\ln t\\\frac{\partial\lambda_i(t)}{\partial\lambda}=&\hat{\beta}t^{\hat{\beta}-1}\end{aligned}$$

CROW BOUNDS

The 2-sided confidence bounds on the instantaneous failure intensity (IFI) are given by:

$$\begin{aligned}IFI_U=&\frac{1}{IMTBF_L}\\IFI_L=&\frac{1}{IMTBF_U}\end{aligned}$$

where $IMTBF_L$ and $IMTBF_U$ are calculated using the process for calculating the confidence bounds on the instantaneous MTBF.

Time Given Cumulative Failure Intensity (Grouped)

FISHER MATRIX BOUNDS

The time, T, must be positive, thus $\ln T$ is treated as being normally distributed.

$$\frac{\ln \hat{T} - \ln T}{\sqrt{Var(\ln \hat{T})}} \sim N(0,1)$$

Confidence bounds on the time are given by:

$$CB = \hat{T} e^{\pm z_\alpha \sqrt{Var(\hat{T})}/\hat{T}}$$

where:

$$Var(\hat{T}) = \left(\frac{\partial T}{\partial \beta}\right)^2 Var(\hat{\beta}) + \left(\frac{\partial T}{\partial \lambda}\right)^2 Var(\hat{\lambda}) + 2\left(\frac{\partial T}{\partial \beta}\right)\left(\frac{\partial T}{\partial \lambda}\right) cov(\hat{\beta},\ \hat{\lambda})$$

The variance calculation is the same as given above in the confidence bounds on Beta. And:

$$\frac{\partial T}{\partial \beta} = \frac{-\left(\frac{\lambda_c(T)}{\lambda}\right)^{1/(\beta-1)} \ln\left(\frac{\lambda_c(T)}{\lambda}\right)}{(1-\beta)^2}$$

$$\frac{\partial T}{\partial \lambda} = \left(\frac{\lambda_c(T)}{\lambda}\right)^{1/(\beta-1)} \frac{1}{\lambda(1-\beta)}$$

CROW BOUNDS

The 2-sided confidence bounds on time given cumulative failure intensity (CFI) are presented below. Let:

$$\hat{t} = \left(\frac{CFI}{\hat{\lambda}}\right)^{\frac{1}{\hat{\beta}-1}}$$

Then estimate the number of failures:

$$N = \hat{\lambda}\hat{T}^{\hat{\beta}}$$

The confidence bounds on time given the cumulative failure intensity are then given by:

$$t_L = \frac{\chi^2_{\frac{\alpha}{2}, 2N}}{2 \cdot CFI}$$
$$t_U = \frac{\chi^2_{1-\frac{\alpha}{2}, 2N+2}}{2 \cdot CFI}$$

Time Given Cumulative MTBF (Grouped)

FISHER MATRIX BOUNDS

The time, T, must be positive, thus $\ln T$ is treated as being normally distributed.

$$\frac{\ln \hat{T} - \ln T}{\sqrt{Var(\ln \hat{T})}} \sim N(0,1)$$

Confidence bounds on the time are given by:

$$CB = \hat{T} e^{\pm z_\alpha \sqrt{Var(\hat{T})}/\hat{T}}$$

where:

$$Var(\hat{T}) = \left(\frac{\partial T}{\partial \beta}\right)^2 Var(\hat{\beta}) + \left(\frac{\partial T}{\partial \lambda}\right)^2 Var(\hat{\lambda}) + 2\left(\frac{\partial T}{\partial \beta}\right)\left(\frac{\partial T}{\partial \lambda}\right) cov(\hat{\beta}, \ \hat{\lambda})$$

The variance calculation is the same as given above in the confidence bounds on Beta. And:

$$\hat{T} = (\lambda \cdot m_c)^{1/(1-\beta)}$$

$$\frac{\partial T}{\partial \beta} = \frac{(\lambda \cdot m_c)^{1/(1-\beta)} \ln(\lambda \cdot \ m_c)}{(1-\beta)^2}$$
$$\frac{\partial T}{\partial \lambda} = \frac{(\lambda \cdot m_c)^{1/(1-\beta)}}{\lambda(1-\beta)}$$

CROW BOUNDS

The 2-sided confidence bounds on time given cumulative MTBF $(CMTBF)$ are estimated using the process for calculating the confidence bounds on time given cumulative failure intensity (CFI) where $CFI = \frac{1}{CMTBF}$.

Time Given Instantaneous MTBF (Grouped)

FISHER MATRIX BOUNDS

The time, T, must be positive, thus $\ln T$ is treated as being normally distributed.

$$\frac{\ln \hat{T} - \ln T}{\sqrt{Var(\ln \hat{T})}} \sim N(0,1)$$

Confidence bounds on the time are given by:

$$CB = \hat{T} e^{\pm z_{\alpha} \sqrt{Var(\hat{T})}/\hat{T}}$$

where:

$$Var(\hat{T}) = \left(\frac{\partial T}{\partial \beta}\right)^2 Var(\hat{\beta}) + \left(\frac{\partial T}{\partial \lambda}\right)^2 Var(\hat{\lambda}) + 2\left(\frac{\partial T}{\partial \beta}\right)\left(\frac{\partial T}{\partial \lambda}\right) cov(\hat{\beta}, \ \hat{\lambda})$$

The variance calculation is the same as given above in the confidence bounds on Beta. And:

$$\hat{T} = (\lambda \beta \cdot m_i(T))^{1/(1-\beta)}$$

$$\frac{\partial T}{\partial \beta} = (\lambda \beta \cdot \ m_i(T))^{1/(1-\beta)} \left[\frac{1}{(1-\beta)^2} \ln(\lambda \beta \cdot m_i(T)) + \frac{1}{\beta(1-\beta)}\right]$$

$$\frac{\partial T}{\partial \lambda} = \frac{(\lambda \beta \cdot \ m_i(T))^{1/(1-\beta)}}{\lambda(1-\beta)}$$

CROW BOUNDS

Failure Terminated

Calculate the constants p_1 and p_2 using procedures described for the confidence bounds on instantaneous MTBF. The lower and upper confidence bounds on time are then given by:

$$\hat{t}_L = \left(\frac{\lambda \beta \cdot IMTBF}{p_1}\right)^{\frac{1}{1-\beta}}$$

$$\hat{t}_U = \left(\frac{\lambda \beta \cdot IMTBF}{p_2}\right)^{\frac{1}{1-\beta}}$$

Time Terminated

Calculate the constants Π_1 and Π_2 using procedures described for the confidence bounds on instantaneous MTBF. The lower and upper confidence bounds on time are then given by:

$$\hat{t}_L = \left(\frac{\lambda\beta \cdot IMTBF}{\Pi_1}\right)^{\frac{1}{1-\beta}}$$

$$\hat{t}_U = \left(\frac{\lambda\beta \cdot IMTBF}{\Pi_2}\right)^{\frac{1}{1-\beta}}$$

Time Given Instantaneous Failure Intensity (Grouped)

FISHER MATRIX BOUNDS

The time, T, must be positive, thus $\ln T$ is treated as being normally distributed.

$$\frac{\ln \hat{T} - \ln T}{\sqrt{Var(\ln \hat{T})}} \sim N(0,1)$$

Confidence bounds on the time are given by:

$$CB = \hat{T} e^{\pm z_\alpha \sqrt{Var(\hat{T})}/\hat{T}}$$

where:

$$Var(\hat{T}) = \left(\frac{\partial T}{\partial \beta}\right)^2 Var(\hat{\beta}) + \left(\frac{\partial T}{\partial \lambda}\right)^2 Var(\hat{\lambda}) + 2\left(\frac{\partial T}{\partial \beta}\right)\left(\frac{\partial T}{\partial \lambda}\right) cov(\hat{\beta},\ \hat{\lambda})$$

The variance calculation is the same as given above in the confidence bounds on Beta. And:

$$\hat{T} = \left(\frac{\lambda_i(T)}{\lambda\beta}\right)^{1/(\beta-1)}$$

$$\frac{\partial T}{\partial \beta} = \left(\frac{\lambda_i(T)}{\lambda\beta}\right)^{1/(\beta-1)} \left[-\frac{\ln(\frac{\lambda_i(T)}{\lambda\beta})}{(\beta-1)^2} + \frac{1}{\beta(1-\beta)}\right]$$

$$\frac{\partial T}{\partial \lambda} = \left(\frac{\lambda_i(T)}{\lambda\beta}\right)^{1/(\beta-1)} \frac{1}{\lambda(1-\beta)}$$

CROW BOUNDS

The 2-sided confidence bounds on time given instantaneous failure intensity (IFI) are estimated using the process for calculating the confidence bounds on time given instantaneous MTBF where $IMTBF=\frac{1}{IFI}$.

Crow Extended Confidence Bounds

In this appendix, we will present the two methods used in the Weibull++ software to estimate the confidence bounds for the Crow extended model when applied to developmental testing data. The Fisher Matrix approach is based on the Fisher Information Matrix and is commonly employed in the reliability field. The Crow bounds were developed by Dr. Larry Crow.

Bounds on Demonstrated Failure Intensity

Fisher Matrix Bounds

If there are no BC failure modes, the demonstrated failure intensity is

$$\widehat{\lambda}_D(T)=\frac{N_A+N_{BD}}{T}\ .$$

Thus:

$$Var(\hat{\lambda}_D(t))=\frac{N_A}{T^2}+\frac{N_{BD}}{T^2}=\frac{\lambda_D(t)}{T}$$

and:

$$\sqrt{T}\left(\frac{\hat{\lambda}_D(T)-\lambda_D(T)}{\sqrt{\lambda_D(T)}}\right)\sim N(0,1)$$

$$\lambda_D(T)=\hat{\lambda}_D(T)+\frac{C^2}{2}\pm\sqrt{\hat{\lambda}_D(T)C^2+\frac{C^4}{4}}$$

where $C=\frac{z_{1-\alpha/2}}{\sqrt{T}}$.

If there are BC failure modes, the demonstrated failure intensity, $\widehat{\lambda}_D(T)=\widehat{\lambda}_{CA}$, is actually the instantaneous failure intensity based on all of the data. $\lambda_{CA}(T)$ must be positive; thus, $\ln\lambda_{CA}(T)$ is approximately treated as being normally distributed.

$$\frac{\ln \hat{\lambda}_{CA}(T) - \ln \lambda_{CA}(T)}{\sqrt{Var(\ln \hat{\lambda}_{CA}(T))}} \sim N(0,1)$$

The approximate confidence bounds on the instantaneous failure intensity are then estimated from:

$$CB = \hat{\lambda}_{CA}(T) e^{\pm z_{\alpha} \sqrt{Var(\hat{\lambda}_{CA}(T))}/\hat{\lambda}_{CA}(T)}$$

where $\lambda_{CA}(t) = \lambda \beta T^{\beta-1}$.

$$Var(\hat{\lambda}_{CA}(T)) = \left(\frac{\partial \lambda_{CA}(T)}{\partial \beta}\right)^2 Var(\hat{\beta}) + \left(\frac{\partial \lambda_{CA}(T)}{\partial \lambda}\right)^2 Var(\hat{\lambda}) + 2\left(\frac{\partial \lambda_{CA}(T)}{\partial \beta}\right)\left(\frac{\partial \lambda_{CA}(T)}{\partial \lambda}\right) cov(\hat{\beta}, \hat{\lambda})$$

The variance calculation is the same as described in the Crow-AMSAA Confidence Bounds appendix.

Crow Bounds

If there are no BC failure modes then:

$$[\lambda_D(T)]_l = \widehat{\lambda}_D(T) \frac{\chi^2_{(2N,1-\alpha/2)}}{2N}$$
$$[\lambda_D(T)]_u = \widehat{\lambda}_D(T) \frac{\chi^2_{(2N,\alpha/2)}}{2N}$$

where $\widehat{\lambda}_D(T) = \widehat{\lambda}_{CA}$.

If there are BC modes then the confidence bounds on the demonstrated failure intensity are calculated as presented in the Crow-AMSAA Confidence Bounds appendix.

Bounds on Demonstrated MTBF

Fisher Matrix Bounds

$$MTBF_{D_L} = \frac{1}{[\lambda_D(T)]_U}$$
$$MTBF_{D_U} = \frac{1}{[\lambda_D(T)]_L}$$

where $[\lambda_D(T)]_L$ and $[\lambda_D(T)]_U$ can be obtained from the equation given above for Bounds on Demonstrated Failure Intensity.

Crow Bounds

$$MTBF_{D_L} = \frac{1}{[\lambda_D(T)]_U}$$
$$MTBF_{D_U} = \frac{1}{[\lambda_D(T)]_L}$$

where $[\lambda_D(T)]_L$ and $[\lambda_D(T)]_U$ can be obtained from the equation given above for Bounds on Demonstrated Failure Intensity.

Bounds on Projected Failure Intensity

Fisher Matrix Bounds

The projected failure intensity $\lambda_P(T)$ must be positive; thus, $\ln \lambda_P(T)$ is approximately treated as being normally distributed as well:

$$\frac{\ln \hat{\lambda}_P(T) - \ln \lambda_P(t)}{\sqrt{Var(\ln \hat{\lambda}_P(T))}} \sim N(0,1)$$

$$CB = \hat{\lambda}_P(T) e^{\pm z_\alpha \sqrt{Var(\hat{\lambda}_P(T))} / \hat{\lambda}_P(T)}$$

where:

- $\hat{\lambda}_P(T) = \frac{N_A}{T} + \sum_{i=1}^{M} (1-d_i)\frac{N_i}{T} + \bar{d}\frac{M}{T}\bar{\beta}$ when there are no BC modes.
- $\hat{\lambda}_P(T) = \widehat{\lambda}_{EM} = \widehat{\lambda}_{CA} - \widehat{\lambda}_{BD} + \sum_{i=1}^{M} (1-d_i)\frac{N_i}{T} + \bar{d}\widehat{h}(T|BD)$ when there are BC modes.
- N_i is the total failure number of the i^{th} distinct BD mode.

You can then get:

$$Var(\lambda_P(T)) \approx Var(\hat{\gamma}_{GP}) + \mu_d^2 Var(h(T)) \approx \frac{\hat{r}_{GP}}{T} + \mu_d^2 Var(h(T))$$

where:

$$\hat{h}(T) = \frac{M}{T}\bar{\beta}$$
$$Var(\hat{h}(T)) = \left(\frac{M}{T}\right)^2 Var(\bar{\beta}) = \left(\frac{M}{T}\right)^2 \left(\frac{M}{M-1}\right)^2 Var(\hat{\beta}) = \frac{M^4}{T^2(M-1)^2} Var(\hat{\beta})$$

The $Var(\hat{\beta})$ can be obtained from Fisher Matrix based on M distinct BD modes.

Crow Bounds

$$[\lambda_P(T)]_L = \hat{\lambda}_P(T) + \frac{C^2}{2} - \sqrt{\hat{\lambda}_P(T) \cdot C^2 + \frac{C^4}{4}}$$
$$[\lambda_P(T)]_U = \hat{\lambda}_P(T) + \frac{C^2}{2} + \sqrt{\hat{\lambda}_P(T) \cdot \ C^2 + \frac{C^4}{4}}$$

where $C = \frac{z_{1-\alpha/2}}{\sqrt{T}}$.

Bounds on Projected MTBF

Fisher Matrix Bounds

$$MTBF_{P_L} = \frac{1}{[\lambda_P(T)]_U}$$
$$MTBF_{P_U} = \frac{1}{[\lambda_P(T)]_L}$$

$[\lambda_P(T)]_U$ and $[\lambda_P(T)]_L$ can be obtained from the equation given above for Bounds on Projected Failure Intensity.

Crow Bounds

$$MTBF_{P_L} = \frac{1}{[\lambda_P(T)]_U}$$
$$MTBF_{P_U} = \frac{1}{[\lambda_P(T)]_L}$$

$[\lambda_P(T)]_U$ and $[\lambda_P(T)]_L$ can be obtained from the equation given above for Bounds on Projected Failure Intensity.

Bounds on Growth Potential Failure Intensity

Fisher Matrix Bounds

If there are no BC failure modes, the growth potential failure intensity is

$$\hat{r}_{GP}(T) = \frac{N_A}{T} + \sum_{i=1}^{M}(1-d_i)\frac{N_i}{T} .$$

Then:

$$\begin{aligned} Var(\hat{r}_{GP}) &= \frac{1}{T}\left[\frac{N_A}{T} + \sum_{i=1}^{M}(1-d_i)^2\frac{N_i}{T}\right] \\ &\leq \frac{1}{T}\left[\frac{N_A}{T} + \sum_{i=1}^{M}(1-d_i)\frac{N_i}{T}\right] \\ &= \frac{r_{GP}}{T} \end{aligned}$$

If there are BC failure modes, the growth potential failure intensity is

$$\hat{r}_{GP}(T) = \widehat{\lambda}_{CA} - \widehat{\lambda}_{BD} + \sum_{i=1}^{M}(1-d_i)\tfrac{N_i}{T}, \quad Var(\hat{r}_{GP}) \approx \tfrac{r_{GP}}{T}.$$

Therefore:

$$\sqrt{T}\left(\frac{\hat{r}_{GP} - r_{GP}}{\sqrt{r_{GP}}}\right) \sim N(0,1)$$

The confidence bounds on the growth potential failure intensity are as follows:

$$\begin{aligned} r_L &= \hat{r}_{GP} + \frac{C^2}{2} - \sqrt{\hat{r}_{GP}\,C^2 + \frac{C^4}{4}} \\ r_U &= \hat{r}_{GP} + \frac{C^2}{2} + \sqrt{\hat{r}_{GP}\,C^2 + \frac{C^4}{4}} \end{aligned}$$

where $C = \frac{z_{1-\alpha/2}}{\sqrt{T}}$.

Crow Bounds

The Crow bounds for the growth potential failure intensity are the same as the Fisher Matrix bounds.

Bounds on Growth Potential MTBF

Fisher Matrix Bounds

$$\begin{aligned} MTBF_{GP_L} &= \frac{1}{r_U} \\ MTBF_{GP_U} &= \frac{1}{r_L} \end{aligned}$$

where r_U and r_L can be obtained from the equation given above for Bounds on Growth Potential Failure Intensity.

Crow Bounds

The Crow bounds for the growth potential MTBF are the same as the Fisher Matrix bounds.

Confidence Bounds for Repairable Systems Analysis

In this appendix, we will present the two methods used in the Weibull++ software to estimate the confidence bounds for Repairable Systems Analysis. The Fisher Matrix approach is based on the Fisher Information Matrix and is commonly employed in the reliability field. The Crow bounds were developed by Dr. Larry Crow.

Beta

Fisher Matrix Bounds

The parameter β must be positive, thus $\ln\beta$ is approximately treated as being normally distributed.

$$\frac{\ln(\hat{\beta})-\ln(\beta)}{\sqrt{Var\left[\ln(\hat{\beta})\right]}} \sim N(0,1)$$

$$CB_{\beta}=\hat{\beta}e^{\pm z_{\alpha}\sqrt{Var(\hat{\beta})}/\hat{\beta}}$$

$$\hat{\beta}=\frac{\sum^{K}{}_{q=1}N_q}{\hat{\lambda}\sum^{K}{}_{q=1}\left[(T_q^{\hat{\beta}}\ln(T_q)-S_q^{\hat{\beta}}\ln(S_q)\right]-\sum^{K}{}_{q=1}\sum^{N_q}{}_{i=1}\ln(X_{iq})}$$

All variance can be calculated using the Fisher Information Matrix. Λ is the natural log-likelihood function.

$$\Lambda=\sum_{q=1}^{K}\left[N_q(\ln(\lambda)+\ln(\beta))-\lambda(T_q^{\beta}-S_q^{\beta})+(\beta-1)\sum_{i=1}^{N_q}\ln(x_{iq})\right]$$

$$\frac{\partial^2\Lambda}{\partial\lambda^2}=-\frac{\sum^{K}{}_{q=1}N_q}{\lambda^2}$$

$$\frac{\partial^2\Lambda}{\partial\lambda\partial\beta}=-\sum_{q=1}^{K}\left[T_q^{\beta}\ln(T_q)-S_q^{\beta}\ln(S_q)\right]$$

$$\frac{\partial^2\Lambda}{\partial\beta^2}=-\frac{\sum^{K}{}_{q=1}N_q}{\beta^2}-\lambda\sum_{q=1}^{K}\left[T_q^{\beta}(\ln(T_q))^2-S_q^{\beta}(\ln(S_q))^2\right]$$

Crow Bounds

Calculate the conditional maximum likelihood estimate of $\tilde{\beta}$:

$$\tilde{\beta} = \frac{\sum^{K}{}_{q=1} M_q}{\sum^{K}{}_{q=1} \sum^{M}{}_{i=1} \ln\left(\frac{T_q}{X_{iq}}\right)}$$

The Crow 2-sided $(1-a)$ 100% confidence bounds on β are:

$$\beta_L = \tilde{\beta}\frac{\chi^2_{\frac{\alpha}{2},2M}}{2M}$$
$$\beta_U = \tilde{\beta}\frac{\chi^2_{1-\frac{\alpha}{2},2M}}{2M}$$

Lambda

Fisher Matrix Bounds

The parameter λ must be positive, thus $\ln \lambda$ is approximately treated as being normally distributed. These bounds are based on:

$$\frac{\ln(\hat{\lambda}) - \ln(\lambda)}{\sqrt{Var\left[\ln(\hat{\lambda})\right]}} \sim N(0,1)$$

The approximate confidence bounds on λ are given as:

$$CB_\lambda = \hat{\lambda}e^{\pm z_\alpha \sqrt{Var(\hat{\lambda})}/\hat{\lambda}}$$

where $\hat{\lambda} = \frac{n}{T_K^{\hat{\beta}}}$.

The variance calculation is the same the equations given in the confidence bounds on Beta.

Crow Bounds

Failure Terminated

The confidence bounds on λ for failure terminated data are calculated using:

$$\lambda_L = \frac{\chi^2_{\frac{\alpha}{2},2N}}{2 \cdot \sum^K_{q=1} T_q^\beta}$$
$$\lambda_U = \frac{\chi^2_{1-\frac{\alpha}{2},2N}}{2 \cdot \sum^K_{q=1} T_q^\beta}$$

where:

- N = total number of failures.
- K = number of systems.
- T_q = end time for the q^{th} system.

Time Terminated

The confidence bounds on λ for time terminated data are calculated using:

$$\lambda_L = \frac{\chi^2_{\frac{\alpha}{2},2N}}{2 \cdot \sum^K_{q=1} T_q^\beta}$$
$$\lambda_U = \frac{\chi^2_{1-\frac{\alpha}{2},2N+2}}{2 \cdot \sum^K_{q=1} T_q^\beta}$$

where:

- N = total number of failures.
- K = number of systems.
- T_q = end time for the q^{th} system.

Cumulative Number of Failures

Fisher Matrix Bounds

The cumulative number of failures, $N(t)$. must be positive, thus $\ln(N(t))$ is approximately treated as being normally distributed.

$$\frac{\ln(\hat{N}(t)) - \ln(N(t))}{\sqrt{Var\left[\ln \hat{N}(t)\right]}} \sim N(0,1)$$

$$N(t) = \hat{N}(t) e^{\pm z_\alpha \sqrt{Var(\hat{N}(t))}/\hat{N}(t)}$$

where:

$$\hat{N}(t) = \hat{\lambda}t^{\hat{\beta}}$$

$$Var(\hat{N}(t)) = \left(\frac{\partial N(t)}{\partial \beta}\right)^2 Var(\hat{\beta}) + \left(\frac{\partial N(t)}{\partial \lambda}\right)^2 Var(\hat{\lambda}) + 2\left(\frac{\partial N(t)}{\partial \beta}\right)\left(\frac{\partial N(t)}{\partial \lambda}\right) cov(\hat{\beta}, \hat{\lambda})$$

The variance calculation is the same as the calculations in the confidence bounds on Beta.

$$\frac{\partial N(t)}{\partial \beta} = \hat{\lambda}t^{\hat{\beta}} \ln(t)$$
$$\frac{\partial N(t)}{\partial \lambda} = t\hat{\beta}$$

Crow Bounds

The 2-sided confidence bounds on the cumulative number of failures are given by:

$$N(t)_L = \frac{\chi^2_{\frac{\alpha}{2},2N}}{2 \cdot S}$$

$$N(t)_U = \frac{\chi^2_{1-\frac{\alpha}{2},2N+2}}{2 \cdot S}$$

where:

- N = total number of failures across all systems. This is not the number of failures up to time t.
- $S = \frac{\left(\frac{N}{\hat{\lambda}}\right)}{t^{\hat{\beta}}}$
- t = time at which calculations are being conducted.

Cumulative Failure Intensity

Fisher Matrix Bounds

The cumulative failure intensity, $\lambda_c(t)$ must be positive, thus $\ln \lambda_c(t)$ is approximately treated as being normally distributed.

$$\frac{\ln(\hat{\lambda}_c(t)) - \ln(\lambda_c(t))}{\sqrt{Var\left[\ln(\hat{\lambda}_c(t))\right]}} \sim N(0,1)$$

The approximate confidence bounds on the cumulative failure intensity are then estimated using:

$$CB = \hat{\lambda}_c(t)e^{\pm z_\alpha \sqrt{Var(\hat{\lambda}_c(t))}/\hat{\lambda}_c(t)}$$

where:

$$\hat{\lambda}_c(t) = \hat{\lambda}t^{\hat{\beta}-1}$$

and:

$$Var(\hat{\lambda}_c(t)) = \left(\frac{\partial \lambda_c(t)}{\partial \beta}\right)^2 Var(\hat{\beta}) + \left(\frac{\partial \lambda_c(t)}{\partial \lambda}\right)^2 Var(\hat{\lambda}) + 2\left(\frac{\partial \lambda_c(t)}{\partial \beta}\right)\left(\frac{\partial \lambda_c(t)}{\partial \lambda}\right) cov(\hat{\beta}, \hat{\lambda})$$

The variance calculation is the same as the calculations in the confidence bounds on Beta.

$$\frac{\partial \lambda_c(t)}{\partial \beta} = \hat{\lambda}t^{\hat{\beta}-1}\ln(t)$$
$$\frac{\partial \lambda_c(t)}{\partial \lambda} = t^{\hat{\beta}-1}$$

Crow Bounds

The 2-sided confidence bounds on the cumulative failure intensity are given by:

$$CFI_L = \frac{\chi^2_{\frac{\alpha}{2},2N}}{2 \cdot t \cdot S}$$

$$CFI_U = \frac{\chi^2_{1-\frac{\alpha}{2},2N+2}}{2 \cdot t \cdot S}$$

where:

- N = total number of failures across all systems. This is not the number of failures up to time t.
- $S = \frac{\left(\frac{N}{\hat{\lambda}}\right)}{t^{\hat{\beta}}}$
- t = time at which calculations are being conducted.

Cumulative MTBF

Fisher Matrix Bounds

The cumulative MTBF, $m_c(t)$. must be positive, thus $\ln m_c(t)$ is approximately treated as being normally distributed.

$$\frac{\ln(\hat{m}_c(t)) - \ln(m_c(t))}{\sqrt{Var\left[\ln(\hat{m}_c(t))\right]}} \sim N(0,1)$$

The approximate confidence bounds on the cumulative MTBF are then estimated from:

$$CB = \hat{m}_c(t)e^{\pm z_\alpha \sqrt{Var(\hat{m}_c(t))}/\hat{m}_c(t)}$$

where:

$$\hat{m}_c(t) = \frac{1}{\hat{\lambda}}t^{1-\hat{\beta}}$$

$$Var(\hat{m}_c(t)) = \left(\frac{\partial m_c(t)}{\partial \beta}\right)^2 Var(\hat{\beta}) + \left(\frac{\partial m_c(t)}{\partial \lambda}\right)^2 Var(\hat{\lambda}) + 2\left(\frac{\partial m_c(t)}{\partial \beta}\right)\left(\frac{\partial m_c(t)}{\partial \lambda}\right) cov(\hat{\beta},\hat{\lambda})$$

The variance calculation is the same as the calculations given in the confidence bounds on Beta.

$$\frac{\partial m_c(t)}{\partial \beta} = -\frac{1}{\hat{\lambda}}t^{1-\hat{\beta}}\ln(t)$$

$$\frac{\partial m_c(t)}{\partial \lambda} = -\frac{1}{\hat{\lambda}^2}t^{1-\hat{\beta}}$$

Crow Bounds

The 2-sided confidence bounds on the cumulative MTBF $(CMTBF)$ are given by:

$$CMTBF_L = \frac{1}{CFI_U}$$

$$CMTBF_U = \frac{1}{CFI_L}$$

where CFI_L and CFI_U are calculated using the process for the confidence bounds on cumulative failure intensity.

Instantaneous MTBF

Fisher Matrix Bounds

The instantaneous MTBF, $m_i(t)$. must be positive, thus $\ln m_i(t)$ is approximately treated as being normally distributed.

$$\frac{\ln(\hat{m}_i(t)) - \ln(m_i(t))}{\sqrt{Var\left[\ln(\hat{m}_i(t))\right]}} \sim N(0,1)$$

The approximate confidence bounds on the instantaneous MTBF are then estimated from:

$$CB = \hat{m}_i(t)e^{\pm z_\alpha \sqrt{Var(\hat{m}_i(t))}/\hat{m}_i(t)}$$

where:

$$\hat{m}_i(t) = \frac{1}{\lambda\beta t^{\beta-1}}$$

$$Var(\hat{m}_i(t)) = \left(\frac{\partial m_i(t)}{\partial\beta}\right)^2 Var(\hat{\beta}) + \left(\frac{\partial m_i(t)}{\partial\lambda}\right)^2 Var(\hat{\lambda}) + 2\left(\frac{\partial m_i(t)}{\partial\beta}\right)\left(\frac{\partial m_i(t)}{\partial\lambda}\right) cov(\hat{\beta},\hat{\lambda})$$

The variance calculation is the same as the calculations given in the confidence bounds on Beta.

$$\frac{\partial m_i(t)}{\partial\beta} = -\frac{1}{\hat{\lambda}\hat{\beta}^2}t^{1-\hat{\beta}} - \frac{1}{\hat{\lambda}\hat{\beta}}t^{1-\hat{\beta}}\ln(t)$$

$$\frac{\partial m_i(t)}{\partial\lambda} = -\frac{1}{\hat{\lambda}^2\hat{\beta}}t^{1-\hat{\beta}}$$

Crow Bounds

Failure Terminated

For failure terminated data and the 2-sided confidence bounds on instantaneous MTBF $(IMTBF)$, consider the following equation:

$$G(\mu|n) = {}^{\infty}_{0} \frac{e^{-x}x^{n-2}}{(n-2)!}\sum_{i=0}^{n-1}\frac{1}{i!}\left(\frac{\mu}{x}\right)^i \exp(-\frac{\mu}{x})\,dx$$

Find the values p_1 and p_2 by finding the solution $G\left(\frac{n^2}{c}\middle|n\right)=\frac{\alpha}{2}$ and $G\left(\frac{n^2}{c}\middle|n\right)=1-\frac{\alpha}{2}$ for the lower and upper bounds, respectively.

If using the biased parameters, $\hat{\beta}$ and $\hat{\lambda}$, then the upper and lower confidence bounds are:

$$\begin{aligned}IMTBF_L &= IMTBF\cdot p_1\\ IMTBF_U &= IMTBF\cdot p_2\end{aligned}$$

where $IMTBF=\frac{1}{\hat{\lambda}\hat{\beta}t^{\hat{\beta}-1}}$.

If using the unbiased parameters, $\bar{\beta}$ and $\bar{\lambda}$, then the upper and lower confidence bounds are:

$$\begin{aligned}IMTBF_L &= IMTBF\cdot\left(\frac{N-2}{N}\right)\cdot p_1\\ IMTBF_U &= IMTBF\cdot\left(\frac{N-2}{N}\right)\cdot p_2\end{aligned}$$

where $IMTBF=\frac{1}{\bar{\lambda}\bar{\beta}t^{\bar{\beta}-1}}$.

Time Terminated

Consider the following equation where $I_1(.)$ is the modified Bessel function of order one:

$$H(x|k)=\sum_{j=1}^{k}\frac{x^{2j-1}}{2^{2j-1}(j-1)!j!I_1(x)}$$

Find the values Π_1 and Π_2 by finding the solution x to $H(x|k)=\frac{\alpha}{2}$ and $H(x|k)=1-\frac{\alpha}{2}$ in the cases corresponding to the lower and upper bounds, respectively. Calculate $\Pi=\frac{4n^2}{x^2}$ for each case.

If using the biased parameters, $\hat{\beta}$ and $\hat{\lambda}$, then the upper and lower confidence bounds are:

$$\begin{aligned}IMTBF_L &= IMTBF\cdot \Pi_1\\ IMTBF_U &= IMTBF\cdot \Pi_2\end{aligned}$$

where $IMTBF=\frac{1}{\hat{\lambda}\hat{\beta}t^{\hat{\beta}-1}}$.

If using the unbiased parameters, $\bar{\beta}$ and $\bar{\lambda}$, then the upper and lower confidence bounds are:

$$IMTBF_L = IMTBF \cdot \left(\frac{N-1}{N}\right) \cdot \Pi_1$$
$$IMTBF_U = IMTBF \cdot \left(\frac{N-1}{N}\right) \cdot \Pi_2$$

where $IMTBF = \frac{1}{\lambda\beta t^{\beta-1}}$.

Instantaneous Failure Intensity

Fisher Matrix Bounds

The instantaneous failure intensity, $\lambda_i(t)$. must be positive, thus $\ln \lambda_i(t)$ is approximately treated as being normally distributed.

$$\frac{\ln(\hat{\lambda}_i(t)) - \ln(\lambda_i(t))}{\sqrt{Var\left[\ln(\hat{\lambda}_i(t))\right]}} \sim N(0,1)$$

The approximate confidence bounds on the instantaneous failure intensity are then estimated from:

$$CB = \hat{\lambda}_i(t) e^{\pm z_\alpha \sqrt{Var(\hat{\lambda}_i(t))}/\hat{\lambda}_i(t)}$$

where $\lambda_i(t) = \lambda\beta t^{\beta-1}$ and:

$$Var(\hat{\lambda}_i(t)) = \left(\frac{\partial\lambda_i(t)}{\partial\beta}\right)^2 Var(\hat{\beta}) + \left(\frac{\partial\lambda_i(t)}{\partial\lambda}\right)^2 Var(\hat{\lambda}) + 2\left(\frac{\partial\lambda_i(t)}{\partial\beta}\right)\left(\frac{\partial\lambda_i(t)}{\partial\lambda}\right) cov(\hat{\beta},\hat{\lambda})$$

The variance calculation is the same as the calculations in the confidence bounds on Beta.

$$\frac{\partial\lambda_i(t)}{\partial\beta} = \hat{\lambda} t^{\hat{\beta}-1} + \hat{\lambda}\hat{\beta} t^{\hat{\beta}-1}\ln(t)$$
$$\frac{\partial\lambda_i(t)}{\partial\lambda} = \hat{\beta} t^{\hat{\beta}-1}$$

Crow Bounds

The 2-sided confidence bounds on the instantaneous failure intensity (IFI) are given by:

$$IFI_L = \frac{1}{IMTBF_U}$$
$$IFI_U = \frac{1}{IMTBF_L}$$

where $IMTBF_L$ and $IMTBF_U$ are calculated using the process presented for the confidence bounds on the instantaneous MTBF.

Time Given Cumulative Failure Intensity

Fisher Matrix Bounds

The time, T. must be positive, thus $\ln T$ is approximately treated as being normally distributed.

$$\frac{\ln(\hat{T})-\ln(T)}{\sqrt{Var\left[\ln\hat{T}\right]}} \sim N(0,1)$$

The confidence bounds on the time are given by:

$$CB=\hat{T}e^{\pm z_{\alpha}\sqrt{Var(\hat{T})}/\hat{T}}$$

where:

$$Var(\hat{T})=\left(\frac{\partial T}{\partial\beta}\right)^2 Var(\hat{\beta})+\left(\frac{\partial T}{\partial\lambda}\right)^2 Var(\hat{\lambda})+2\left(\frac{\partial T}{\partial\beta}\right)\left(\frac{\partial T}{\partial\lambda}\right)cov(\hat{\beta},\hat{\lambda})$$

The variance calculation is the same as the calculations given in the confidence bounds on Beta.

$$\hat{T}=\left(\frac{\lambda_c(T)}{\lambda}\right)^{1/(\beta-1)}$$

$$\frac{\partial T}{\partial\beta}=\frac{-\left(\frac{\lambda_c(T)}{\lambda}\right)^{1/(\beta-1)}\ln\left(\frac{\lambda_c(T)}{\lambda}\right)}{(1-\beta)^2}$$

$$\frac{\partial T}{\partial\lambda}=\left(\frac{\lambda_c(T)}{\lambda}\right)^{1/(\beta-1)}\frac{1}{\lambda(1-\beta)}$$

Crow Bounds

The 2-sided confidence bounds on time given cumulative failure intensity (CFI) are given by:

$$\hat{t}=\left(\frac{CFI}{\hat{\lambda}}\right)^{\frac{1}{\hat{\beta}-1}}$$

Then estimate, the number of failures, N, such that:

$$N=\hat{\lambda}\hat{t}^{\hat{\beta}}$$

The lower and upper confidence bounds on time are then estimated using:

$$t_L = \frac{\chi^2_{\frac{\alpha}{2},2N}}{2 \cdot CFI}$$
$$t_U = \frac{\chi^2_{1-\frac{\alpha}{2},2N+2}}{2 \cdot CFI}$$

Time Given Cumulative MTBF

Fisher Matrix Bounds

The time, T. must be positive, thus $\ln T$ is approximately treated as being normally distributed.

$$\frac{\ln(\hat{T}) - \ln(T)}{\sqrt{Var\left[\ln(\hat{T})\right]}} \sim N(0,1)$$

The confidence bounds on the time are given by:

$$CB = \hat{T} e^{\pm z_\alpha \sqrt{Var(\hat{T})}/\hat{T}}$$

where:

$$Var(\hat{T}) = \left(\frac{\partial T}{\partial \beta}\right)^2 Var(\hat{\beta}) + \left(\frac{\partial T}{\partial \lambda}\right)^2 Var(\hat{\lambda}) + 2\left(\frac{\partial T}{\partial \beta}\right)\left(\frac{\partial T}{\partial \lambda}\right) cov(\hat{\beta}, \hat{\lambda})$$

The variance calculation is the same as the calculations in the confidence bounds on Beta.

$$\hat{T} = (\lambda \cdot m_c)^{1/(1-\beta)}$$

$$\frac{\partial T}{\partial \beta} = \frac{(\lambda \cdot m_c)^{1/(1-\beta)} \ln(\lambda \cdot m_c)}{(1-\beta)^2}$$
$$\frac{\partial T}{\partial \lambda} = \frac{(\lambda \cdot m_c)^{1/(1-\beta)}}{\lambda(1-\beta)}$$

Crow Bounds

The 2-sided confidence bounds on time given cumulative MTBF $(CMTBF)$ are estimated using the process for the confidence bounds on time given cumulative failure intensity (CFI) where $CFI = \frac{1}{CMTBF}$.

Time Given Instantaneous MTBF

Fisher Matrix Bounds

The time, T. must be positive, thus $\ln T$ is approximately treated as being normally distributed.

$$\frac{\ln(\hat{T})-\ln(T)}{\sqrt{Var\left[\ln(\hat{T})\right]}} \sim N(0,1)$$

The confidence bounds on the time are given by:

$$CB=\hat{T}e^{\pm z_{\alpha}\sqrt{Var(\hat{T})}/\hat{T}}$$

where:

$$Var(\hat{T})=\left(\frac{\partial T}{\partial \beta}\right)^{2}Var(\hat{\beta})+\left(\frac{\partial T}{\partial \lambda}\right)^{2}Var(\hat{\lambda})+2\left(\frac{\partial T}{\partial \beta}\right)\left(\frac{\partial T}{\partial \lambda}\right)cov(\hat{\beta},\hat{\lambda})$$

The variance calculation is the same as the calculations in the confidence bounds on Beta.

$$\hat{T}=(\lambda\beta\cdot MTBF_{i})^{1/(1-\beta)}$$

$$\frac{\partial T}{\partial \beta}=(\lambda\beta\cdot MTBF_{i})^{1/(1-\beta)}[\frac{1}{(1-\beta)^{2}}\ln(\lambda\beta\cdot MTBF_{i})+\frac{1}{\beta(1-\beta)}]$$

$$\frac{\partial T}{\partial \lambda}=\frac{(\lambda\beta\cdot MTBF_{i})^{1/(1-\beta)}}{\lambda(1-\beta)}$$

Crow Bounds

Failure Terminated

Calculate the constants p_1 and p_2 using procedures described for the confidence bounds on instantaneous MTBF. The lower and upper confidence bounds on time are then given by:

$$\hat{t}_{L}=\left(\frac{\lambda\beta\cdot IMTBF}{p_{1}}\right)^{\frac{1}{1-\beta}}$$

$$\hat{t}_{U}=\left(\frac{\lambda\beta\cdot IMTBF}{p_{2}}\right)^{\frac{1}{1-\beta}}$$

Time Terminated

Calculate the constants Π_1 and Π_2 using procedures described for the confidence bounds on instantaneous MTBF. The lower and upper confidence bounds on time are then given by:

$$\hat{t}_L = \left(\frac{\lambda\beta \cdot IMTBF}{\Pi_1}\right)^{\frac{1}{1-\beta}}$$

$$\hat{t}_U = \left(\frac{\lambda\beta \cdot IMTBF}{\Pi_2}\right)^{\frac{1}{1-\beta}}$$

Time Given Instantaneous Failure Intensity

Fisher Matrix Bounds

These bounds are based on:

$$\frac{\ln(\hat{T}) - \ln(T)}{\sqrt{Var\left[\ln(\hat{T})\right]}} \sim N(0,1)$$

The confidence bounds on the time are given by:

$$CB = \hat{T}e^{\pm z_\alpha \sqrt{Var(\hat{T})}/\hat{T}}$$

where:

$$Var(\hat{T}) = \left(\frac{\partial T}{\partial \beta}\right)^2 Var(\hat{\beta}) + \left(\frac{\partial T}{\partial \lambda}\right)^2 Var(\hat{\lambda}) + 2\left(\frac{\partial T}{\partial \beta}\right)\left(\frac{\partial T}{\partial \lambda}\right) cov(\hat{\beta},\hat{\lambda})$$

The variance calculation is the same as the calculations given in the confidence bounds on Beta.

$$\hat{T} = \left(\frac{\lambda_i(T)}{\lambda \cdot \beta}\right)^{1/(\beta-1)}$$

$$\frac{\partial T}{\partial \beta} = \left(\frac{\lambda_i(T)}{\lambda \cdot \beta}\right)^{1/(\beta-1)} [-\frac{\ln(\frac{\lambda_i(T)}{\lambda \cdot \beta})}{(\beta-1)^2} + \frac{1}{\beta(1-\beta)}]$$

$$\frac{\partial T}{\partial \lambda} = \left(\frac{\lambda_i(T)}{\lambda \cdot \beta}\right)^{1/(\beta-1)} \frac{1}{\lambda(1-\beta)}$$

Crow Bounds

The 2-sided confidence bounds on time given instantaneous failure intensity (IFI) are estimated using the process for the confidence bounds on time given instantaneous MTBF where $IMTBF = \frac{1}{IFI}$.

Reliability

Fisher Matrix Bounds

These bounds are based on:

$$\log it(\hat{R}(t)) \sim N(0,1)$$

$$\log it(\hat{R}(t)) = \ln\left\{\frac{\hat{R}(t)}{1-\hat{R}(t)}\right\}$$

The confidence bounds on reliability are given by:

$$CB = \frac{\hat{R}(t)}{\hat{R}(t) + (1-\hat{R}(t))e^{\pm z_{\alpha}\sqrt{Var(\hat{R}(t))}/[\hat{R}(t)(1-\hat{R}(t))]}}$$

$$Var(\hat{R}(t)) = \left(\frac{\partial R}{\partial \beta}\right)^2 Var(\hat{\beta}) + \left(\frac{\partial R}{\partial \lambda}\right)^2 Var(\hat{\lambda}) + 2\left(\frac{\partial R}{\partial \beta}\right)\left(\frac{\partial R}{\partial \lambda}\right) cov(\hat{\beta},\hat{\lambda})$$

The variance calculation is the same as the calculations in the confidence bounds on Beta.

$$\frac{\partial R}{\partial \beta} = e^{-[\hat{\lambda}(t+d)^{\hat{\beta}} - \hat{\lambda}t^{\hat{\beta}}]}[\lambda t^{\hat{\beta}}\ln(t) - \lambda(t+d)^{\hat{\beta}}\ln(t+d)]$$

$$\frac{\partial R}{\partial \lambda} = e^{-[\hat{\lambda}(t+d)^{\hat{\beta}} - \hat{\lambda}t^{\hat{\beta}}]}[t^{\hat{\beta}} - (t+d)^{\hat{\beta}}]$$

Crow Bounds

Failure Terminated

For failure terminated data, the 100($1-\alpha$)% confidence interval on the current reliability at time t for a specified mission duration d is:

$$\left([\hat{R}(d)]^{\frac{1}{p_1}}, [\hat{R}(d)]^{\frac{1}{p_2}}\right)$$

where:

- $\hat{R}(d) = e^{-\left[\hat{\lambda}(t+d)^{\hat{\beta}} - \hat{\lambda}t^{\hat{\beta}}\right]}$
- p_1 and p_2 are obtained from the confidence bounds on instantaneous MTBF for failure terminated data.

Time Terminated

For time terminated data, the 100($1-\alpha$)% confidence interval on the current reliability at time t for a specified mission duration d is:

$$\left(\left[\hat{R}(d)\right]^{\frac{1}{\Pi_1}}, \left[\hat{R}(d)\right]^{\frac{1}{\Pi_2}}\right)$$

where:

- $\hat{R}(d) = e^{-\left[\hat{\lambda}(t+d)^{\hat{\beta}} - \hat{\lambda}t^{\hat{\beta}}\right]}$
- Π_1 and Π_2 are obtained from the confidence bounds on instantaneous MTBF for time terminated data.

Time Given Reliability and Mission Time

Fisher Matrix Bounds

The time, t. must be positive, thus $\ln t$ is approximately treated as being normally distributed.

$$\frac{\ln(\hat{t}) - \ln(t)}{\sqrt{Var\left[\ln(\hat{t})\right]}} \sim N(0,1)$$

The confidence bounds on time are calculated by using:

$$CB = \hat{t}\, e^{\pm z_\alpha \sqrt{Var(\hat{t})}/\hat{t}}$$

where:

$$Var(\hat{t}) = \left(\frac{\partial t}{\partial \beta}\right)^2 Var(\hat{\beta}) + \left(\frac{\partial t}{\partial \lambda}\right)^2 Var(\hat{\lambda}) + 2\left(\frac{\partial t}{\partial \beta}\right)\left(\frac{\partial t}{\partial \lambda}\right) cov(\hat{\beta}, \hat{\lambda})$$

$\hat{t}$ is calculated numerically from:

$$\hat{R}(d) = e^{-[\hat{\lambda}(\hat{t}+d)^{\hat{\beta}} - \hat{\lambda}\hat{t}^{\hat{\beta}}]}\ ;\ d = \text{mission time}$$

The variance calculations are done by:

$$\frac{\partial t}{\partial \beta} = \frac{\hat{t}^{\hat{\beta}}\ln(\hat{t}) - (\hat{t}+d)^{\hat{\beta}}\ln(\hat{t}+d)}{\hat{\beta}(\hat{t}+d)^{\hat{\beta}-1} - \hat{\beta}\hat{t}^{\hat{\beta}-1}}$$

$$\frac{\partial t}{\partial \lambda} = \frac{\hat{t}^{\hat{\beta}} - (\hat{t}+d)^{\hat{\beta}}}{\hat{\lambda}\hat{\beta}(\hat{t}+d)^{\hat{\beta}-1} - \hat{\lambda}\hat{\beta}\hat{t}^{\hat{\beta}-1}}$$

Crow Bounds

Failure Terminated

For failure terminated data, the 2-sided confidence bounds on time given reliability and mission time estimated by calculating:

$$(\hat{R}_L, \hat{R}_U) = \left(R^{\frac{1}{p_1}}, R^{\frac{1}{p_2}}\right)$$

where p_1 and p_2 are obtained from the confidence bounds on instantaneous MTBF for failure terminated data.

Let $R = \hat{R}_L$ and solve numerically for t_1 using $R = e^{-[\hat{\lambda}(\hat{t}_1+d)^{\hat{\beta}} - \hat{\lambda}\hat{t}_1^{\hat{\beta}}]}$.

Let $R = \hat{R}_U$ and solve numerically for t_2 using $R = e^{-[\hat{\lambda}(\hat{t}_2+d)^{\hat{\beta}} - \hat{\lambda}\hat{t}_2^{\hat{\beta}}]}$.

If $t_1 < t_2$ then $t_L = t_1$ and $t_U = t_2$. If $t_1 > t_2$ then $t_L = t_2$ and $t_U = t_1$.

Time Terminated

For time terminated data, the 2-sided confidence bounds on time given reliability and mission time estimated by calculating:

$$(\hat{R}_L, \hat{R}_U) = \left(R^{\frac{1}{\Pi_1}}, R^{\frac{1}{\Pi_2}}\right).$$

where Π_1 and Π_2 are obtained from the confidence bounds on instantaneous MTBF for time terminated data.

Let $R = \hat{R}_L$ and solve numerically for t_1 using $R = e^{-[\hat{\lambda}(\hat{t}_1+d)^{\hat{\beta}} - \hat{\lambda}\hat{t}_1^{\hat{\beta}}]}$.

Let $R = \hat{R}_U$ and solve numerically for t_2 using $R = e^{-[\hat{\lambda}(\hat{t}_2+d)^{\hat{\beta}} - \hat{\lambda}\hat{t}_2^{\hat{\beta}}]}$.

If $t_1 < t_2$. then $t_L = t_1$ and $t_U = t_2$. If $t_1 > t_2$. then $t_L = t_2$ and $t_U = t_1$.

Mission Time Given Reliability and Time

Fisher Matrix Bounds

The mission time, d. must be positive, thus $\ln(d)$ is approximately treated as being normally distributed.

$$\frac{\ln(\hat{d})-\ln(d)}{\sqrt{Var\left[\ln(\hat{d})\right]}}\sim N(0,1)$$

The confidence bounds on mission time are given by using:

$$CB=\hat{d}\,e^{\pm z_{\alpha}\sqrt{Var(\hat{d})}/\hat{d}}$$

where:

$$Var(\hat{d})=\left(\frac{\partial d}{\partial\beta}\right)^{2}Var(\hat{\beta})+\left(\frac{\partial d}{\partial\lambda}\right)^{2}Var(\hat{\lambda})+2\left(\frac{\partial td}{\partial\beta}\right)\left(\frac{\partial d}{\partial\lambda}\right)cov(\hat{\beta},\hat{\lambda})$$

Calculate $\hat{d}$ from:

$$\hat{d}=\left[t^{\hat{\beta}}-\frac{\ln(R)}{\hat{\lambda}}\right]^{\frac{1}{\hat{\beta}}}-t$$

The variance calculations are done by:

$$\frac{\partial d}{\partial\beta}=\left[\frac{t^{\hat{\beta}}\ln(t)}{(t+\hat{d})^{\hat{\beta}}}-\ln(t+\hat{d})\right]\cdot\frac{t+\hat{d}}{\hat{\beta}}$$

$$\frac{\partial d}{\partial\lambda}=\frac{t^{\hat{\beta}}-(t+\hat{d})^{\hat{\beta}}}{\hat{\lambda}\hat{\beta}(t+\hat{d})^{\hat{\beta}-1}}$$

Crow Bounds

Failure Terminated

Step 1: Calculate $(\hat{R}_{lower},\hat{R}_{upper})=(R^{\frac{1}{p_1}},R^{\frac{1}{p_2}})$.

Step 2: Let $R=\hat{R}_{lower}$ and solve for d_1 such that:

$$d_1=\left(t^{\hat{\beta}}-\frac{\ln(R_{lower})}{\hat{\lambda}}\right)^{\frac{1}{\hat{\beta}}}-t$$

Step 3: Let $R = \hat{R}_{upper}$ and solve for d_2 such that:

$$d_2 = \left(t^{\hat{\beta}} - \frac{\ln(R_{upper})}{\hat{\lambda}}\right)^{\frac{1}{\hat{\beta}}} - t$$

Step 4: If $d_1 < d_2$. then $d_{lower} = d_1$ and $d_{upper} = d_2$. If $d_1 > d_2$. then $d_{lower} = d_2$ and $d_{upper} = d_1$.

Time Terminated

Step 1: Calculate $(\hat{R}_{lower}, \hat{R}_{upper}) = (R^{\frac{1}{\Pi_1}}, R^{\frac{1}{\Pi_2}})$.

Step 2: Let $R = \hat{R}_{lower}$ and solve for d_1 using the same equation given for the failure terminated data.

Step 3: Let $R = \hat{R}_{upper}$ and solve for d_2 using the same equation given for the failure terminated data.

Step 4: If $d_1 < d_2$. then $d_{lower} = d_1$ and $d_{upper} = d_2$. If $d_1 > d_2$. then $d_{lower} = d_2$ and $d_{upper} = d_1$.

Reliability Growth Analysis Glossary

- **"A" Mode:** A failure mode for which a corrective action will not be implemented.
- **Actual Growth Potential Factor:** The failure intensity of the M modes (total number of distinct unfixed BD modes) after corrective actions have been implemented for them, using the actual values for the effectiveness factors. This metric is used in the Crow Extended - Continuous Evaluation model.
- **Actual Growth Potential Failure Intensity:** The minimum attainable failure intensity based on the current management strategy. This metric is used in the Crow Extended - Continuous Evaluation model.
- **Actual Growth Potential MTBF:** The maximum attainable MTBF based on the current management strategy. This metric is used in the Crow Extended - Continuous Evaluation model.
- **Actual Idealized Growth Curve:** The reliability growth planning curve for the Crow extended model that takes into account the fix delay.

- **Actual Projected Failure Intensity:** The projected failure intensity based on the current management strategy. This metric is used in the Crow Extended - Continuous Evaluation model.
- **Actual Projected MTBF:** The projected MTBF based on the current management strategy.
- **Allowable Failures:** Maximum number of failures that can occur during the demonstration test and still pass the test. Used in the design of reliability tests for repairable systems.
- **AMSAA:** Army Materiel Systems Analysis Activity.
- **Average Actual EF:** An average of the effectiveness factors (EF) for the BD modes that takes into account the point at which the corrective action will be implemented. If the fix is going to be implemented at a later time or during another phase, then the EF value for the mode at the current analysis point is set to zero. If the delayed fix is going to be implemented at the current analysis point, then the EF value is set to the specified value for the mode. This metric is used in the Crow Extended - Continuous Evaluation model.
- **Average Fix Delay:** The average test time required to incorporate corrective actions into the configuration. A fix delay can be specified for each phase of the reliability growth plan.
- **Average Nominal EF:** An average of the effectiveness factors (EF) for the BD modes assuming all fixes for the seen BD modes will be implemented.
- **"BC" Mode:** For the Crow Extended model, it is a failure mode for which a corrective action will be implemented during the test. For the Crow Extended - Continuous Evaluation model, it is a failure mode for which a corrective action will be implemented at the time of failure.
- **"BD" Mode:** For the Crow Extended model, it is a failure mode for which a corrective action will be delayed until the end of the test. For the Crow Extended - Continuous Evaluation model, a failure mode for which a corrective action will be implemented at some point in time after the failure time.
- **BD Mode Failure Intensity:** See Discovery Rate Failure Intensity.
- **Beta:** The shape parameter for the Crow-AMSAA (NHPP) and Crow Extended models. Beta must be greater than zero.
- **Chi-Squared GOF Test:** A goodness-of-fit test that evaluates the hypothesis that the data follows a non-homogeneous Poisson process (NHPP). This goodness-of-fit test is used in grouped data types.

- **Common Beta Hypothesis Test:** Tests the hypothesis that all systems in the data set have similar values of beta.
- **Cramer-von Mises:** A goodness-of-fit test that evaluates the hypothesis that the data follows a non-homogeneous Poisson process (NHPP). This goodness-of-fit test is used for non-grouped data types.
- **Discovery Rate Beta:** Indicates whether the interarrival times between unique BD modes are getting larger or smaller. In most cases, we want this value to be less than 1 assuming that most failures will be identified early on and that their inter-arrival times will become larger as the test progresses.
- **Discovery Rate Failure Intensity:** The failure intensity of the unseen BD modes. This is represented by $h(t)$.
- **Discovery Rate MTBF:** The rate at which the next new unique BD mode will be observed. This is represented by the inverse of $h(t)$.
- **Demonstrated/Achieved Failure Intensity:** The Instantaneous failure intensity at the termination time.
- **Demonstrated/Achieved MTBF:** The Instantaneous MTBF at the termination time.
- **Effectiveness Factor:** The portion of the BD mode's failure intensity that is expected to be removed based on the planned corrective action. This applies only to delayed fixes.
- **Failure Intensity:** The probability of failure within the next $\Delta(t)$ given that the system may or may not have failed.
- **Failure Intensity Modes Seen:** The failure intensity for the failure modes that have been seen during the test. This is calculated by subtracting the failure intensity of the modes that have not been seen from the total system failure intensity.
- **Failure Truncated:** Indicates that the test was stopped at a failure time. A test can either be failure truncated or time truncated.
- **Goal MTBF:** The MTBF requirement when determining a reliability growth plan.
- **Growth Potential Design Margin (GPDM):** A safety margin when setting target MTBF values for the reliability growth plan. It is common for systems to degrade in terms of reliability when a prototype product is going into full manufacturing. The degradation is due to variations in material, processes, etc. Furthermore, the in-house reliability growth testing usually overestimates the actual product reliability because the field usage conditions may not be perfectly simulated during growth testing. Typical values for the GPDM are around

1.2. Higher values yield less risk for the program, but require a more rigorous reliability growth test plan. Lower values imply higher program risk, with less "safety margin."

- **Growth Potential Failure Intensity:** Minimum failure intensity that can be attained with the current management strategy. The minimum failure intensity will be attained when all unique BD modes have been observed and fixed. This metric is used in the Crow Extended model.
- **Growth Potential MTBF:** Maximum MTBF that can be attained with the current management strategy. The maximum MTBF will be attained when all unique BD modes have been observed and fixed.
- **Growth Rate:** Represents the rate at which the MTBF of the system is increasing. The growth rate for the Crow-AMSAA (NHPP) model is equal to $1-\beta$ and is therefore a value greater than or equal to zero,but less than 1. For the Duane Model, the growth rate (α) is the negative of the slope of the line on the cumulative failure intensity vs. time plot. A growth rate that is equal to 1 implies an infinite MTBF growth.
- **Homogeneous Poisson Process (HPP):** A homogeneous process is equivalent to the widely used Poisson distribution. In this scenario, the system's failure intensity is not affected by age. Therefore, the expected number of failures associated with the system is accumulated at a constant rate. It is a special case of the Crow-AMSAA (NHPP) model when $\beta = 1$.
- **Initial Failure Intensity:** The failure intensity inherent to the system prior to any testing.
- **Initial MTBF:** The MTBF inherent to the system prior to any testing.
- **Integrated Reliability Growth Testing (IRGT):** A process where reliability growth testing is conducted as part of existing testing procedures. This is less expensive than conducting a formal reliability growth test. IRGT is usually implemented at the same time as the basic reliability tasks. The test conditions under IRGT are usually less than the actual customer use conditions.
- **Lambda:** The scale parameter for the Crow-AMSAA (NHPP) and Crow Extended models. Lambda must be greater than zero.
- **Laplace Trend Test:** Tests the hypothesis that a trend does not exist within the data. The Laplace trend test can determine if the system is deteriorating, improving or if a trend does not exist.
- **Likelihood Value**: The log-likelihood value returned by the likelihood function given the estimated parameters. This value is returned when maximum likelihood estimation (MLE) is used to estimate the parameters.

- **Management Strategy Ratio:** Defines the portion of the system's failure intensity that will be addressed by corrective actions. A management strategy ratio equal to 1 indicates a perfect management strategy. In this case, a corrective action will be implemented for every mode that is seen. A typical value is 0.95, which indicates that 5% of the system's failure intensity will not be addressed (i.e., 5% will be "A" Modes).
- **Maturity Factor:** The ratio of the initial MTBF to the final MTBF.
- **MIL-HDBK-189:** Military handbook for reliability growth management.
- **Minimal Repair:** This is a situation where the repair of a failed system is just enough to get the system operational again (i.e., no renewal).
- **MTBF Modes Seen:** The inverse of the failure intensity for the modes that have been seen during the test. See Failure Intensity Modes Seen.
- **Nominal Growth Potential Factor:** The failure intensity of the M modes (total number of distinct unfixed BD modes) after corrective actions have been implemented for them, using the nominal values for the effectiveness factors. This metric is used in the Crow Extended - Continuous Evaluation model.
- **Nominal Growth Potential Failure Intensity:** The minimum attainable failure intensity if all delayed corrective actions are implemented for the modes that have been seen, and delayed corrective actions are also implemented for the unseen BD modes, assuming testing would continue until all unseen BD modes are revealed. This metric is used in the Crow Extended - Continuous Evaluation model.
- **Nominal Growth Potential MTBF:** The maximum attainable MTBF if all delayed corrective actions are implemented for the modes that have been seen, and delayed corrective actions are also implemented for the unseen BD modes, assuming testing would continue until all unseen BD modes are revealed. This metric is used in the Crow Extended - Continuous Evaluation model.
- **Nominal Idealized Growth Curve:** The reliability growth planning curve for the Crow Extended model that does not take into account the fix delay. The nominal curve assumes that all fixes are implemented instantaneously. Reliability growth is realized instantaneously on this curve since there is not a delay associated with the implemented fixes.
- **Nominal Projected Failure Intensity:** The projected failure intensity assuming all delayed fixes for the modes that have been seen are implemented.

- **Nominal Projected MTBF:** The projected MTBF assuming all delayed fixes for the modes that have been seen are implemented. This metric is used in the Crow Extended - Continuous Evaluation model.

- **Non-Homogeneous Poisson Process (NHPP):** The NHPP is generally used to model the reliability of a repairable system. An NHPP is an extension of the HPP process. The NHPP allows for the system failure intensity to change with system age. Therefore, the expected number of failures associated with the system is not accumulated at a constant rate.

- **p ratio:** The probability of not incorporating a corrective action by time T.

- **Planned Growth:** MTBF or failure intensity specified during a phase for the Crow Extended model for reliability growth planning.

- **Projected Failure Intensity:** The estimated failure intensity that will be reached if the proposed delayed corrective actions are implemented, with the specified effectiveness factors. This metric is used in the Crow Extended model.

- **Projected MTBF:** The estimated MTBF that will be reached if the proposed delayed corrective actions are implemented with the specified effectiveness factors. This metric is used in the Crow Extended model.

- **Reliability Growth:** The positive improvement in a parameter or metric over a period of time due to changes in the system's design or manufacturing process.

- **Statistical Test for Effectiveness of Corrective Actions:** A statistical test that can determine if the average failure intensity for phase 2 is statistically less than the average failure intensity for phase 1. This test can also determine if the average failure intensity for phase 2 is statistically less than the demonstrated failure intensity at the end of phase 1. This test is used with multi-phase data sets.

- **Stochastic Process:** A sequence of interdependent random events.

- **Termination Time:** In developmental testing, this is equal to the total accumulated test time of all systems. For repairable systems, this is equal to the age of the oldest system.

- **Test-Fix-Test:** A testing procedure where all corrective actions are implemented during the test.

- **Test-Find-Test:** A testing procedure where all corrective actions are delayed until the end of the test.

- **Test-Fix-Find-Test:** A testing procedure where some corrective actions are implemented during the test, while others are delayed until the end of the test.

- **Time Truncated:** Indicates that the test was stopped after a specific amount of test time. A test can either be failure truncated or time truncated.

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