Reference

Accelerated Life Testing Data Analysis

ReliaSoft

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Introduction to Accelerated Life Testing

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What is Accelerated Life Testing?

Traditional life data analysis involves analyzing times-to-failure data obtained under normal operating conditions in order to quantify the life characteristics of a product, system or component. For many reasons, obtaining such life data (or times-to-failure data) may be very difficult or impossible. The reasons for this difficulty can include the long life times of today's products, the small time period between design and release, and the challenge of testing products that are used continuously under normal conditions. Given these difficulties and the need to observe failures of products to better understand their failure modes and life characteristics, reliability practitioners have attempted to devise methods to force these products to fail more quickly than they would under normal use conditions. In other words, they have attempted to accelerate their failures. Over the years, the phrase *accelerated life testing* has been used to describe all such practices.

As we use the phrase in this reference, accelerated life testing involves the acceleration of failures with the single purpose of quantifying the life characteristics of the product at normal use conditions. More specifically, accelerated life testing can be divided into two areas: *qualitative accelerated testing* and *quantitative accelerated life testing*. In qualitative accelerated testing, the engineer is mostly interested in identifying failures and failure modes without attempting to make any predictions as to the product's life under normal use conditions. In quantitative accelerated life testing, the engineer is interested in predicting the life of the product (or more specifically, life characteristics such as MTTF, B(10) life, etc.) at normal use conditions, from data obtained in an accelerated life test.

Qualitative vs. Quantitative Accelerated Tests

Each type of test that has been called an accelerated test provides different information about the product and its failure mechanisms. These tests can be divided into two types: qualitative tests (HALT, HAST, torture tests, shake and bake tests, etc.) and quantitative accelerated life tests. This reference addresses and quantifies the models and procedures associated with quantitative accelerated life tests (QALT).

Qualitative Accelerated Testing

Qualitative tests are tests which yield failure information (or failure modes) only. They have been referred to by many names including:

- Elephant tests
- Torture tests
- HALT
- Shake & bake tests

Qualitative tests are performed on small samples with the specimens subjected to a single severe level of stress, to multiple stresses, or to a time-varying stress (e.g., stress cycling, cold to hot, etc.). If the specimen survives, it passes the test. Otherwise, appropriate actions will be taken to improve the product's design in order to eliminate the cause(s) of failure. Qualitative tests are used primarily to reveal probable failure modes. However, if not designed properly, they may cause the product to fail due to modes that would never have been encountered in real life. A good qualitative test is one that quickly reveals those failure modes that will occur during the life of the product under normal use conditions. In general, qualitative tests are not designed to yield life data that can be used in subsequent quantitative accelerated life data analysis as described in this reference. In general, qualitative tests do not quantify the life (or reliability) characteristics of the product under normal use conditions, however they provide valuable information as to the types and levels of stresses one may wish to employ during a subsequent quantitative test.

BENEFITS AND DRAWBACKS OF QUALITATIVE TESTS

- **Benefits:**
    - Increase reliability by revealing probable failure modes.
    - Provide valuable feedback in designing quantitative tests, and in many cases are a pre-cursor to a quantitative test.

- **Drawbacks:**
    - Do not quantify the reliability of the product at normal use conditions.

Quantitative Accelerated Life Testing

Quantitative accelerated life testing (QALT), unlike the qualitative testing methods described previously, consists of tests designed to quantify the life characteristics of the product, component or system under normal use conditions, and thereby provide reliability information. Reliability information can include the probability of failure of the product under use conditions, mean life under use conditions, and projected returns and warranty costs. It can also be used to assist in the performance of risk assessments, design comparisons, etc.

Quantitative accelerated life testing can take the form of usage rate acceleration or overstress acceleration. Both accelerated life test methods are described next. Because usage rate acceleration test data can be analyzed with typical life data analysis methods, the overstress acceleration method is the testing method relevant to both ALTA folios in Weibull++ and the remainder of this reference.

Quantitative Accelerated Life Tests

For all life tests, some time-to-failure information (or time-to-an-event) for the product is required since the failure of the product is the event we want to understand. In other words, if we wish to understand, measure and predict any event, we must observe how that event occurs!

Most products, components or systems are expected to perform their functions successfully for long periods of time (often years). Obviously, for a company to remain competitive, the time required to obtain times-to-failure data must be considerably less than the expected life of the product. Two methods of acceleration, usage rate acceleration and overstress acceleration, have been devised to obtain times-to-failure data at an accelerated pace. For products that do not operate continuously, one can accelerate the time it takes to induce/observe failures by continuously testing these products. This is called usage rate acceleration. For products for which usage rate acceleration is impractical, one can apply stress(es) at levels which exceed the levels that a product will encounter under normal use conditions and use the times-to-failure data obtained in this manner to extrapolate to use conditions. This is called overstress acceleration.

Usage Rate Acceleration

For products which do not operate continuously under normal conditions, if the test units are operated continuously, failures are encountered earlier than if the units were tested at normal usage. For example, a microwave oven operates for small periods of time every day. One can accelerate a test on microwave ovens by operating them more frequently until failure. The same could be said of washers. If we assume an average washer use of 6 hours a week, one could conceivably reduce the testing time 28-fold by testing these washers continuously.

Data obtained through usage acceleration can be analyzed with the same methods used to analyze regular times-to-failure data. These typical life data analysis techniques are thoroughly described in ReliaSoft's Life Data Analysis Reference.

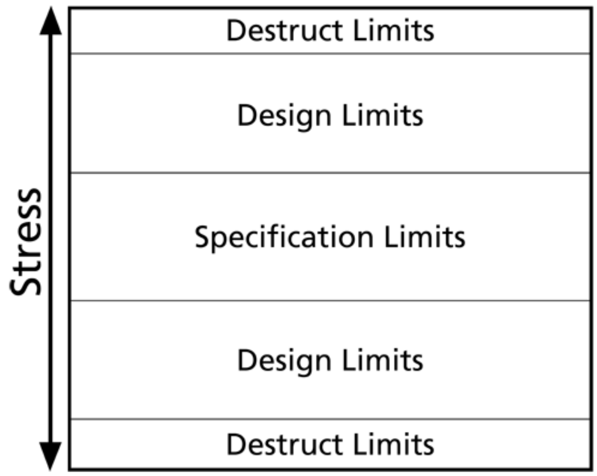
The limitation of usage rate acceleration arises when products, such as computer servers and peripherals, maintain a very high or even continuous usage. In such cases, usage acceleration, even though desirable, is not a feasible alternative. In these cases the practitioner must stimulate the product to fail, usually through the application of stress(es). This method of accelerated life testing is called overstress acceleration and is described next.

Overstress Acceleration

For products with very high or continuous usage, the accelerated life testing practitioner must stimulate the product to fail in a life test. This is accomplished by applying stress(es) that exceed the stress(es) that a product will encounter under normal use conditions. The times-to-failure data obtained under these conditions are then used to extrapolate to use conditions. Accelerated life tests can be performed at high or low temperature, humidity, voltage, pressure, vibration, etc. in order to accelerate or stimulate the failure mechanisms. They can also be performed at a combination of these stresses.

Stresses & Stress Levels

Accelerated life test stresses and stress levels should be chosen so that they accelerate the failure modes under consideration but do not introduce failure modes that would never occur under use conditions. Normally, these stress levels will fall outside the product specification limits but inside the design limits as illustrated next:



This choice of stresses/stress levels and of the process of setting up the experiment is extremely important. Consult your design engineer(s) and material scientist(s) to determine what stimuli (stresses) are appropriate as well as to identify the appropriate limits (or stress levels). If these stresses or limits are unknown, qualitative tests should be performed in order to ascertain the appropriate stress(es) and stress levels. Proper use of design of experiments (DOE) methodology is also crucial at this step. In addition to proper stress selection, the application of the stresses must be accomplished in some logical, controlled and quantifiable fashion. Accurate data on the stresses applied, as well as the observed behavior of the test specimens, must be maintained.

Clearly, as the stress used in an accelerated test becomes higher, the required test duration decreases (because failures will occur more quickly). However, as the stress level moves farther away from the use conditions, the uncertainty in the extrapolation increases. Confidence intervals provide a measure of this uncertainty in extrapolation. (Confidence Intervals are presented in Appendix A).

Understanding Quantitative Accelerated Life Data Analysis

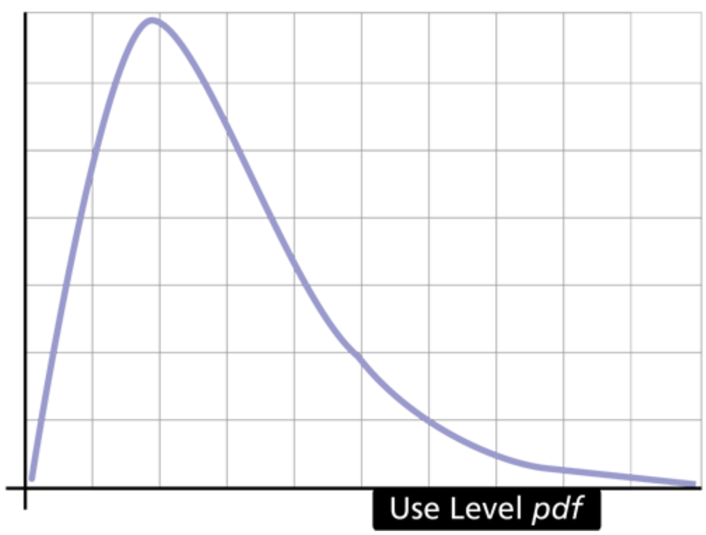
In typical life data analysis one determines, through the use of statistical distributions, a life distribution that describes the times-to-failure of a product. Statistically speaking, one wishes to determine the use level probability density function, or *pdf*, of the times-to-failure. Appendix A of this reference presents these statistical concepts and provides a basic statistical background as it applies to life data analysis.

Once this *pdf* has been obtained, all other desired reliability results can be easily determined, including:

- Percentage failing under warranty.
- Risk assessment.
- Design comparison.
- Wear-out period (product performance degradation).

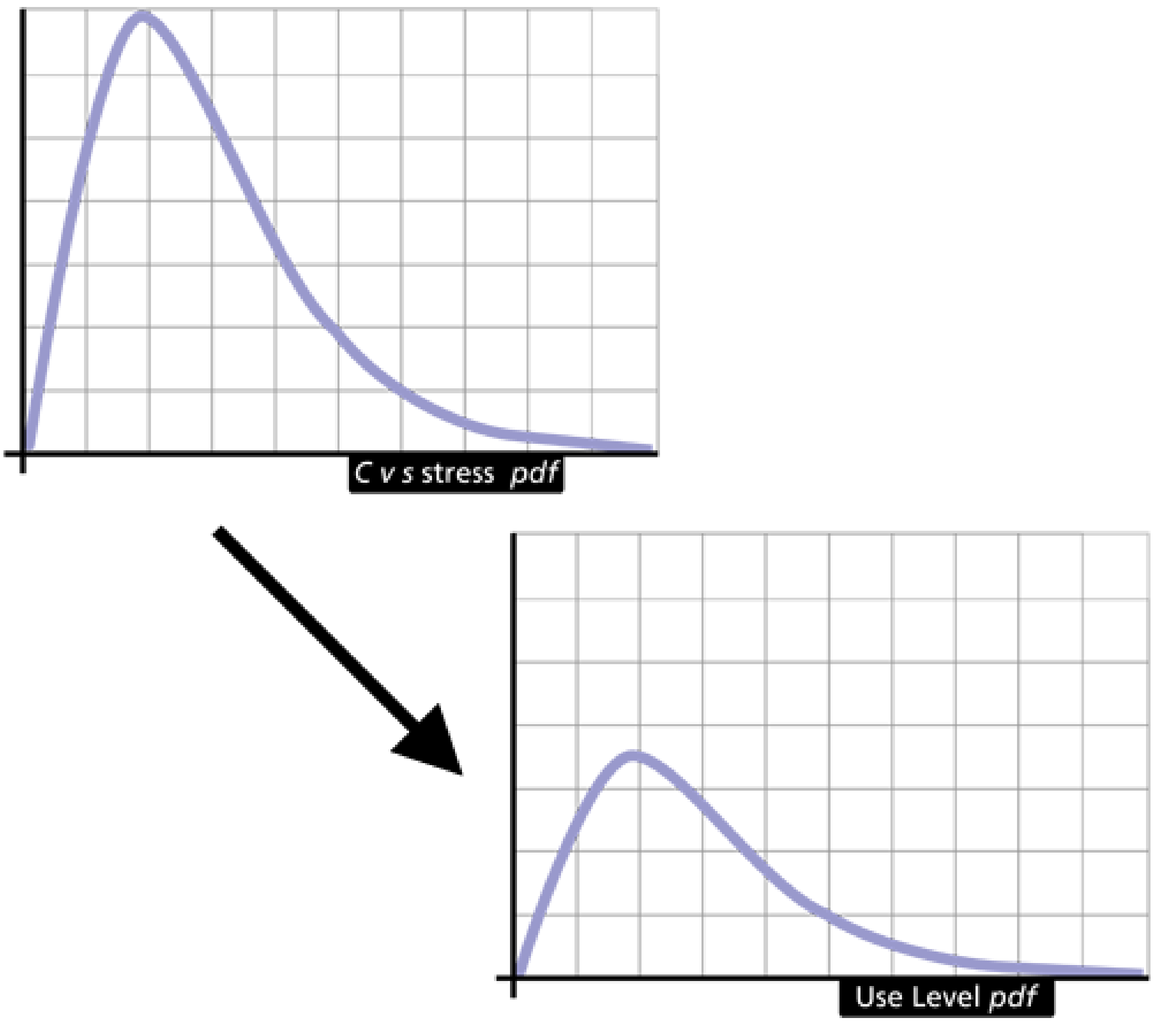
In typical life data analysis, this use level probability density function, or *pdf*, of the times-to-failure can be easily determined using regular times-to-failure/suspension data and an underlying distribution such as the Weibull, exponential or lognormal distribution. These lifetime distributions are presented in greater detail in the Distributions Used in Accelerated Testing chapter of this reference.

In accelerated life data analysis, however, we face the challenge of determining the use level *pdf* from accelerated life test data, rather than from times-to-failure data obtained under use conditions. To accomplish this, we must develop a method that allows us to extrapolate from data collected at accelerated conditions to arrive at an estimation of use level characteristics.

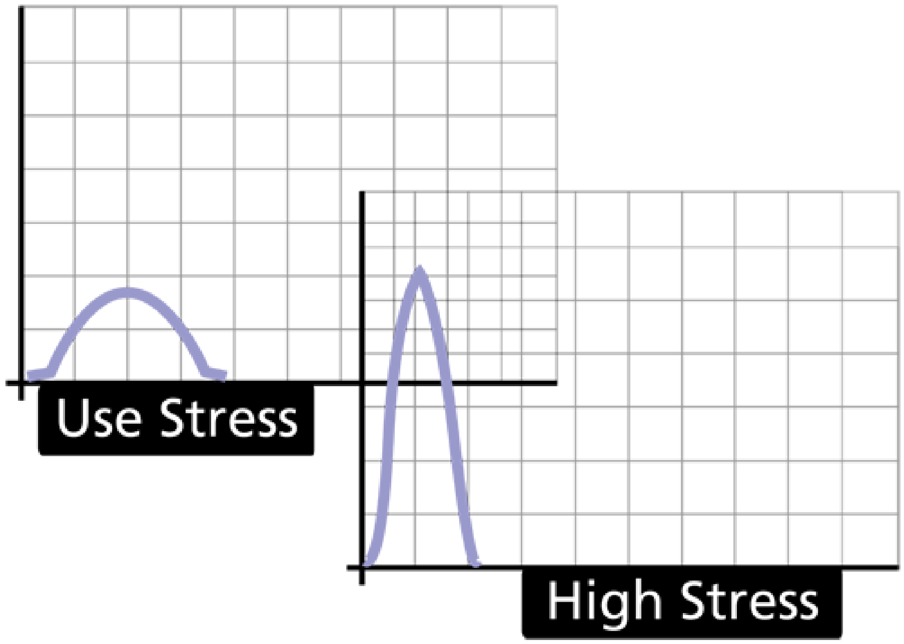


Looking at a Single Constant Stress Accelerated Life Test

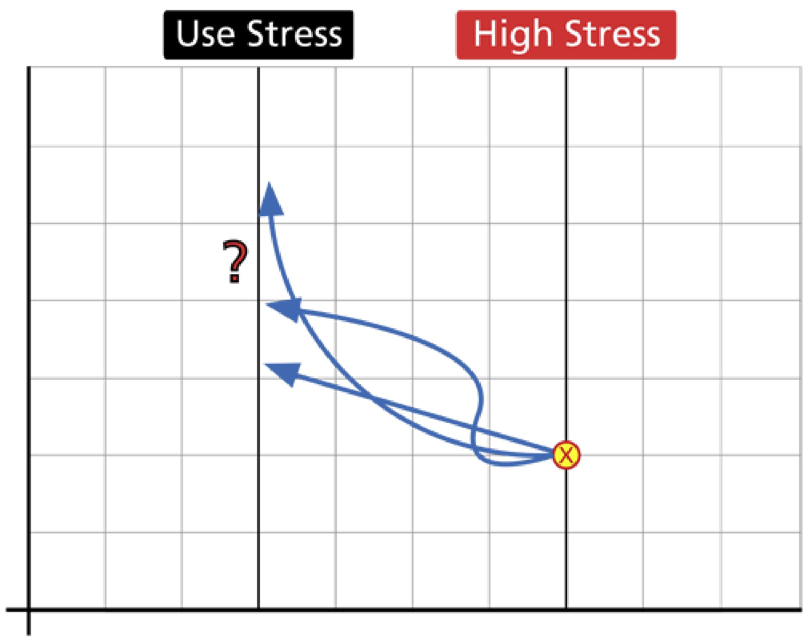
To understand the process involved with extrapolating from overstress test data to use level conditions, let's look closely at a simple accelerated life test. For simplicity we will assume that the product was tested under a single stress at a single constant stress level. We will further assume that times-to-failure data have been obtained at this stress level. The times-to-failure at this stress level can then be easily analyzed using an underlying life distribution. A *pdf* of the times-to-failure of the product can be obtained at that single stress level using traditional approaches. This *pdf*, the overstress *pdf*, can likewise be used to make predictions and estimates of life measures of interest at that particular stress level. The objective in an accelerated life test, however, is not to obtain predictions and estimates at the particular elevated stress level at which the units were tested, but to obtain these measures at another stress level, the use stress level.



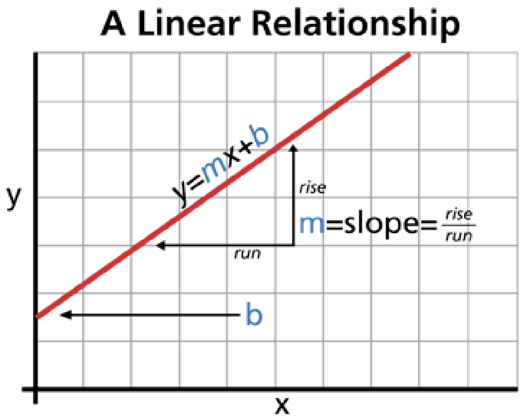
To accomplish this objective, we must devise a method to traverse the path from the overstress *pdf* to extrapolate a use level *pdf*. The next figure illustrates a typical behavior of the *pdf* at the high stress (or overstress level) and the *pdf* at the use stress level.

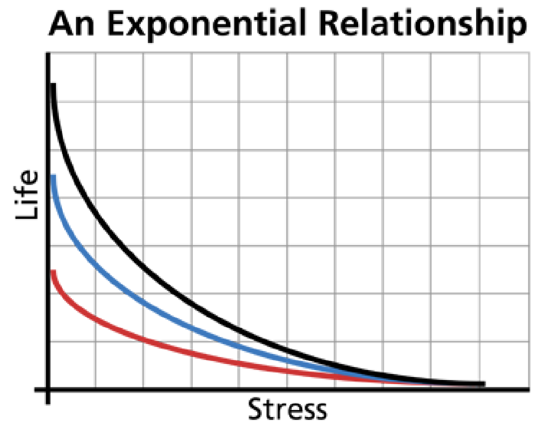


To further simplify the scenario, let's assume that the *pdf* for the product at any stress level can be described by a single point. The next figure illustrates such a simplification where we need to determine a way to project (or map) this single point from the high stress to the use stress.

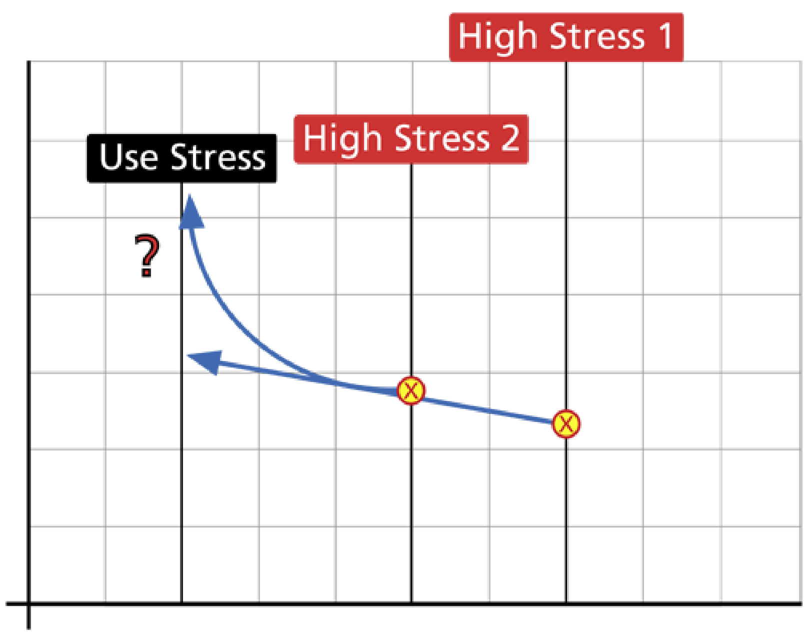


Obviously, there are infinite ways to map a particular point from the high stress level to the use stress level. We will assume that there is some model (or a function) that maps our point from the high stress level to the use stress level. This model or function can be described mathematically and can be as simple as the equation for a line. The next figure demonstrates some simple models or relationships.





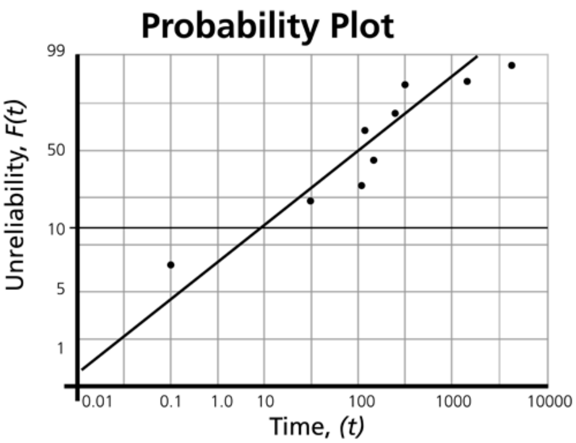
Even when a model is assumed (e.g., linear, exponential, etc.), the mapping possibilities are still infinite since they depend on the parameters of the chosen model or relationship. For example, a simple linear model would generate different mappings for each slope value because we can draw an infinite number of lines through a point. If we tested specimens of our product at two different stress levels, we could begin to fit the model to the data. Clearly, the more points we have, the better off we are in correctly mapping this particular point or fitting the model to our data.

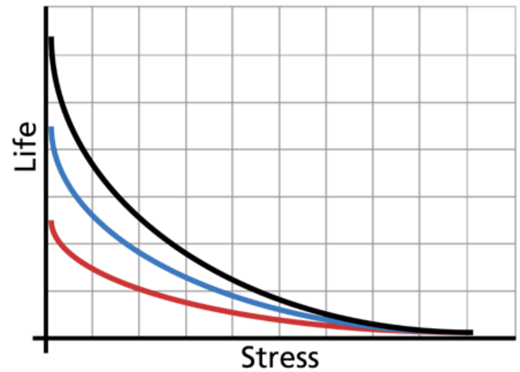


The above figure illustrates that you need a minimum of two higher stress levels to properly map the function to a use stress level.

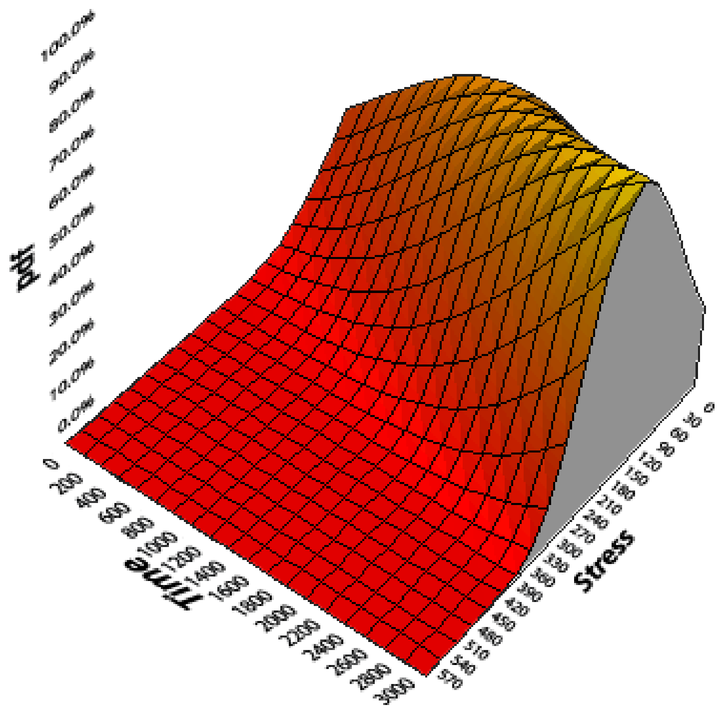
Life Distributions and Life-Stress Models

The analysis of accelerated life test data consists of (1) an underlying life distribution that describes the product at different stress levels and (2) a life-stress relationship (or model) that quantifies the manner in which the life distribution changes across different stress levels. These elements of analysis are graphically shown next:





The combination of both an underlying life distribution and a life-stress model can be best seen in the next figure where a *pdf* is plotted against both time and stress.



The assumed underlying life distribution can be any life distribution. The most commonly used life distributions include the Weibull, exponential and lognormal distribution. Along with the life distribution, a life-stress relationship is also used. These life-stress relationships have been empirically derived and fitted to data. An overview of some of these life-stress relationships is presented in the Analysis Method subchapter.

Analysis Method

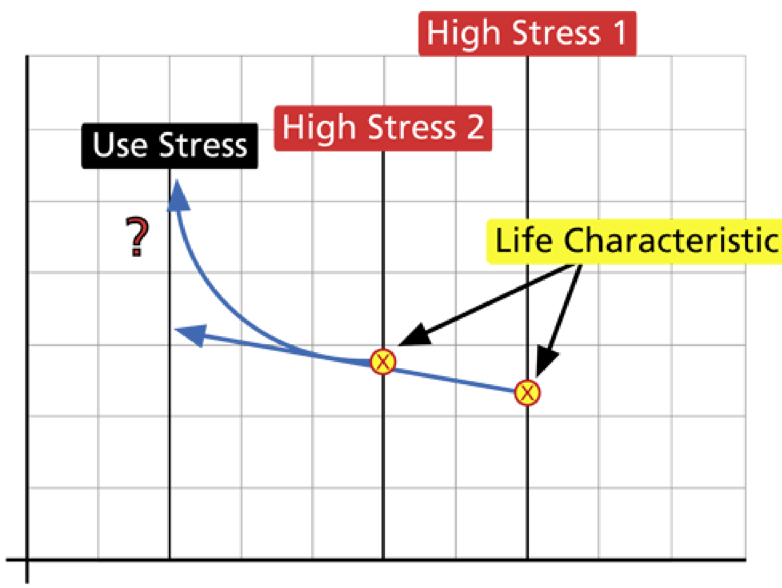
With our current understanding of the principles behind accelerated life testing analysis, we will continue with a discussion of the steps involved in analyzing life data collected from accelerated life tests like those described in the Quantitative Accelerated Life Tests section.

Select a Life Distribution

The first step in performing an accelerated life data analysis is to choose an appropriate life distribution. Although it is rarely appropriate, the exponential distribution has in the past been widely used as the underlying life distribution because of its simplicity. The Weibull and lognormal distributions, which require more involved calculations, are more appropriate for most uses. The underlying life distributions available in Weibull++ are presented in detail in the Distributions Used in Accelerated Testing chapter of this reference.

Select a Life-Stress Relationship

After you have selected an underlying life distribution appropriate to your data, the second step is to select (or create) a model that describes a characteristic point or a life characteristic of the distribution from one stress level to another.



The life characteristic can be any life measure such as the mean, median, R(x), F(x), etc. This life characteristic is expressed as a function of stress. Depending on the assumed underlying life distribution, different life characteristics are considered. Typical life characteristics for some distributions are shown in the next table.

Distribution	Parameters	Life Characteristic
Weibull	β^*, η	Scale parameter, η
Exponential	λ	Mean life ($1/\lambda$)
Lognormal	$\bar{T}, \sigma^*$	Median, $\breve{T}$

*Usually assumed constant

For example, when considering the Weibull distribution, the scale parameter, η, is chosen to be the life characteristic that is stress dependent, while β is assumed to remain constant across different stress levels. A life-stress relationship is then assigned to η. Eight common life-stress models are presented later in this reference. Click a topic to go directly to that page.

- Arrhenius Relationship
- Eyring Relationship
- Inverse Power Law Relationship
- Temperature-Humidity Relationship
- Temperature Non-Thermal Relationship
- Multivariable Relationships: General Log-Linear and Proportional Hazards
- Time-Varying Stress Models

Parameter Estimation

Once you have selected an underlying life distribution and life-stress relationship model to fit your accelerated test data, the next step is to select a method by which to perform parameter estimation. Simply put, parameter estimation involves fitting a model to the data and solving for the parameters that describe that model. In our case, the model is a combination of the life distribution and the life-stress relationship (model). The task of parameter estimation can vary from trivial (with ample data, a single constant stress, a simple distribution and simple model) to impossible. Available methods for estimating the parameters of a model include the graphical method, the least squares method and the maximum likelihood estimation method. Parameter estimation methods are presented in detail in Appendix B of this reference. Greater emphasis will be given to the MLE method because it provides a more robust solution, and is the one employed in Weibull++.

Derive Reliability Information

Once the parameters of the underlying life distribution and life-stress relationship have been estimated, a variety of reliability information about the product can be derived such as:

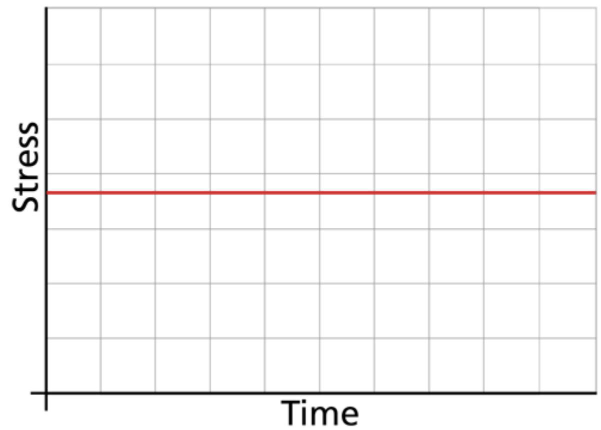
- Warranty time.
- The instantaneous failure rate, which indicates the number of failures occurring per unit time.
- The mean life which provides a measure of the average time of operation to failure.
- B(X) life, which is the time by which X% of the units will fail.
- etc.

Stress Loading

The discussion of accelerated life testing analysis thus far has included the assumption that the stress loads applied to units in an accelerated test have been constant with respect to time. In real life, however, different types of loads can be considered when performing an accelerated test. Accelerated life tests can be classified as constant stress, step stress, cycling stress, random stress, etc. These types of loads are classified according to the dependency of the stress with respect to time. There are two possible stress loading schemes, loadings in which the stress is time-independent and loadings in which the stress is time-dependent. The mathematical treatment, models and assumptions vary depending on the relationship of stress to time. Both of these loading schemes are described next.

Stress Is Time-Independent (Constant Stress)

When the stress is time-independent, the stress applied to a sample of units does not vary. In other words, if temperature is the thermal stress, each unit is tested under the same accelerated temperature, (e.g., 100° C), and data are recorded. This is the type of stress load that has been discussed so far.



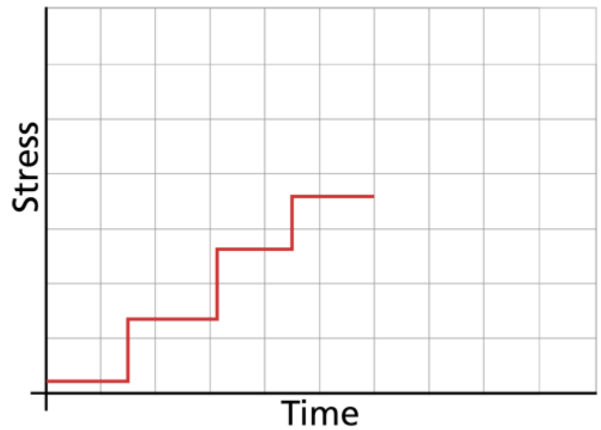
This type of stress loading has many advantages over time-dependent stress loadings. Specifically:

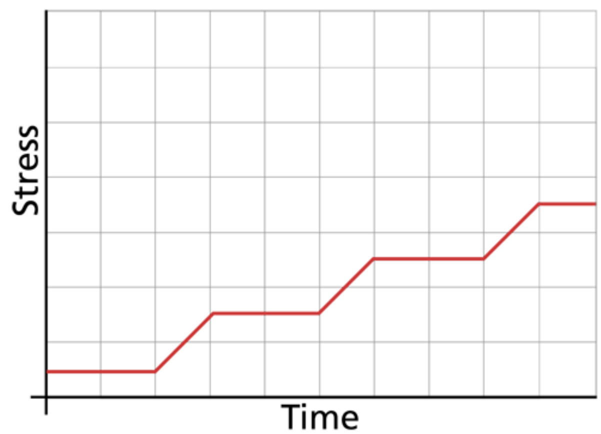
- Most products are assumed to operate at a constant stress under normal use.
- It is far easier to run a constant stress test (e.g., one in which the chamber is maintained at a single temperature).
- It is far easier to quantify a constant stress test.
- Models for data analysis exist, are widely publicized and are empirically verified.
- Extrapolation from a well-executed constant stress test is more accurate than extrapolation from a time-dependent stress test.

Stress Is Time-Dependent

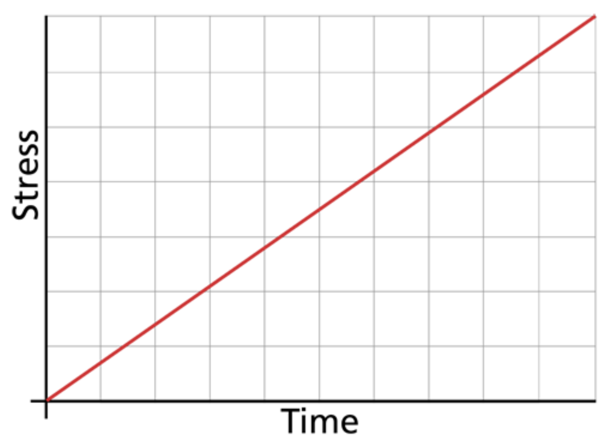
When the stress is time-dependent, the product is subjected to a stress level that varies with time. Products subjected to time-dependent stress loadings will yield failures more quickly, and models that fit them are thought by many to be the "holy grail" of accelerated life testing. The cumulative damage model allows you to analyze data from accelerated life tests with time-dependent stress profiles.

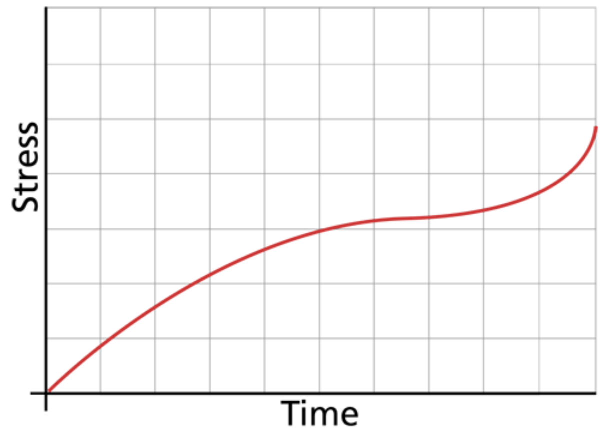
The step-stress model, as discussed in [31], and the related ramp-stress model are typical cases of time-dependent stress tests. In these cases, the stress load remains constant for a period of time and then is stepped/ramped into a different stress level, where it remains constant for another time interval until it is stepped/ramped again. There are numerous variations of this concept.





The same idea can be extended to include a stress as a continuous function of time.





Summary of Accelerated Life Testing Analysis

In summary, accelerated life testing analysis can be conducted on data collected from carefully designed quantitative accelerated life tests. Well-designed accelerated life tests will apply stress (es) at levels that exceed the stress level the product will encounter under normal use conditions in order to accelerate the failure modes that would occur under use conditions. An underlying life distribution (like the exponential, Weibull and lognormal lifetime distributions) can be chosen to fit the life data collected at each stress level to derive overstress *pdf*s for each stress level. A life-stress relationship (Arrhenius, Eyring, etc.) can then be chosen to quantify the path from the overstress *pdf*s in order to extrapolate a use level *pdf*. From the extrapolated use level *pdf*, a variety of functions can be derived, including reliability, failure rate, mean life, warranty time etc.

Accelerated Life Testing and Weibull++

IN THIS CHAPTER

This chapter presents issues relevant to using the Weibull++ software package to analyze data collected in accelerated life tests. These issues include the types of data that can be analyzed and the types of plots that can be created to display analysis results.

Data and Data Types

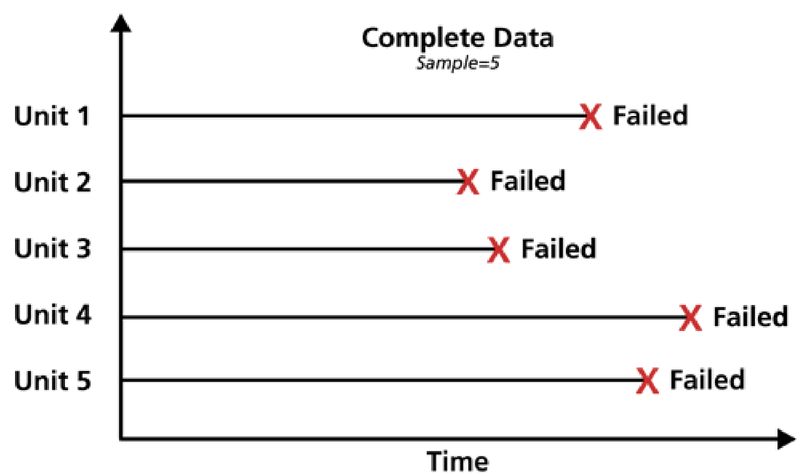
Statistical models rely extensively on data to make predictions. In life data analysis, the models are the *statistical distributions* and the data are the *life data* or *times-to-failure data* of our product. In the case of accelerated life data analysis, the models are the *life-stress relationships* and the data are the *times-to-failure data at a specific stress level*.The accuracy of any prediction is directly proportional to the quality, accuracy and completeness of the supplied data.

Good data, along with the appropriate model choice, usually results in good predictions. Bad or insufficient data will almost always result in bad predictions.

In the analysis of life data, we want to use all available data sets, which sometimes are incomplete or include uncertainty as to when a failure occurred. Life data can therefore be separated into two types: *complete data* (all information is available) or *censored data* (some of the information is missing). Each type is explained next.

Complete Data

Complete data means that the value of each sample unit is observed or known. For example, if we had to compute the average test score for a sample of ten students, complete data would consist of the known score for each student. Likewise in the case of life data analysis, our data set (if complete) would be composed of the times-to-failure of all units in our sample. For example, if we tested five units and they all failed (and their times-to-failure were recorded), we would then have complete information as to the time of each failure in the sample.

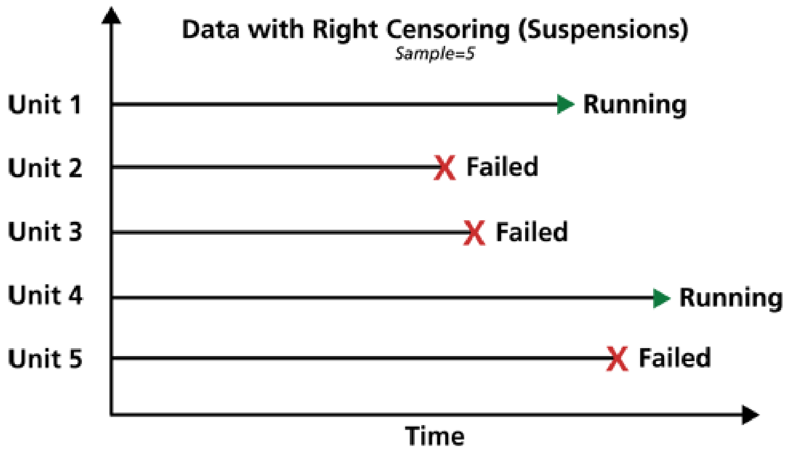


Censored Data

In many cases, all of the units in the sample may not have failed (i.e., the event of interest was not observed) or the exact times-to-failure of all the units are not known. This type of data is commonly called *censored data*. There are three types of possible censoring schemes, right censored (also called suspended data), interval censored and left censored.

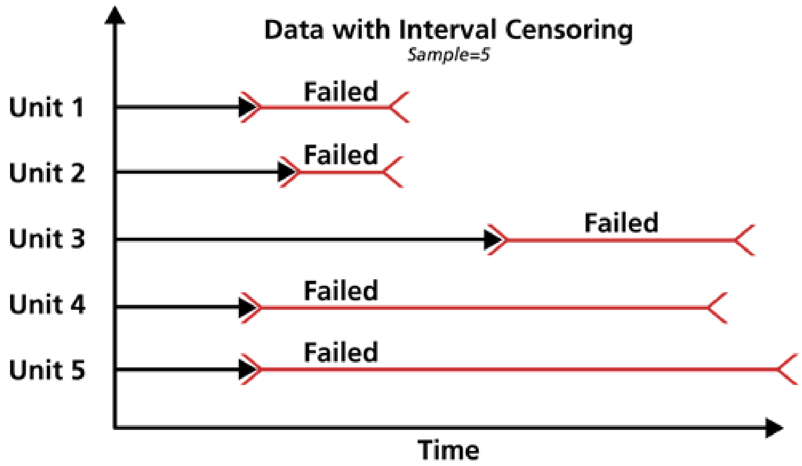
Right Censored (Suspension) Data

The most common case of censoring is what is referred to as *right censored data*, or *suspended data*. In the case of life data, these data sets are composed of units that did not fail. For example, if we tested five units and only three had failed by the end of the test, we would have right censored data (or suspension data) for the two units that did not failed. The term *right censored* implies that the event of interest (i.e., the time-to-failure) is to the right of our data point. In other words, if the units were to keep on operating, the failure would occur at some time after our data point (or to the right on the time scale).



Interval Censored Data

The second type of censoring is commonly called *interval censored data*. Interval censored data reflects uncertainty as to the exact times the units failed within an interval. This type of data frequently comes from tests or situations where the objects of interest are not constantly monitored. For example, if we are running a test on five units and inspecting them every 100 hours, we only know that a unit failed or did not fail between inspections. Specifically, if we inspect a certain unit at 100 hours and find it operating, and then perform another inspection at 200 hours to find that the unit is no longer operating, then the only information we have is that the unit failed at some point in the interval between 100 and 200 hours. This type of censored data is also called *inspection data* by some authors.

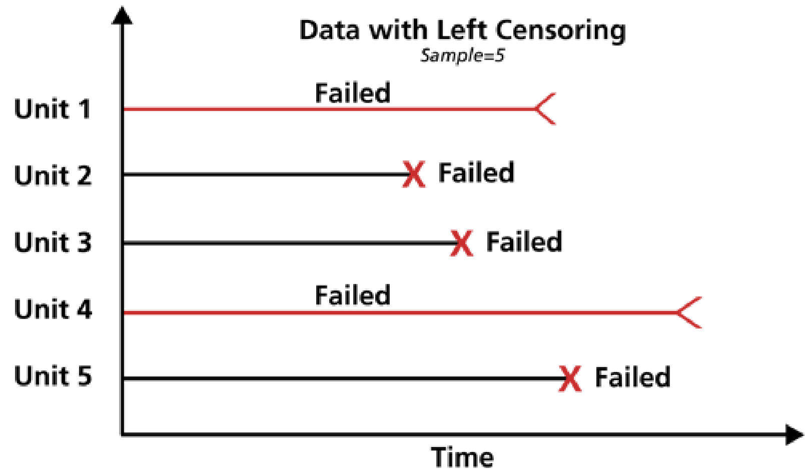


It is generally recommended to avoid interval censored data because they are less informative compared to complete data. However, there are cases when interval data are unavoidable due to the nature of the product, the test and the test equipment. In those cases, caution must be taken to set the inspection intervals to be short enough to observe the spread of the failures. For example, if the inspection interval is too long, all the units in the test may fail within that interval, and thus no failure distribution could be obtained.

In the case of accelerated life tests, the data set affects the accuracy of the fitted life-stress relationship, and subsequently, the extrapolation to Use Stress conditions. In this case, inspection intervals should be chosen according to the expected acceleration factor at each stress level, and therefore these intervals will be of different lengths for each stress level.

Left Censored Data

The third type of censoring is similar to the interval censoring and is called *left censored data*. In left censored data, a failure time is only known to be before a certain time. For instance, we may know that a certain unit failed sometime before 100 hours but not exactly when. In other words, it could have failed any time between 0 and 100 hours. This is identical to *interval censored data* in which the starting time for the interval is zero.



Grouped Data Analysis

In the standard folio, data can be entered individually or in groups. Grouped data analysis is used for tests in which groups of units possess the same time-to-failure or in which groups of units were suspended at the same time. We highly recommend entering redundant data in groups. Grouped data speeds data entry by the user and significantly speeds up the calculations.

A Note about Complete and Suspension Data

Depending on the event that we want to measure, data type classification (i.e., complete or suspension) can be open to interpretation. For example, under certain circumstances, and depending on the question one wishes to answer, a specimen that has failed might be classified as a suspension for analysis purposes. To illustrate this, consider the following times-to-failure data for a product that can fail due to modes A, B and C:

Time-to-Failure, hr	Mode of Failure
105	A
125	B
134	A
167	C
212	C
345	A
457	B
541	C
623	B

If the objective of the analysis is to determine the probability of failure of the product, regardless of the mode responsible for the failure, we would analyze the data with all data entries classified as failures (complete data). However, if the objective of the analysis is to determine the probability of failure of the product due to Mode A only, we would then choose to treat failures due to Modes B or C as suspension (right censored) data. Those data points would be treated as suspension data with respect to Mode A because the product operated until the recorded time without failure due to Mode A.

Fractional Failures

After the completion of a reliability test or after failures are observed in the field, a redesign can be implemented to improve a product's reliability. After the redesign, and before new failure data become available, it is often times desirable to"adjust" the reliability that was calculated from the previous design and take "credit" for this theoretical improvement. This can be achieved with fractional failures. Using past experience to estimate the effectiveness of a corrective action or redesign, an analysis can take credit for this improvement by adjusting the failure count. Therefore, if a corrective action on a failure mode is believed to be 70% effective, then the failure count can be reduced from 1 to 0.3 to reflect the effectiveness of the corrective action.

For example, consider the following data set.

Number in State	State F or S	State End Time (Hr)
1	F	105
0.4	F	168
1	F	220
1	F	290
1	F	410

In this case, a design change has been implemented for the failure mode that occurred at 168 hours and is assumed to be 60% effective. In the background, Weibull++ converts this data set to:

Number in State	State F or S	State End Time (Hr)
1	F	105
0.4	F	168
0.6	S	168

1	F	220
1	F	290
1	F	410

If Rank Regression is used to estimate distribution parameters, the median ranks for the previous data set are calculated as follows:

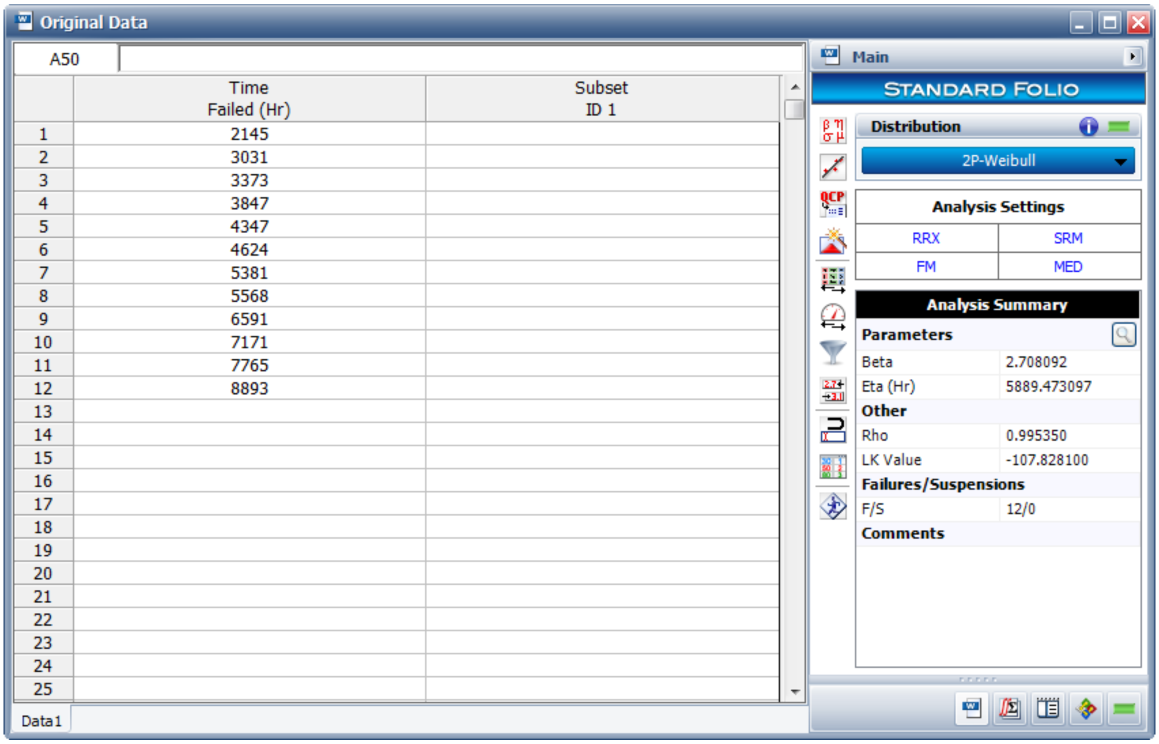
Number in State	State F or S	State End Time (Hr)	MON	Median Rank (%)
1	F	105	1	12.945
0.4	F	168	20.267	
0.6	S	168	-	-
1	F	220	2.55	41.616
1	F	290	3.7	63.039
1	F	410	4.85	84.325

Given this information, the standard Rank Regression procedure is then followed to estimate parameters.

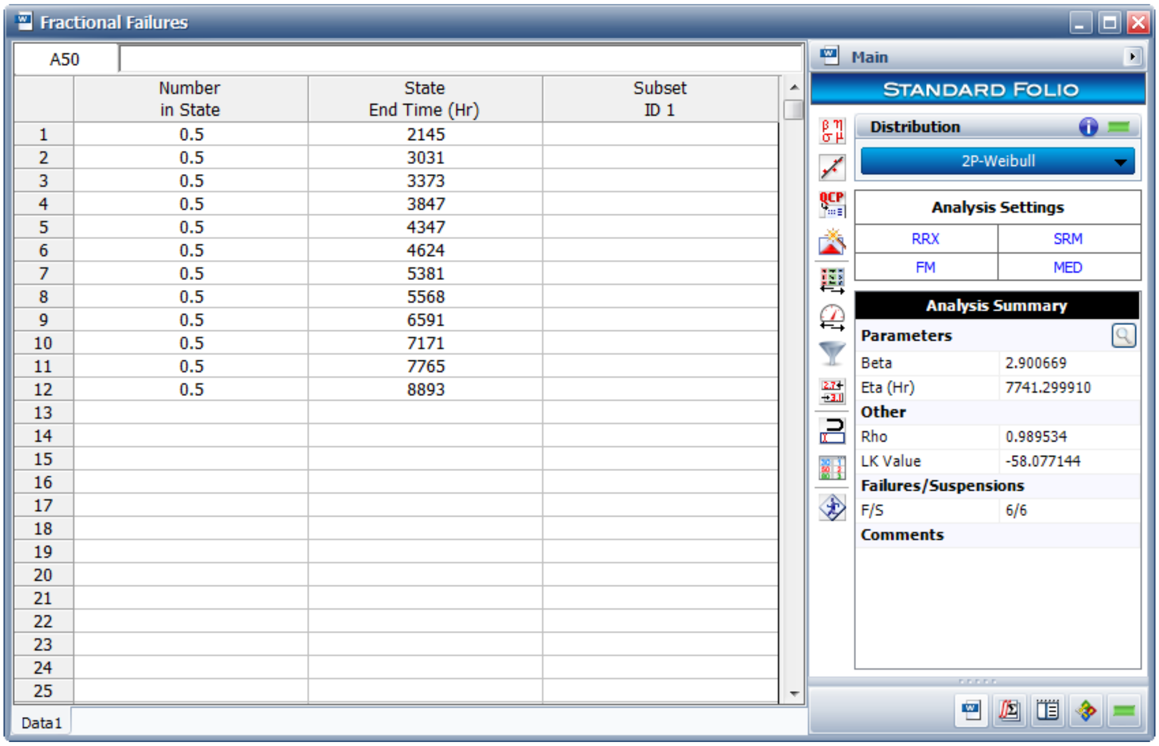
If Maximum Likelihood Estimation (MLE) is used to estimate distribution parameters, the grouped data likelihood function is used with the number in group being a non-integer value.

Example

A component underwent a reliability test. 12 samples were run to failure. The following figure shows the failures and the analysis in a Weibull++ standard folio.

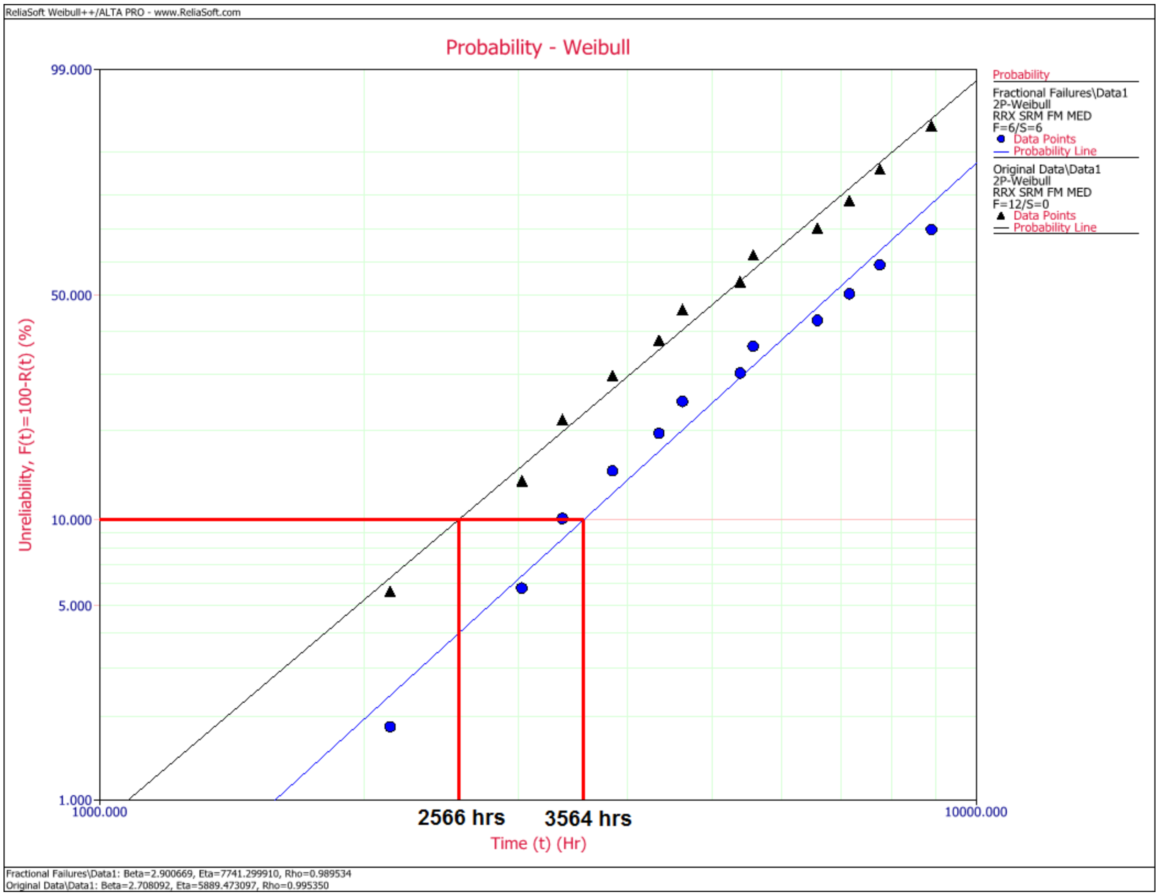


The analysts believe that the planned design improvements will yield 50% effectiveness. To estimate the reliability of the product based on the assumptions about the repair effectiveness, they enter the data in groups, counting a 0.5 failure for each group. The following figure shows the adjusted data set and the calculated parameters.



	Number in State	State End Time (Hr)	Subset ID 1
1	0.5	2145	
2	0.5	3031	
3	0.5	3373	
4	0.5	3847	
5	0.5	4347	
6	0.5	4624	
7	0.5	5381	
8	0.5	5568	
9	0.5	6591	
10	0.5	7171	
11	0.5	7765	
12	0.5	8893	
13			
14			
15			
16			
17			
18			
19			
20			
21			
22			
23			
24			
25			

The following overlay plot of unreliability vs. time shows that by using fractional failures the estimated unreliability of the component has decreased, while the B10 life has increased from 2,566 hours to 3,564 hours.



Plots

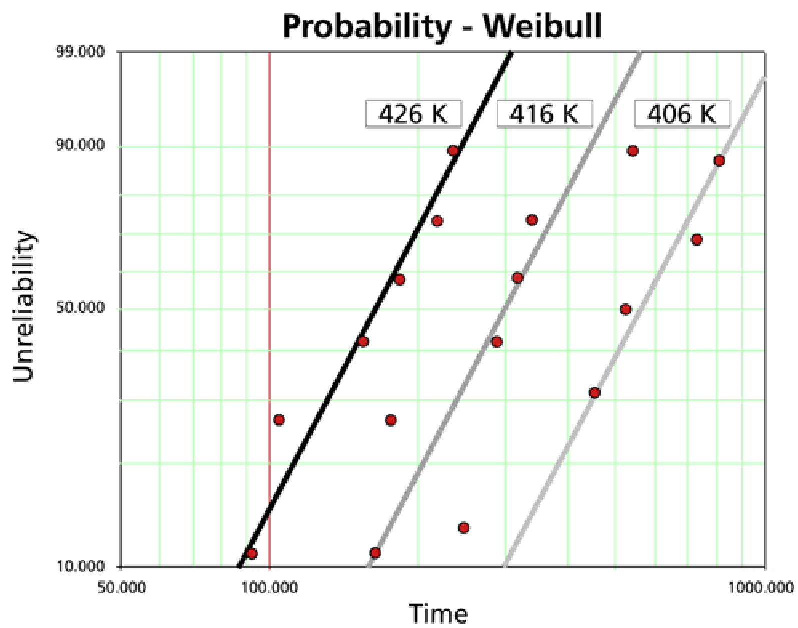
In addition to the probability plots required in life data analysis, accelerated life test data analysis utilizes a variety of stress-related plots. Each plot provides information crucial to performing accelerated life test analyses. The addition of stress dependency into the life equations introduces another dimension into the plots. This generates a whole new family of 3-dimensional (3D) plots. The following table summarizes the types of plots available for ALTA folios in ReliaSoft's Weibull++.

Plot Type	Information Available	Confidence Bounds
Probability	Unreliability vs. Time, Scale and shape parameters at each stress level, on a 2D probability plot	Yes
Use Level Probability	Unreliability vs. Time, Scale and shape parameters at each stress level, on a 2D probability plot	Yes
Reliability vs. Time	Reliability vs. Time (linear) at a user specified stress level in 2D, and across stress levels in 3D	Yes
Unreliability vs. Time	Unreliability vs. Time (linear) at a user specified stress level in 2D, and across stress levels in 3D	Yes
pdf	pdf vs. Time, view Shape, Skewness, Mode at user specified stress level in 2D and across stress levels in 3D	Yes
Failure Rate vs. Time	Instantaneous Failure Rate vs. Time at user specified stress level in 2D and across stress levels in 3D	Yes
Life vs. Stress	Life vs. Stress on a 2D life-stress plot at a user specified stress level	Yes
Standardized Residuals	Probability plot of the Standardized Residuals	No
Cox-Snell Residuals	Probability plot of the Cox-Snell Residuals on exponential paper	No
Standardized vs. Fitted Values	Plot of the Standardized Residuals vs. the scale parameter at each test stress level	No
STD vs. Stress	Standard Deviation vs. Stress on a 2D linear plot at a user specified stress level	No
AF vs. Stress	Acceleration Factor vs. Stress on a 2D linear plot at a user specified stress level	Yes

Considerations relevant to the use of some of the plots available in Weibull++ are discussed in the sections that follow.

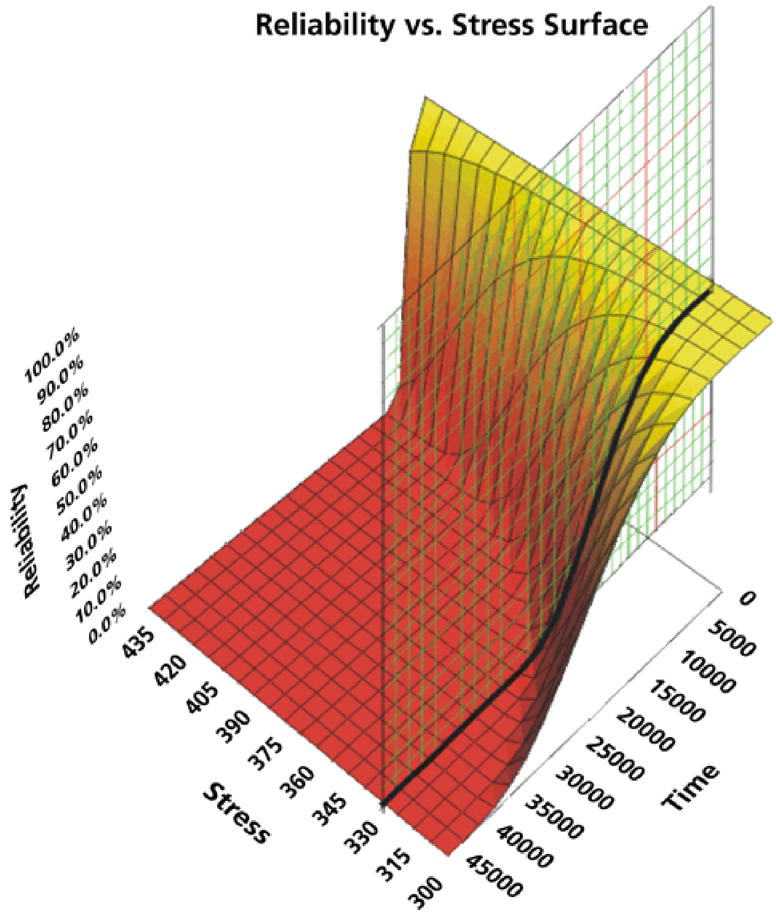
Probability Plots

The probability plots used in accelerated life testing data analysis are similar to those used in life data analysis. The only difference is that each probability plot in accelerated testing is associated with the corresponding stress or stresses. Multiple lines will be plotted on a probability plot in Weibull++, each corresponding to a different stress level. The information that can be obtained from probability plots includes: reliability with confidence bounds, warranty time with confidence bounds, shape and scale parameters, etc.



Reliability and Unreliability Plots

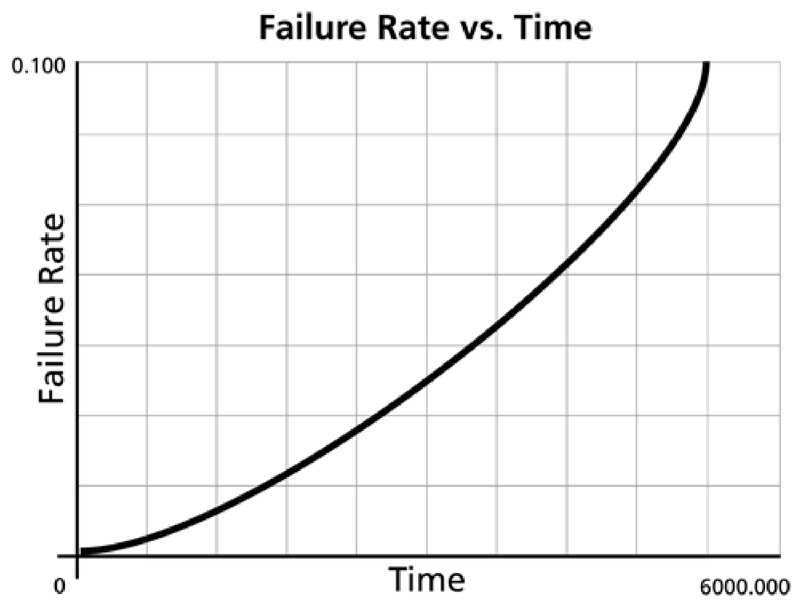
There are two types of reliability plots. The first type is a 2-dimensional plot of Reliability vs. Time for a given stress level. The second type is a 3-dimensional plot of the Reliability vs. Time vs. Stress. The 2-dimensional plot of reliability is just a section of the 3-dimensional plot at the desired stress level, as illustrated in the next figure.

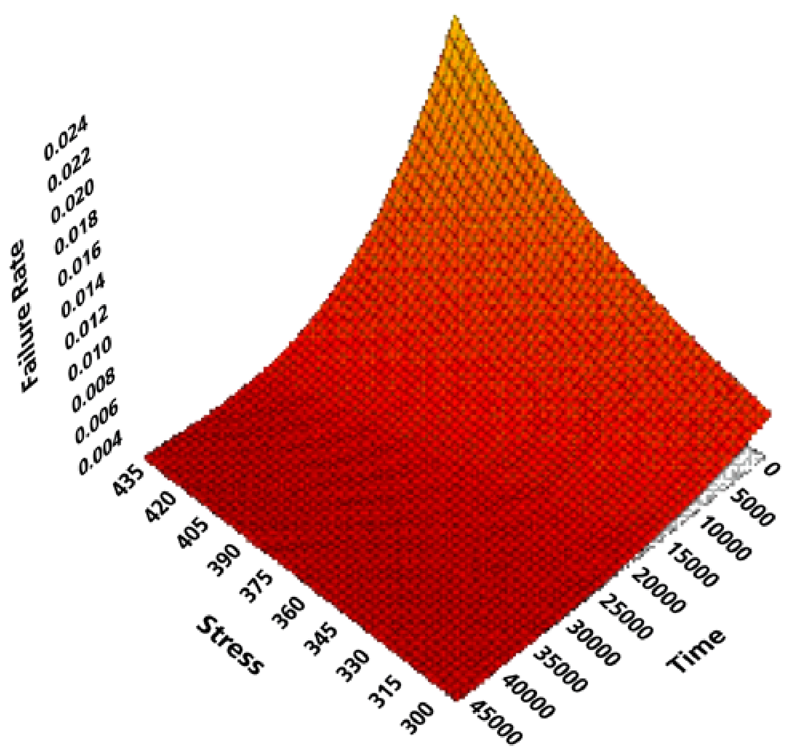


A Reliability vs. Time plot provides reliability values at a given time and time at a given reliability. These can be plotted with or without confidence bounds. The same 2-dimensional and 3-dimensional plots are available for unreliability as well, and they are just the complement of the reliability plots.

Failure Rate Plots

The instantaneous failure rate is a function of time and stress. For this reason, a 2-dimensional plot of Failure Rate vs. Time at a given stress and a 3-dimensional plot of Failure Rate vs. Time and Stress can be obtained in Weibull++.

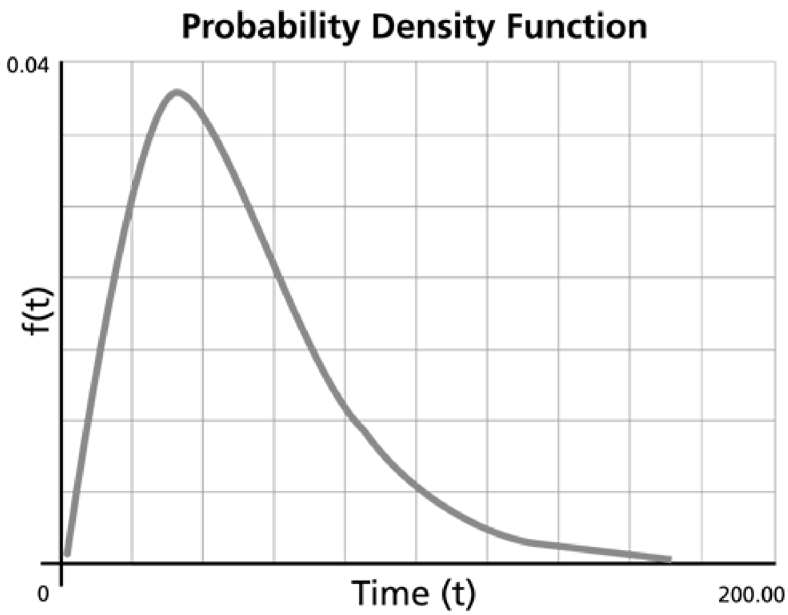


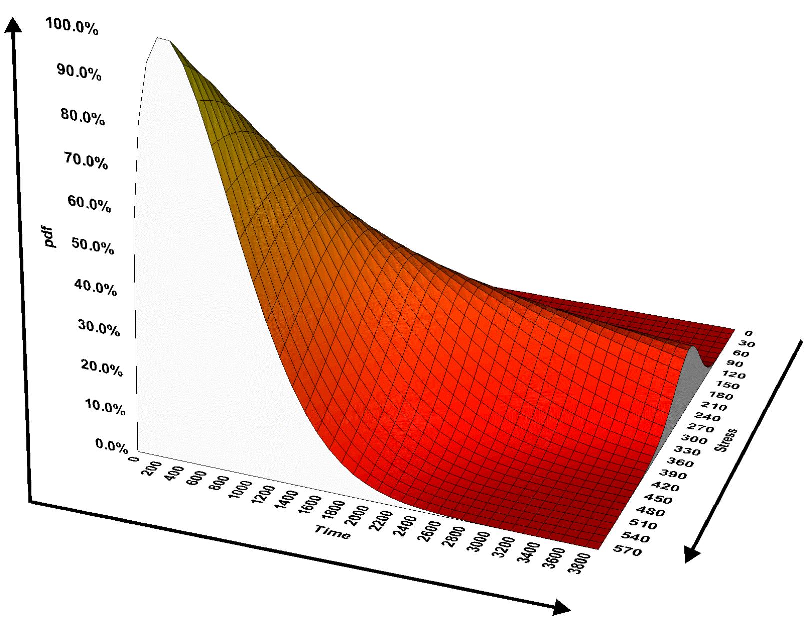


A failure rate plot shows the expected number of failures per unit time at a particular stress level (e.g., failures per hour at 410K).

Pdf Plots

The *pdf* is a function of time and stress. For this reason, a 2-dimensional plot of the *pdf* vs. Time at a given stress and a 3-dimensional plot of the *pdf* vs. Time and Stress can be obtained in Weibull++.



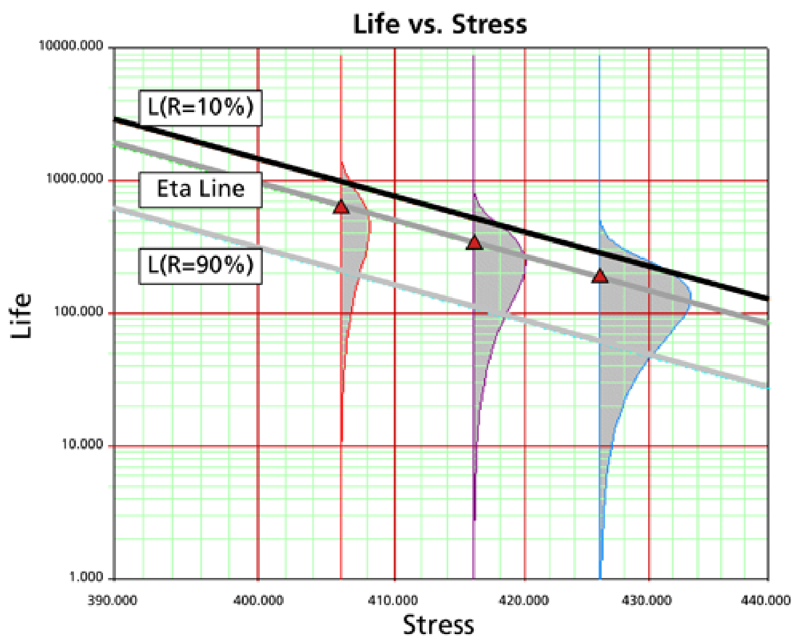


A *pdf* plot represents the relative frequency of failures as a function of time and stress. Although the *pdf* plot is less important in most reliability applications than the other plots available in

Weibull++, it provides a good way of visualizing the distribution and its characteristics such as its shape, skewness, mode, etc.

Life-Stress Plots

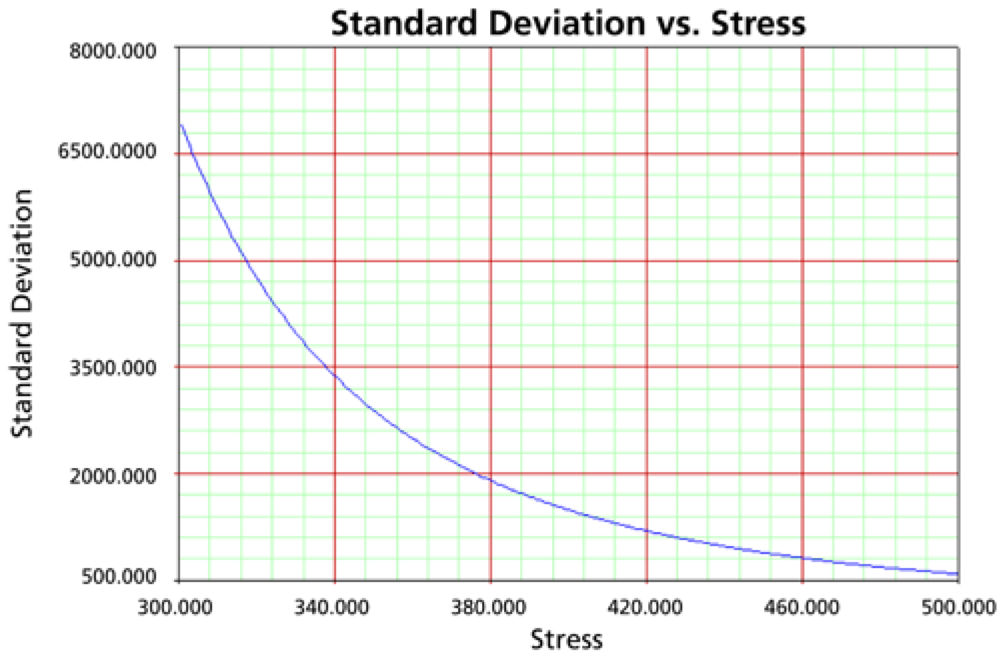
Life vs. Stress plots and Probability plots are the most important plot types in accelerated life testing analysis. Life vs. Stress plots are widely used for estimating the parameters of life-stress relationships. Any life measure can be plotted versus stress in the Life vs. Stress plots available in Weibull++. Confidence bounds information on each life measure can also be plotted. The most widely used Life vs. Stress plots are the Arrhenius and the inverse power law plots. The following figure illustrates a typical Arrhenius Life vs. Stress plot.



Each line in the figure above represents the path for extrapolating a life measure, such as a percentile, from one stress level to another. The slope and intercept of those lines are the parameters of the life-stress relationship (whenever the relationship can be linearized). The imposed *pdfs* represent the distribution of the data at each stress level.

Standard Deviation Plots

Standard Deviation vs. Stress is a useful plot in accelerated life testing analysis and provides information about the spread of the data at each stress level.



Acceleration Factor Plots

The acceleration factor is a unitless number that relates a product's life at an accelerated stress level to the life at the use stress level. It is defined by:

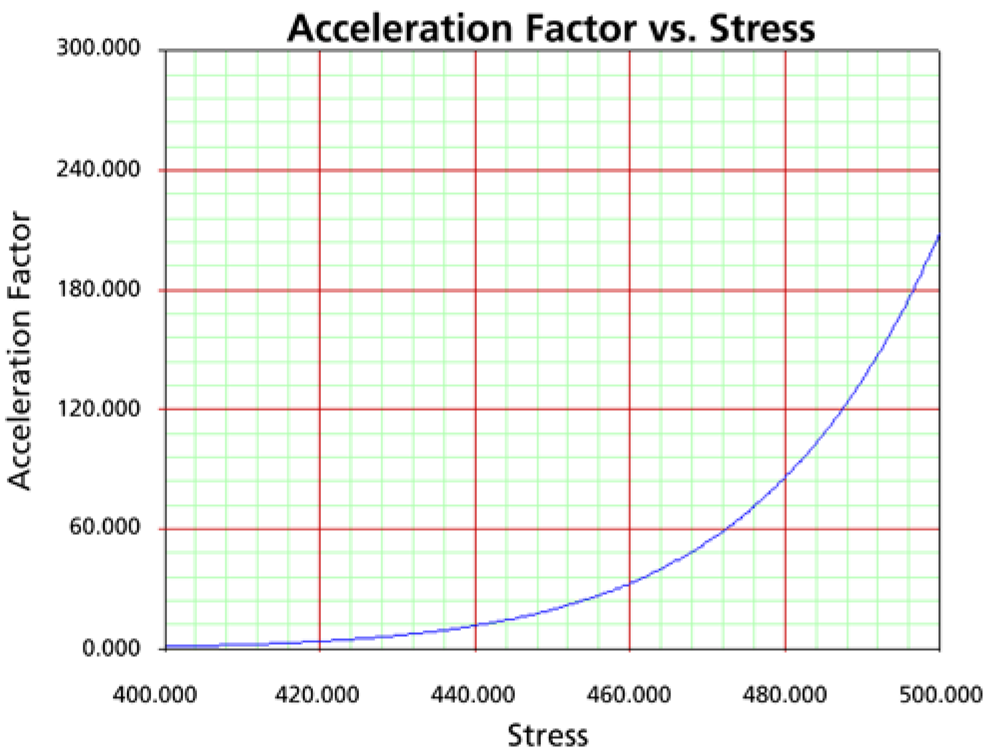
$$A_F = \frac{L_u}{L_A}$$

where:

- L_u is the life at the use stress level.
- L_A is the life at the accelerated level.

As it can be seen, the acceleration factor depends on the life-stress relationship (i.e., Arrhenius, Eyring, etc.) and is thus a function of stress.

The Acceleration Factor vs. Stress plot is generated using the equation above at a constant use stress level and at a varying accelerated stress. In the below figure, the Acceleration Factor vs. Stress was plotted for a constant use level of 300K. Since $L_A = L_u$, the value of the acceleration factor at 300K is equal to 1. The acceleration factor for a temperature of 450K is approximately 8. This means that the life at the use level of 300K is eight times higher than the life at 450K.



Residual Plots

Residual analysis for reliability consists of analyzing the results of a regression analysis by assigning residual values to each data point in the data set. Plotting these residuals provides a very good tool in assessing model assumptions and revealing inadequacies in the model, as well as revealing extreme observations. Three types of residual plots are available in Weibull++.

Standardized Residuals (SR)

The standardized residuals plot for the Weibull and lognormal distributions can be obtained in Weibull++. Each plot type is discussed next.

SR for the Weibull Distribution

Once the parameters have been estimated, the standardized residuals for the Weibull distribution can be calculated by:

$$\hat{e}_i = \widehat{\beta}\left[\ln(T_i) - \ln(\hat{\eta}(V))\right]$$

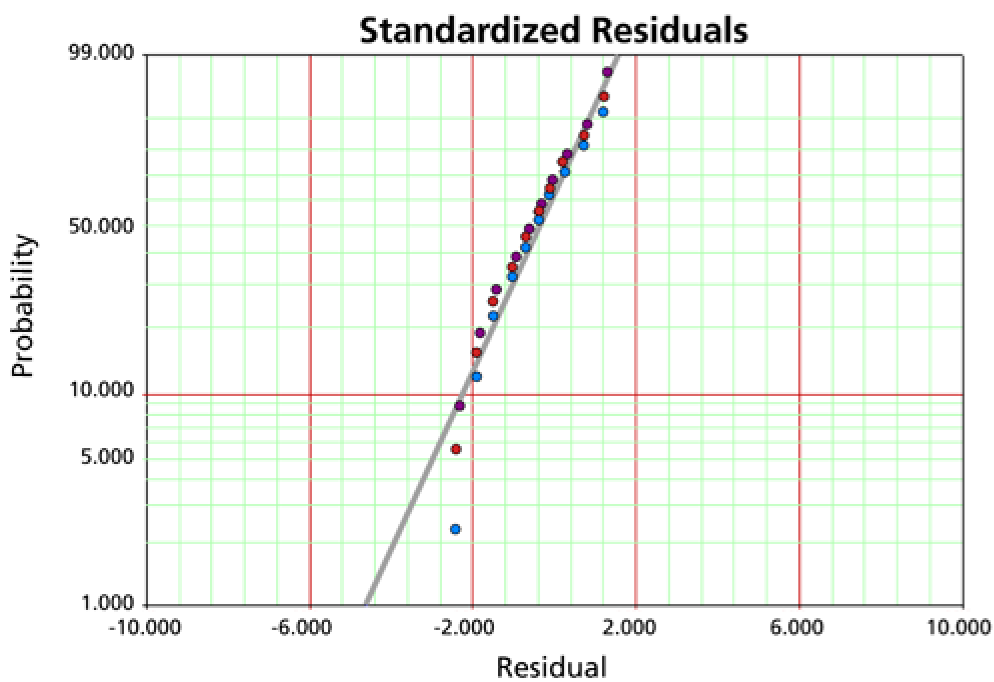
Then, under the assumed model, these residuals should look like a sample from an extreme value distribution with a mean of 0. For the Weibull distribution the standardized residuals are plotted on a smallest extreme value probability paper. If the Weibull distribution adequately describes the data, then the standardized residuals should appear to follow a straight line on such a probability plot. Note that when an observation is censored (suspended), the corresponding residual is also censored.

SR for the Lognormal Distribution

Once the parameters have been estimated, the fitted or calculated responses can be calculated by:

$$\hat{e}_i = \frac{\ln(T_i) - \widehat{\mu}'}{\widehat{\sigma}_{T'}}$$

Then, under the assumed model, the standardized residuals should be normally distributed with a mean of 0 and a standard deviation of 1 ($\sim N(0,1)$). Consequently, the standardized residuals for the lognormal distribution are commonly displayed on a normal probability plot.

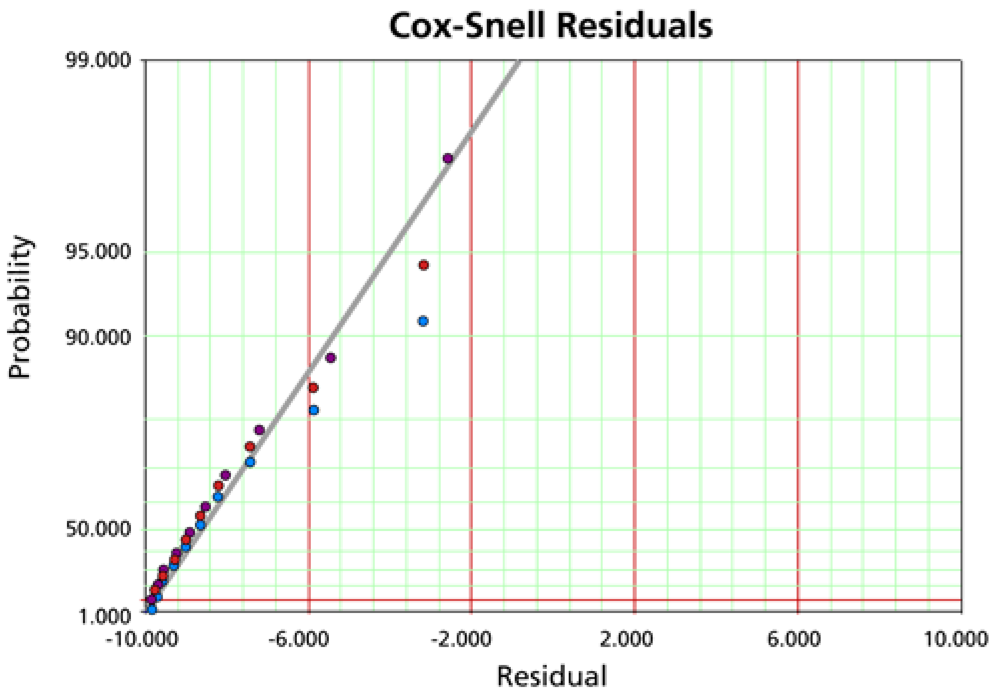


Cox-Snell Residuals

The Cox-Snell residuals are given by:

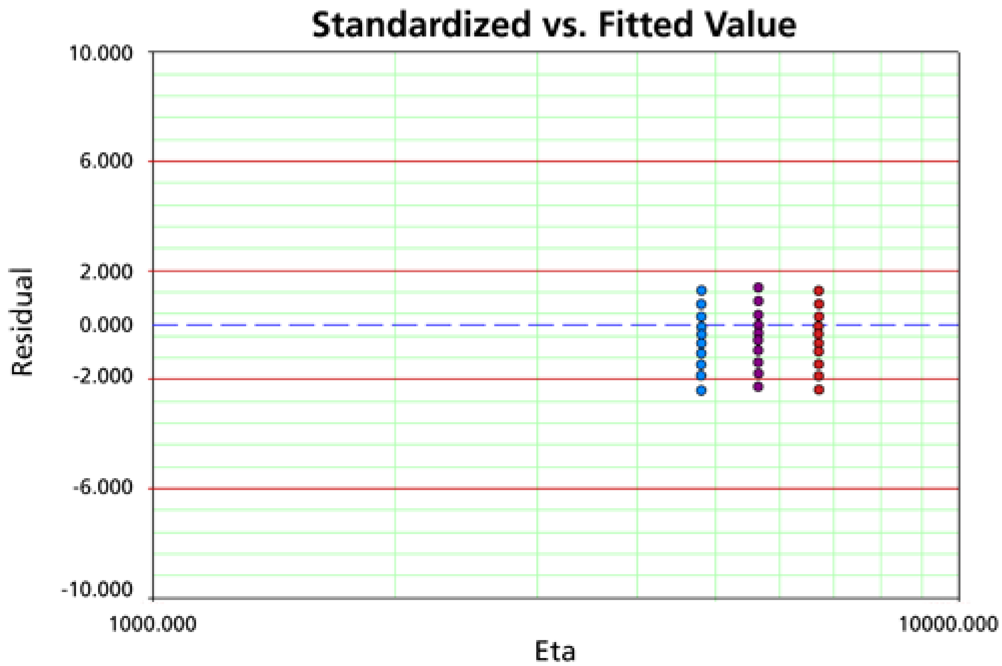
$$\hat{e}_i = -\ln[R(T_i)]$$

where $R(T_i)$ is the calculated reliability value at failure time T_i. The Cox-Snell residuals are plotted on an exponential probability paper.



Standardized vs. Fitted Values

A Standardized vs. Fitted Value plot helps to detect behavior not modeled in the underlying relationship. However, when heavy censoring is present, the plot is more difficult to interpret. In a Standardized vs. Fitted Value plot, the standardized residuals are plotted versus the scale parameter of the underlying life distribution (which is a function of stress) on log-linear paper (linear on the Y-axis). Therefore, the standardized residuals are plotted versus $\eta(V)$ for the Weibull distribution, versus $\mu'(V)$ for the lognormal distribution and versus $m(V)$ for the exponential distribution.


Standardized vs. Fitted Value
10.000
6.000
2.000
0.000
-2.000
-6.000
-10.000
Residual
1000.000
10000.000
Eta

Distributions Used in Accelerated Testing

IN THIS CHAPTER

In this chapter, we will briefly present three lifetime distributions commonly used in accelerated life test analysis: the exponential, the Weibull and the lognormal distributions. Note that although all forms are mentioned below, Weibull++ uses the 1-parameter form of the exponential distribution and the 2-parameter form of the Weibull distribution.

Readers who are interested in a more rigorous overview of these distributions (or for information about other life distributions) can refer to ReliaSoft's Life Data Analysis Reference. For information about the parameter estimation methods, see Appendix B.

The Exponential Distribution

The exponential distribution is commonly used for components or systems exhibiting a *constant failure rate*. Due to its simplicity, it has been widely employed, even in cases where it doesn't apply. In its most general case, the 2-parameter exponential distribution is defined by:

$$f(t) = \lambda e^{-\lambda(t-\gamma)}$$

Where λ is the constant failure rate in failures per unit of measurement (e.g., failures per hour, per cycle, etc.) and γ is the location parameter. In addition, $\lambda = \frac{1}{m}$, where m is the mean time between failures (or to failure).

If the location parameter, γ, is assumed to be zero, then the distribution becomes the 1-parameter exponential or:

$$f(t) = \lambda e^{-\lambda t}$$

For a detailed discussion of this distribution, see The Exponential Distribution.

Exponential Distribution Functions

The Mean or MTTF

The mean, $\overline{T}$, or mean time to failure (MTTF) is given by:

$$\begin{aligned}\bar{T} &= \int_{\gamma}^{\infty} t \cdot f(t)dt \\ &= \int_{\gamma}^{\infty} t \cdot \lambda \cdot e^{-\lambda t}dt \\ &= \gamma + \frac{1}{\lambda} = m\end{aligned}$$

Note that when $\gamma = 0$, the MTTF is the inverse of the exponential distribution's constant failure rate. This is only true for the exponential distribution. Most other distributions do not have a constant failure rate. Consequently, the inverse relationship between failure rate and MTTF does not hold for these other distributions.

The Median

The median, $\breve{T}$, is:

$$\breve{T} = \gamma + \frac{1}{\lambda} \cdot 0.693$$

The Mode

The mode, $\tilde{T}$, is:

$$\tilde{T} = \gamma$$

The Standard Deviation

The standard deviation, σ_T , is:

$$\sigma_T = \frac{1}{\lambda} = m$$

The Exponential Reliability Function

The equation for the 2-parameter exponential cumulative density function, or *cdf*, is given by:

$$F(t) = Q(t) = 1 - e^{-\lambda(t-\gamma)}$$

Recalling that the reliability function of a distribution is simply one minus the *cdf*, the reliability function of the 2-parameter exponential distribution is given by:

$$R(t) = 1 - Q(t) = 1 - \int_0^{t-\gamma} f(x)dx$$

$$R(t) = 1 - \int_0^{t-\gamma} \lambda e^{-\lambda x} dx = e^{-\lambda(t-\gamma)}$$

The 1-parameter exponential reliability function is given by:

$$R(t) = e^{-\lambda t} = e^{-\frac{t}{m}}$$

The Exponential Conditional Reliability Function

The exponential conditional reliability equation gives the reliability for a mission of t duration, having already successfully accumulated T hours of operation up to the start of this new mission. The exponential conditional reliability function is:

$$R(t|T) = \frac{R(T+t)}{R(T)} = \frac{e^{-\lambda(T+t-\gamma)}}{e^{-\lambda(T-\gamma)}} = e^{-\lambda t}$$

which says that the reliability for a mission of t duration undertaken after the component or equipment has already accumulated T hours of operation from age zero is only a function of the mission duration, and not a function of the age at the beginning of the mission. This is referred to as the *memoryless property*.

The Exponential Reliable Life Function

The reliable life, or the mission duration for a desired reliability goal, t_R , for the 1-parameter exponential distribution is:

$$R(t_R) = e^{-\lambda(t_R - \gamma)}$$

$$\ln[R(t_R)] = -\lambda(t_R - \gamma)$$

or:

$$t_R = \gamma - \frac{\ln[R(t_R)]}{\lambda}$$

The Exponential Failure Rate Function

The exponential failure rate function is:

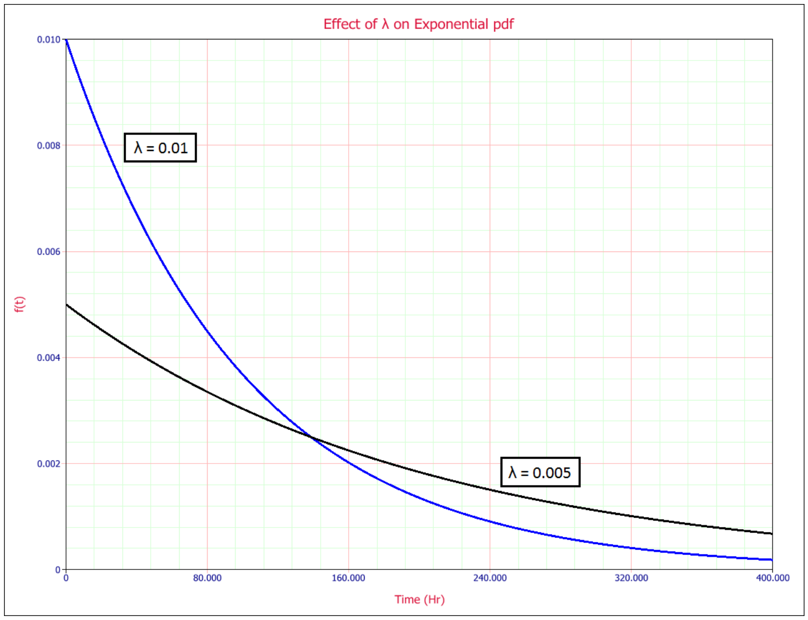
$$\lambda(t) = \frac{f(t)}{R(t)} = \frac{\lambda e^{-\lambda(t-\gamma)}}{e^{-\lambda(t-\gamma)}} = \lambda = \text{constant}$$

Once again, note that the constant failure rate is a characteristic of the exponential distribution, and special cases of other distributions only. Most other distributions have failure rates that are functions of time.

Characteristics of the Exponential Distribution

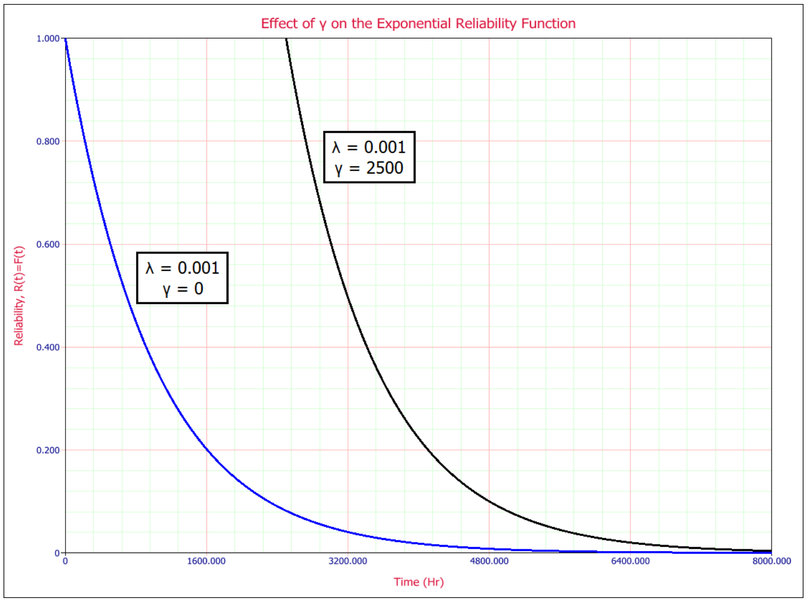
The primary trait of the exponential distribution is that it is used for modeling the behavior of items with a constant failure rate. It has a fairly simple mathematical form, which makes it fairly easy to manipulate. Unfortunately, this fact also leads to the use of this model in situations where it is not appropriate. For example, it would not be appropriate to use the exponential distribution to model the reliability of an automobile. The constant failure rate of the exponential distribution would require the assumption that the automobile would be just as likely to experience a breakdown during the first mile as it would during the one-hundred-thousandth mile. Clearly, this is not a valid assumption. However, some inexperienced practitioners of reliability engineering and life data analysis will overlook this fact, lured by the siren-call of the exponential distribution's relatively simple mathematical models.

The Effect of lambda and gamma on the Exponential *pdf*



- The exponential *pdf* has no shape parameter, as it has only one shape.
- The exponential *pdf* is always convex and is stretched to the right as λ decreases in value.
- The value of the *pdf* function is always equal to the value of λ at $t = 0$ (or $t = \gamma$).
- The location parameter, γ, if positive, shifts the beginning of the distribution by a distance of γ to the right of the origin, signifying that the chance failures start to occur only after γ hours of operation, and cannot occur before this time.
- The scale parameter is $\frac{1}{\lambda} = \bar{T} - \gamma = m - \gamma$.
- As $t \rightarrow \infty$, $f(t) \rightarrow 0$.

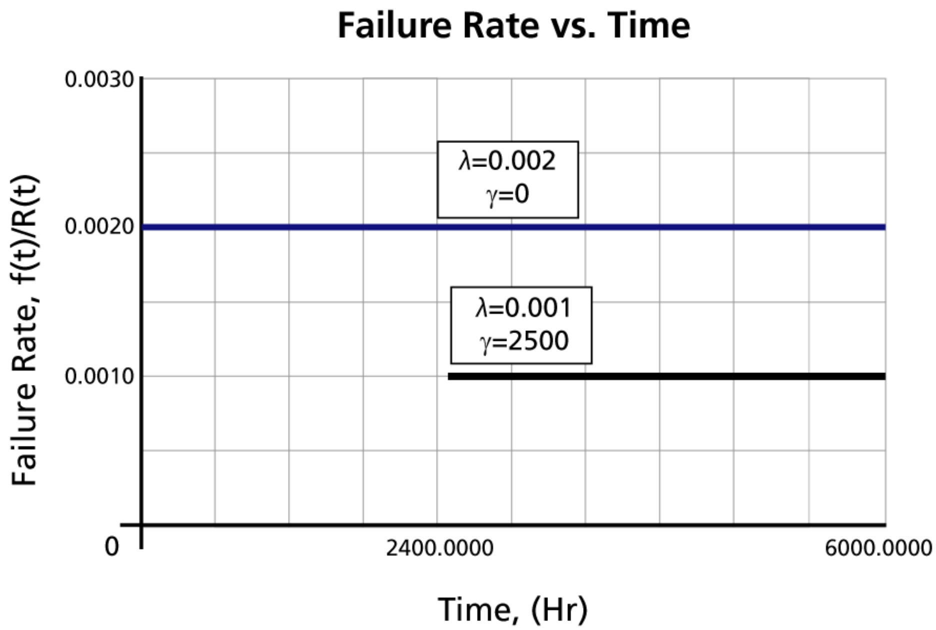
The Effect of lambda and gamma on the Exponential Reliability Function



- The 1-parameter exponential reliability function starts at the value of 100% at $t=0$, decreases thereafter monotonically and is convex.
- The 2-parameter exponential reliability function remains at the value of 100% for $t=0$ up to $t=\gamma$, and decreases thereafter monotonically and is convex.
- As $t\rightarrow\infty$, $R(t\rightarrow\infty)\rightarrow 0$.
- The reliability for a mission duration of $t=m=\frac{1}{\lambda}$, or of one MTTF duration, is always equal to 0.3679 or 36.79%. This means that the reliability for a mission which is as long as one MTTF is relatively low and is not recommended because only 36.8% of the missions will be completed successfully. In other words, of the equipment undertaking such a mission, only 36.8% will survive their mission.

The Effect of lambda and gamma on the Failure Rate Function

- The 1-parameter exponential failure rate function is constant and starts at $t=0$.
- The 2-parameter exponential failure rate function remains at the value of 0 for $t=0$ up to $t=\gamma$, and then keeps at the constant value of λ.



The Weibull Distribution

The Weibull distribution is a general purpose reliability distribution used to model material strength, times-to-failure of electronic and mechanical components, equipment or systems. In its most general case, the 3-parameter Weibull *pdf* is defined by:

$$f(t) = \frac{\beta}{\eta}\left(\frac{t-\gamma}{\eta}\right)^{\beta-1} e^{-(\frac{t-\gamma}{\eta})^{\beta}}$$

where β = shape parameter, η = scale parameter and γ = location parameter.

If the location parameter, γ, is assumed to be zero, then the distribution becomes the 2-parameter Weibull or:

$$f(t) = \frac{\beta}{\eta}(\frac{t}{\eta})^{\beta-1} e^{-(\frac{t}{\eta})^{\beta}}$$

One additional form is the 1-parameter Weibull distribution, which assumes that the location parameter, γ is zero, and the shape parameter is a known constant, or β = constant = C, so:

$$f(t) = \frac{C}{\eta}(\frac{t}{\eta})^{C-1}e^{-(\frac{t}{\eta})^{C}}$$

For a detailed discussion of this distribution, see The Weibull Distribution.

Weibull Distribution Functions

The Mean or MTTF

The mean, $\overline{T}$, (also called *MTTF*) of the Weibull *pdf* is given by:

$$\overline{T} = \gamma + \eta \cdot \Gamma\left(\frac{1}{\beta} + 1\right)$$

where

$$\Gamma\left(\frac{1}{\beta} + 1\right)$$

is the gamma function evaluated at the value of:

$$\left(\frac{1}{\beta} + 1\right)$$

The gamma function is defined as:

$$\Gamma(n) = \int_{0}^{\infty} e^{-x} x^{n-1} dx$$

For the 2-parameter case, this can be reduced to:

$$\overline{T} = \eta \cdot \Gamma\left(\frac{1}{\beta} + 1\right)$$

Note that some practitioners erroneously assume that η is equal to the MTTF, $\overline{T}$. This is only true for the case of: $\beta = 1$ or:

$$\begin{aligned}\overline{T} &= \eta \cdot \Gamma\left(\frac{1}{1}+1\right) \\ &= \eta \cdot \Gamma\left(\frac{1}{1}+1\right) \\ &= \eta \cdot \Gamma(2) \\ &= \eta \cdot 1 \\ &= \eta\end{aligned}$$

The Median

The median, $\breve{T}$, of the Weibull distribution is given by:

$$\breve{T} = \gamma + \eta(\ln 2)^{\frac{1}{\beta}}$$

The Mode

The mode, $\tilde{T}$, is given by:

$$\tilde{T} = \gamma + \eta\left(1 - \frac{1}{\beta}\right)^{\frac{1}{\beta}}$$

The Standard Deviation

The standard deviation, σ_T , is given by:

$$\sigma_T = \eta \cdot \sqrt{\Gamma\left(\frac{2}{\beta}+1\right) - \Gamma\left(\frac{1}{\beta}+1\right)^2}$$

The Weibull Reliability Function

The equation for the 3-parameter Weibull cumulative density function, *cdf*, is given by:

$$F(t) = 1 - e^{-\left(\frac{t-\gamma}{\eta}\right)^{\beta}}$$

This is also referred to as *unreliability* and designated as $Q(t)$ by some authors.

Recalling that the reliability function of a distribution is simply one minus the *cdf*, the reliability function for the 3-parameter Weibull distribution is then given by:

$$R(t)=e^{-\left(\frac{t-\gamma}{\eta}\right)^{\beta}}$$

The Weibull Conditional Reliability Function

The 3-parameter Weibull conditional reliability function is given by:

$$R(t|T)=\frac{R(T+t)}{R(T)}=\frac{e^{-\left(\frac{T+t-\gamma}{\eta}\right)^{\beta}}}{e^{-\left(\frac{T-\gamma}{\eta}\right)^{\beta}}}$$

or:

$$R(t|T)=e^{-\left[\left(\frac{T+t-\gamma}{\eta}\right)^{\beta}-\left(\frac{T-\gamma}{\eta}\right)^{\beta}\right]}$$

These give the reliability for a new mission of t duration, having already accumulated T time of operation up to the start of this new mission, and the units are checked out to assure that they will start the next mission successfully. It is called conditional because you can calculate the reliability of a new mission based on the fact that the unit or units already accumulated hours of operation successfully.

The Weibull Reliable Life

The reliable life, T_R, of a unit for a specified reliability, R, starting the mission at age zero, is given by:

$$T_R=\gamma+\eta\cdot\{-\ln(R)\}^{\frac{1}{\beta}}$$

This is the life for which the unit/item will be functioning successfully with a reliability of R. If $R=0.50$, then $T_R=\breve{T}$, the median life, or the life by which half of the units will survive.

The Weibull Failure Rate Function

The Weibull failure rate function, $\lambda(t)$, is given by:

$$\lambda(t) = \frac{f(t)}{R(t)} = \frac{\beta}{\eta}\left(\frac{t-\gamma}{\eta}\right)^{\beta-1}$$

Characteristics of the Weibull Distribution

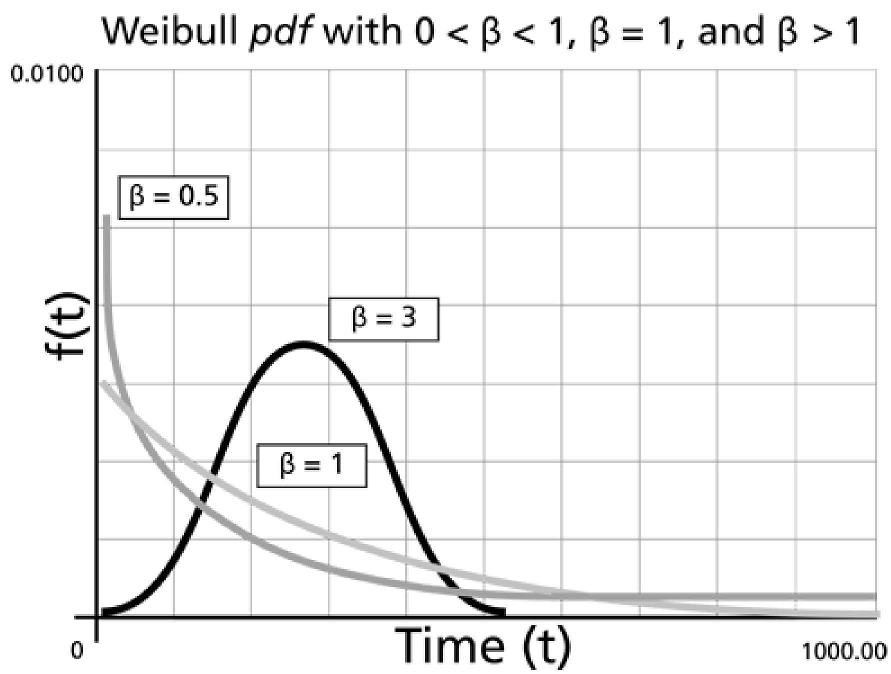
The Weibull distribution is widely used in reliability and life data analysis due to its versatility. Depending on the values of the parameters, the Weibull distribution can be used to model a variety of life behaviors. We will now examine how the values of the shape parameter, β, and the scale parameter, η, affect such distribution characteristics as the shape of the curve, the reliability and the failure rate. Note that in the rest of this section we will assume the most general form of the Weibull distribution, (i.e., the 3-parameter form). The appropriate substitutions to obtain the other forms, such as the 2-parameter form where $\gamma = 0,$ or the 1-parameter form where $\beta = C =$ constant, can easily be made.

Effects of the Shape Parameter, beta

The Weibull shape parameter, β, is also known as the *slope*. This is because the value of β is equal to the slope of the regressed line in a probability plot. Different values of the shape parameter can have marked effects on the behavior of the distribution. In fact, some values of the shape parameter will cause the distribution equations to reduce to those of other distributions. For example, when $\beta = 1$, the *pdf* of the 3-parameter Weibull distribution reduces to that of the 2-parameter exponential distribution or:

$$f(t) = \frac{1}{\eta}e^{-\frac{t-\gamma}{\eta}}$$

where $\frac{1}{\eta} = \lambda =$ failure rate. The parameter β is a pure number, (i.e., it is dimensionless). The following figure shows the effect of different values of the shape parameter, β, on the shape of the *pdf*. As you can see, the shape can take on a variety of forms based on the value of β.



For $0 < \beta \leq 1$:

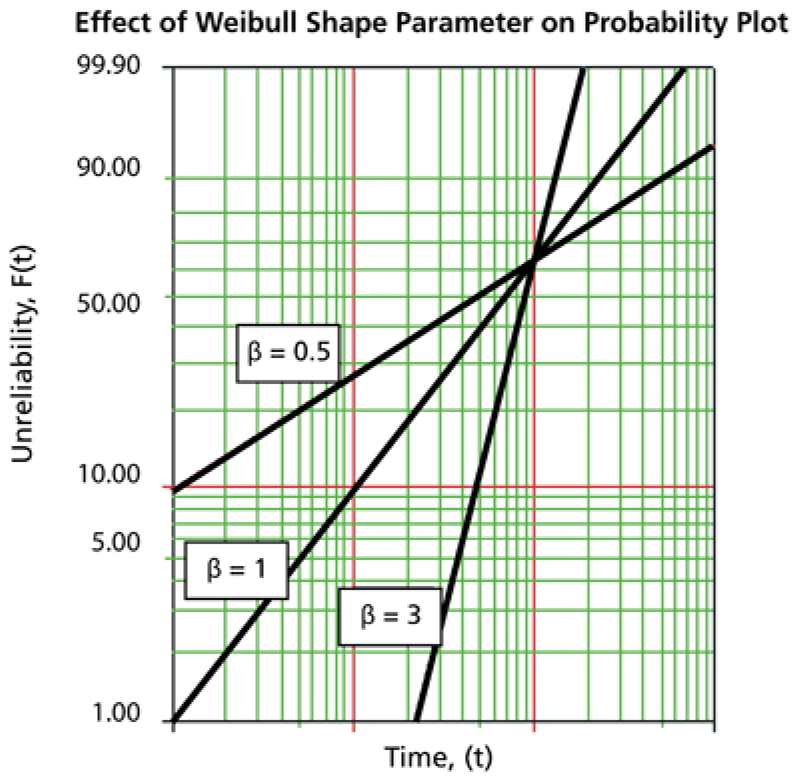
- As $t \rightarrow 0$ (or γ), $f(t) \rightarrow \infty.$
- As $t \rightarrow \infty$, $f(t) \rightarrow 0$.
- $f(t)$ decreases monotonically and is convex as it increases beyond the value of γ .
- The mode is non-existent.

For $\beta > 1$:

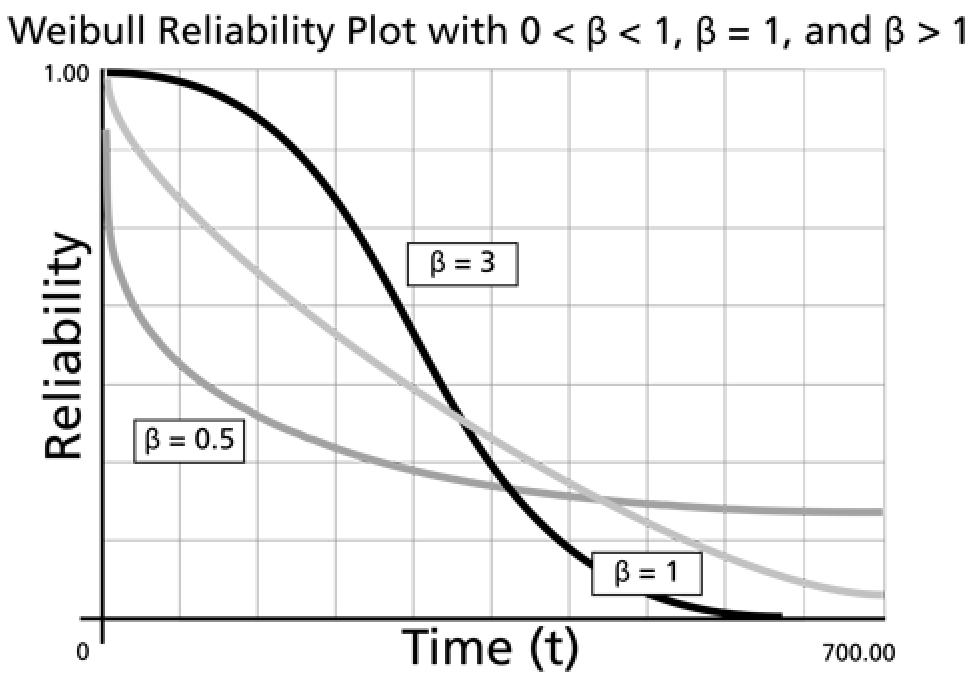
- $f(t) = 0$ at $t = 0$ (or γ).
- $f(t)$ increases as $t \rightarrow \tilde{T}$ (the mode) and decreases thereafter.
- For $\beta < 2.6$ the Weibull *pdf* is positively skewed (has a right tail), for $2.6 < \beta < 3.7$ its coefficient of skewness approaches zero (no tail). Consequently, it may approximate the normal *pdf*, and for $\beta > 3.7$ it is negatively skewed (left tail). The way the value of β relates to the physical behavior of the items being modeled becomes more apparent when we observe how its different values affect the reliability and failure rate functions. Note that for $\beta = 0.999$, $f(0) = \infty$, but for $\beta = 1.001$,

$f(0) = 0$. This abrupt shift is what complicates MLE estimation when β is close to 1.

The Effect of beta on the *cdf* and Reliability Function



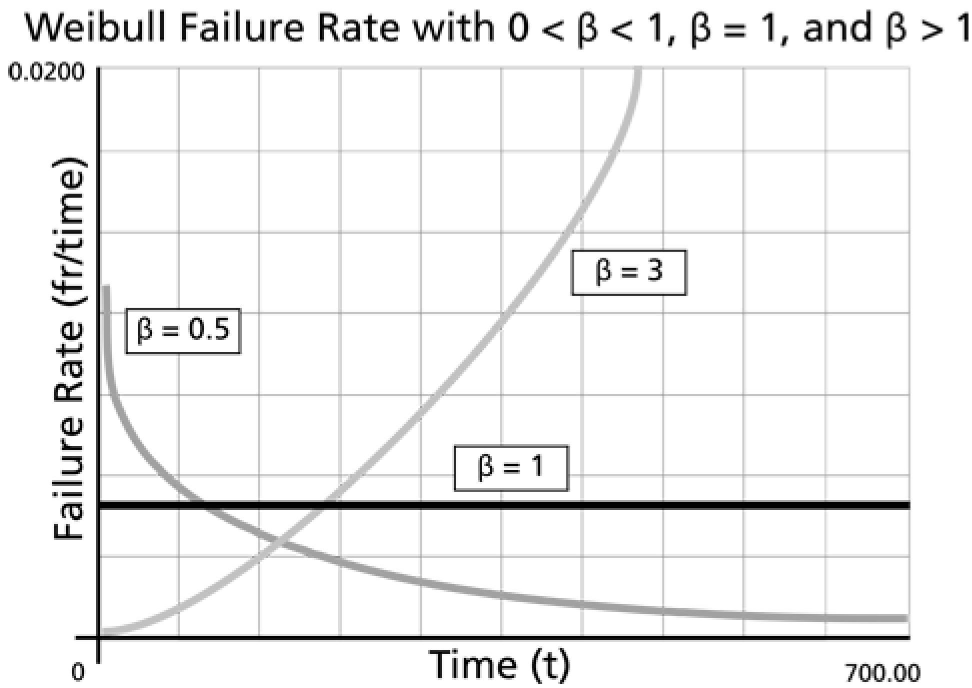
The above figure shows the effect of the value of β on the *cdf*, as manifested in the Weibull probability plot. It is easy to see why this parameter is sometimes referred to as the slope. Note that the models represented by the three lines all have the same value of η. The following figure shows the effects of these varied values of β on the reliability plot, which is a linear analog of the probability plot.



- $R(t)$ decreases sharply and monotonically for $0 < \beta < 1$ and is convex.
- For $\beta = 1$, $R(t)$ decreases monotonically but less sharply than for $0 < \beta < 1$ and is convex.
- For $\beta > 1$, $R(t)$ decreases as increases. As wear-out sets in, the curve goes through an inflection point and decreases sharply.

The Effect of beta on the Weibull Failure Rate

The value of β has a marked effect on the failure rate of the Weibull distribution and inferences can be drawn about a population's failure characteristics just by considering whether the value of β is less than, equal to, or greater than one.



As indicated by above figure, populations with $\beta < 1$ exhibit a failure rate that decreases with time, populations with $\beta = 1$ have a constant failure rate (consistent with the exponential distribution) and populations with $\beta > 1$ have a failure rate that increases with time. All three life stages of the bathtub curve can be modeled with the Weibull distribution and varying values of β. The Weibull failure rate for $0 < \beta < 1$ is unbounded at $T = 0$ (or γ). The failure rate, $\lambda(t),$ decreases thereafter monotonically and is convex, approaching the value of zero as $t \rightarrow \infty$ or $\lambda(\infty) = 0$. This behavior makes it suitable for representing the failure rate of units exhibiting early-type failures, for which the failure rate decreases with age. When encountering such behavior in a manufactured product, it may be indicative of problems in the production process, inadequate burn-in, substandard parts and components, or problems with packaging and shipping. For $\beta = 1$, $\lambda(t)$ yields a constant value of $\frac{1}{\eta}$ or:

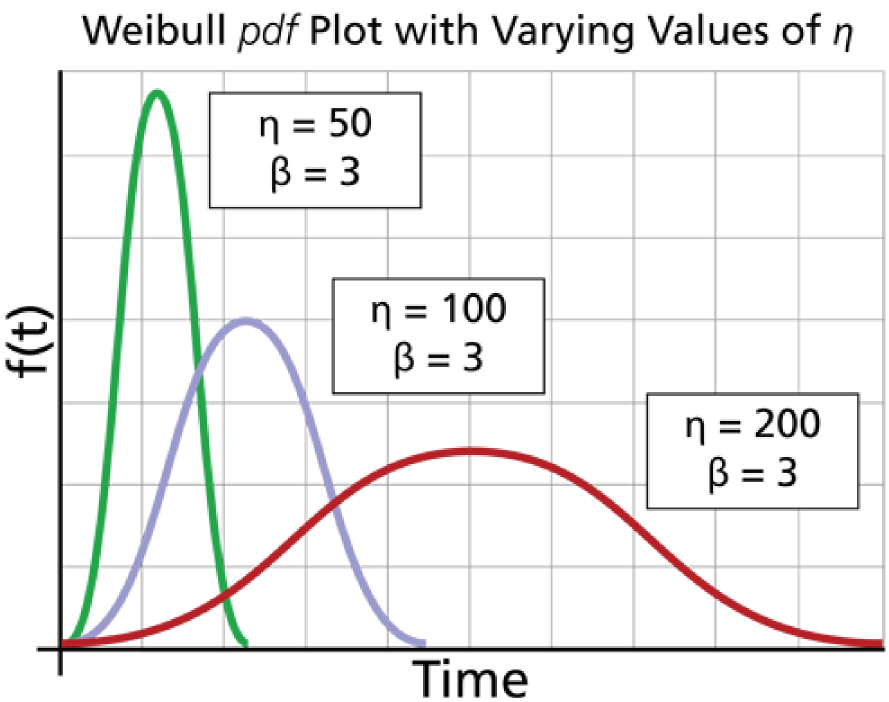
$$\lambda(t) = \lambda = \frac{1}{\eta}$$

This makes it suitable for representing the failure rate of chance-type failures and the useful life period failure rate of units.

For $\beta > 1$, $\lambda(t)$ increases as t increases and becomes suitable for representing the failure rate of units exhibiting wear-out type failures. For $1 < \beta < 2,$ the $\lambda(t)$ curve is concave, consequently the failure rate increases at a decreasing rate as t increases.

For $\beta = 2$ there emerges a straight line relationship between $\lambda(t)$ and t, starting at a value of $\lambda(t) = 0$ at $t = \gamma$, and increasing thereafter with a slope of $\frac{2}{\eta^2}$. Consequently, the failure rate increases at a constant rate as t increases. Furthermore, if $\eta = 1$ the slope becomes equal to 2, and when $\gamma = 0$, $\lambda(t)$ becomes a straight line which passes through the origin with a slope of 2. Note that at $\beta = 2$, the Weibull distribution equations reduce to that of the Rayleigh distribution.

When $\beta > 2,$ the $\lambda(t)$ curve is convex, with its slope increasing as t increases. Consequently, the failure rate increases at an increasing rate as t increases, indicating wearout life.

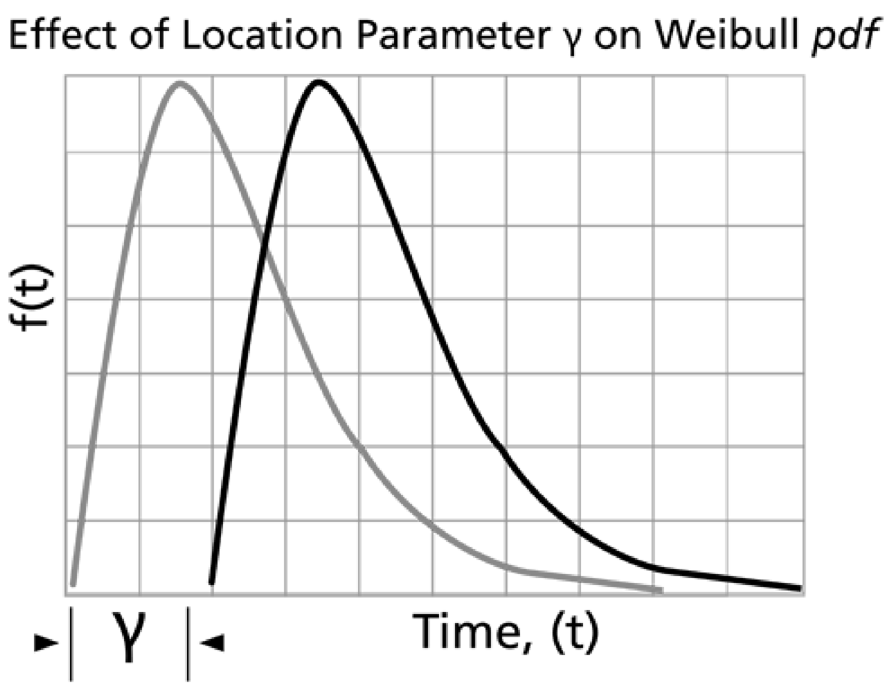


A change in the scale parameter η has the same effect on the distribution as a change of the abscissa scale. Increasing the value of η while holding β constant has the effect of stretching out the *pdf*. Since the area under a *pdf* curve is a constant value of one, the "peak" of the *pdf* curve will also decrease with the increase of η, as indicated in the above figure.

- If η is increased while β and γ are kept the same, the distribution gets stretched out to the right and its height decreases, while maintaining its shape and location.
- If η is decreased while β and γ are kept the same, the distribution gets pushed in towards the left (i.e., towards its beginning or towards 0 or γ), and its height increases.
- η has the same units as t, such as hours, miles, cycles, actuations, etc.

Effects of the Location Parameter, gamma

The location parameter, γ, as the name implies, locates the distribution along the abscissa. Changing the value of γ has the effect of *sliding* the distribution and its associated function either to the right (if $\gamma > 0$) or to the left (if $\gamma < 0$).



- When $\gamma = 0,$ the distribution starts at $t = 0$ or at the origin.
- If $\gamma > 0,$ the distribution starts at the location γ to the right of the origin.
- If $\gamma < 0,$ the distribution starts at the location γ to the left of the origin.
- γ provides an estimate of the earliest time-to-failure of such units.
- The life period 0 to $+\gamma$ is a failure free operating period of such units.

- The parameter γ may assume all values and provides an estimate of the earliest time a failure may be observed. A negative γ may indicate that failures have occurred prior to the beginning of the test, namely during production, in storage, in transit, during checkout prior to the start of a mission, or prior to actual use.
- γ has the same units as t, such as hours, miles, cycles, actuations, etc.

The Lognormal Distribution

The lognormal distribution is commonly used for general reliability analysis, cycles-to-failure in fatigue, material strengths and loading variables in probabilistic design. When the natural logarithms of the times-to-failure are normally distributed, then we say that the data follow the lognormal distribution.

The *pdf* of the lognormal distribution is given by:

$$f(t)=\frac{1}{t\sigma'\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{t'-\mu'}{\sigma'})^2}$$
$$f(t)\geq 0, t>0, \sigma'>0$$
$$t'=\ln(t)$$

where μ' is the mean of the natural logarithms of the times-to-failure and σ' is the standard deviation of the natural logarithms of the times to failure.

For a detailed discussion of this distribution, see The Lognormal Distribution.

Lognormal Distribution Functions

The Mean or MTTF

The mean of the lognormal distribution, μ, is discussed in Kececioglu [19]:

$$\mu=e^{\mu'+\frac{1}{2}\sigma'^2}$$

The mean of the natural logarithms of the times-to-failure, μ', in terms of $\bar{T}$ and σ is given by:

$$\mu' = \ln(\bar{T}) - \frac{1}{2}\ln\left(\frac{\sigma^2}{\bar{T}^2}+1\right)$$

The Median

The median of the lognormal distribution, $\breve{T}$, is discussed in Kececioglu [19]:

$$\breve{T} = e^{\mu'}$$

The Mode

The mode of the lognormal distribution, $\tilde{T}$, is discussed in Kececioglu [19]:

$$\tilde{T} = e^{\mu' - \sigma'^2}$$

The Standard Deviation

The standard deviation of the lognormal distribution, σ_T , is discussed in Kececioglu [19]:

$$\sigma_T = \sqrt{\left(e^{2\mu' + \sigma'^2}\right)\left(e^{\sigma'^2} - 1\right)}$$

The standard deviation of the natural logarithms of the times-to-failure, σ' , in terms of $\bar{T}$ and σ is given by:

$$\sigma' = \sqrt{\ln\left(\frac{\sigma_T^2}{\bar{T}^2}+1\right)}$$

The Lognormal Reliability Function

The reliability for a mission of time t , starting at age 0, for the lognormal distribution is determined by:

$$R(t) = \int_t^{\infty} f(x)dx$$

or:

$$R(t)=\int_{\ln(t)}^{\infty}\frac{1}{\sigma'\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu'}{\sigma'}\right)^{2}}dx$$

As with the normal distribution, there is no closed-form solution for the lognormal reliability function. Solutions can be obtained via the use of standard normal tables. Since the application automatically solves for the reliability we will not discuss manual solution methods. For interested readers, full explanations can be found in the references.

The Lognormal Conditional Reliability Function

The lognormal conditional reliability function is given by:

$$R(t|T)=\frac{R(T+t)}{R(T)}=\frac{\int_{\ln(T+t)}^{\infty}\frac{1}{\sigma'\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu'}{\sigma'}\right)^{2}}ds}{\int_{\ln(T)}^{\infty}\frac{1}{\sigma'\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu'}{\sigma'}\right)^{2}}dx}$$

Once again, the use of standard normal tables is necessary to solve this equation, as no closed-form solution exists.

The Lognormal Reliable Life Function

As there is no closed-form solution for the lognormal reliability equation, no closed-form solution exists for the lognormal reliable life either. In order to determine this value, one must solve the following equation for t :

$$R_t=\int_{\ln(t)}^{\infty}\frac{1}{\sigma'\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu'}{\sigma'}\right)^{2}}dx$$

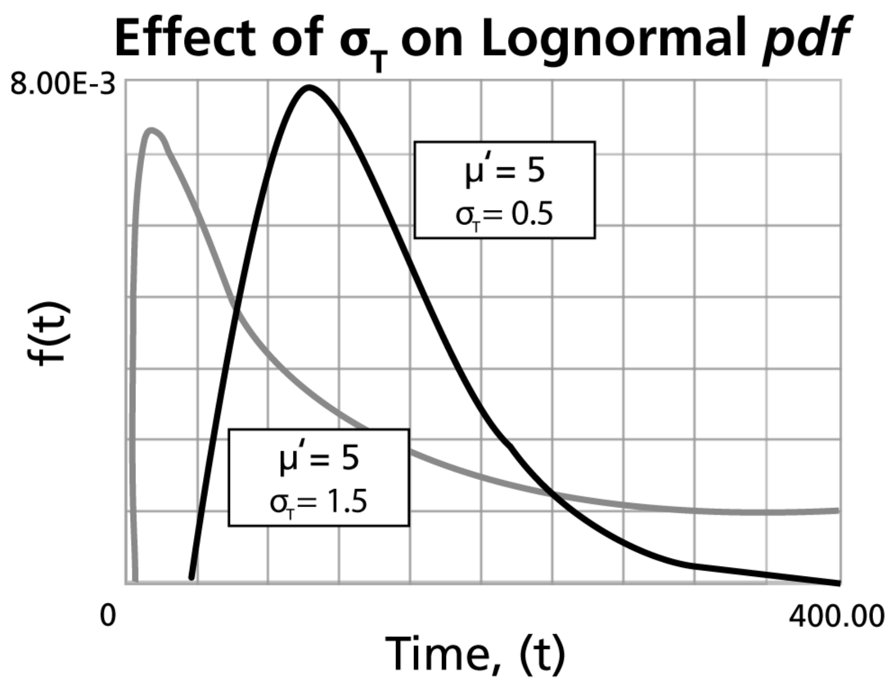
The Lognormal Failure Rate Function

The lognormal failure rate is given by:

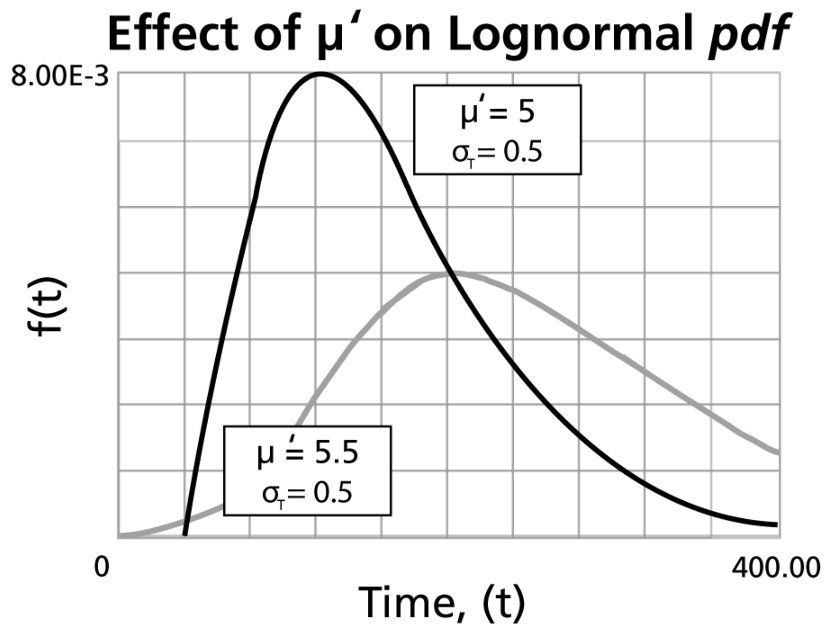
$$\lambda(t)=\frac{f(t)}{R(t)}=\frac{\frac{1}{t\cdot\sigma'\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{t'-\mu'}{\sigma'})^{2}}}{\int_{t'}^{\infty}\frac{1}{\sigma'\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{x-\mu'}{\sigma'})^{2}}dx}$$

As with the reliability equations, standard normal tables will be required to solve for this function.

Characteristics of the Lognormal Distribution



- The lognormal distribution is a distribution skewed to the right.
- The *pdf* starts at zero, increases to its mode, and decreases thereafter.
- The degree of skewness increases as σ' increases, for a given μ'



- For the same σ' , the *pdf* 's skewness increases as μ' increases.

- For σ' values significantly greater than 1, the *pdf* rises very sharply in the beginning, (i.e., for very small values of T near zero), and essentially follows the ordinate axis, peaks out early, and then decreases sharply like an exponential *pdf* or a Weibull *pdf* with $0 < \beta < 1$.

- The parameter, μ' , in terms of the logarithm of the $T's$ is also the scale parameter, and not the location parameter as in the case of the normal *pdf*.

- The parameter σ' , or the standard deviation of the $T's$ in terms of their logarithm or of their T' , is also the shape parameter and not the scale parameter, as in the normal *pdf*, and assumes only positive values.

Lognormal Distribution Parameters in ReliaSoft's Software

In ReliaSoft's software, the parameters returned for the lognormal distribution are always logarithmic. That is: the parameter μ' represents the mean of the natural logarithms of the times-to-failure, while σ' represents the standard deviation of these data point logarithms. Specifically, the returned σ' is the square root of the variance of the natural logarithms of the data points. Even though the application denotes these values as mean and standard deviation, the user is reminded that these are given as the parameters of the distribution, and are thus the mean and standard deviation of the natural logarithms of the data. The mean value of the times-to-failure, not used as a parameter, as well as the standard deviation can be obtained through the QCP or the Function Wizard.

Arrhenius Relationship

IN THIS CHAPTER

The Arrhenius life-stress model (or relationship) is probably the most common life-stress relationship utilized in accelerated life testing. It has been widely used when the stimulus or acceleration variable (or stress) is thermal (i.e., temperature). It is derived from the Arrhenius reaction rate equation proposed by the Swedish physical chemist Svandte Arrhenius in 1887.

Formulation

The Arrhenius reaction rate equation is given by:

$$R(T) = Ae^{-\frac{E_a}{k \cdot T}}$$

where:

- R is the speed of reaction.
- A is an unknown nonthermal constant.
- E_a is the activation energy (eV).
- k is the Boltzmann's constant $(8.6173303 \times 10^{-5} \text{eVK}^{-1})$.
- T is the absolute temperature (K).

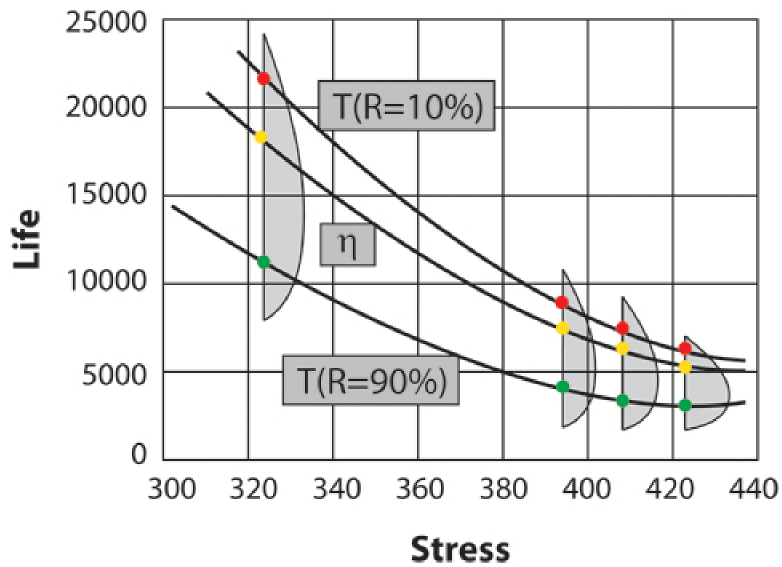
The activation energy is the energy that a molecule must have to participate in the reaction. In other words, the activation energy is a measure of the effect that temperature has on the reaction.

The Arrhenius life-stress model is formulated by assuming that life is proportional to the inverse reaction rate of the process, thus the Arrhenius life-stress relationship is given by:

$$L(V) = Ce^{\frac{B}{V}}$$

where:

- L represents a quantifiable life measure, such as mean life, characteristic life, median life, or $B(x)$ life, etc.
- V represents the stress level (formulated for temperature and **temperature values in absolute units, degrees Kelvin or degrees Rankine**).
- C is one of the model parameters to be determined, $(C > 0)$.
- B is another model parameter to be determined.

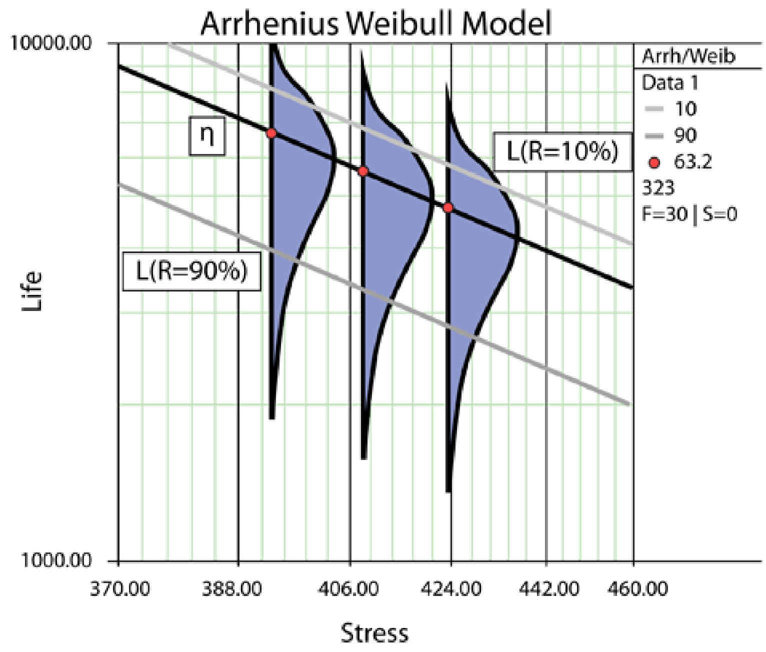


Since the Arrhenius is a physics-based model derived for temperature dependence, it is used for temperature accelerated tests. For the same reason, temperature values must be in absolute units (Kelvin or Rankine), even though the Arrhenius equation is unitless.

Life Stress Plots

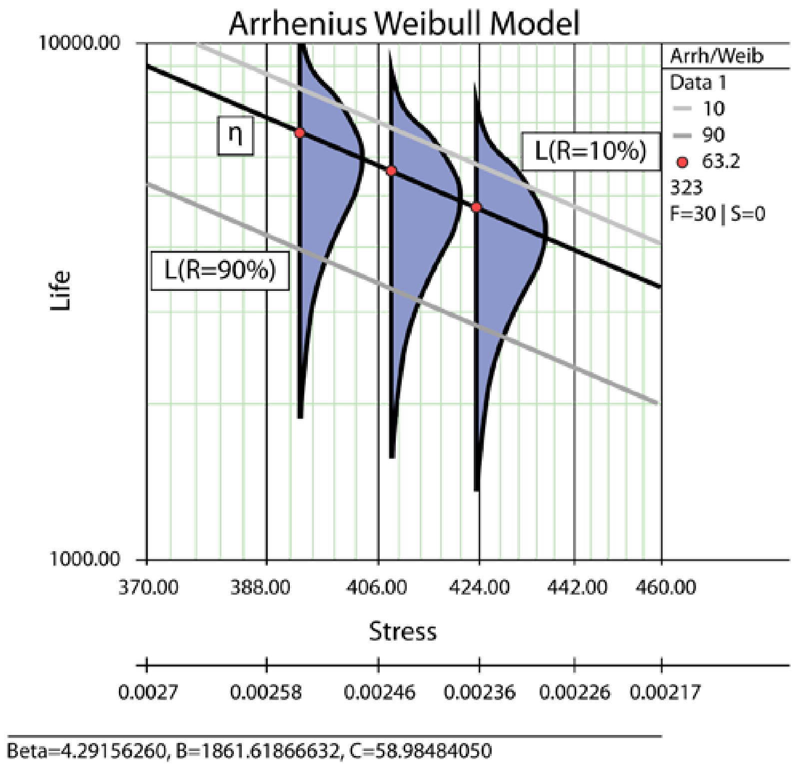
The Arrhenius relationship can be linearized and plotted on a Life vs. Stress plot, also called the Arrhenius plot. The relationship is linearized by taking the natural logarithm of both sides in the Arrhenius equation or:

$$ln(L(V)) = ln(C) + \frac{B}{V}$$

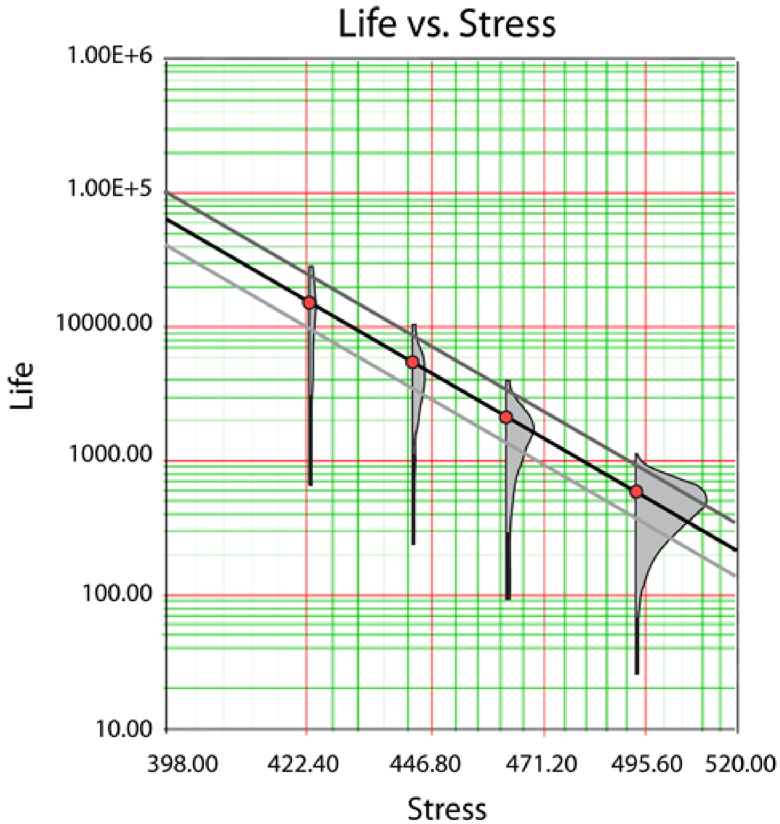


Beta=4.29156260, B=1861.61866632, C=58.98484050

In the linearized Arrhenius equation, $\ln(C)$ is the intercept of the line and B is the slope of the line. Note that the inverse of the stress, and not the stress, is the variable. In the above figure, life is plotted versus stress and not versus the inverse stress. This is because the linearized Arrhenius equation was plotted on a reciprocal scale. On such a scale, the slope B appears to be negative even though it has a positive value. This is because B is actually the slope of the reciprocal of the stress and not the slope of the stress. The reciprocal of the stress is decreasing as stress is increasing ($\frac{1}{V}$ is decreasing as V is increasing). The two different axes are shown in the next figure.



The Arrhenius relationship is plotted on a reciprocal scale for practical reasons. For example, in the above figure it is more convenient to locate the life corresponding to a stress level of 370K than to take the reciprocal of 370K (0.0027) first, and then locate the corresponding life. The shaded areas shown in the above figure are the imposed at each test stress level. From such imposed *pdfs* one can see the range of the life at each test stress level, as well as the scatter in life. The next figure illustrates a case in which there is a significant scatter in life at each of the test stress levels.

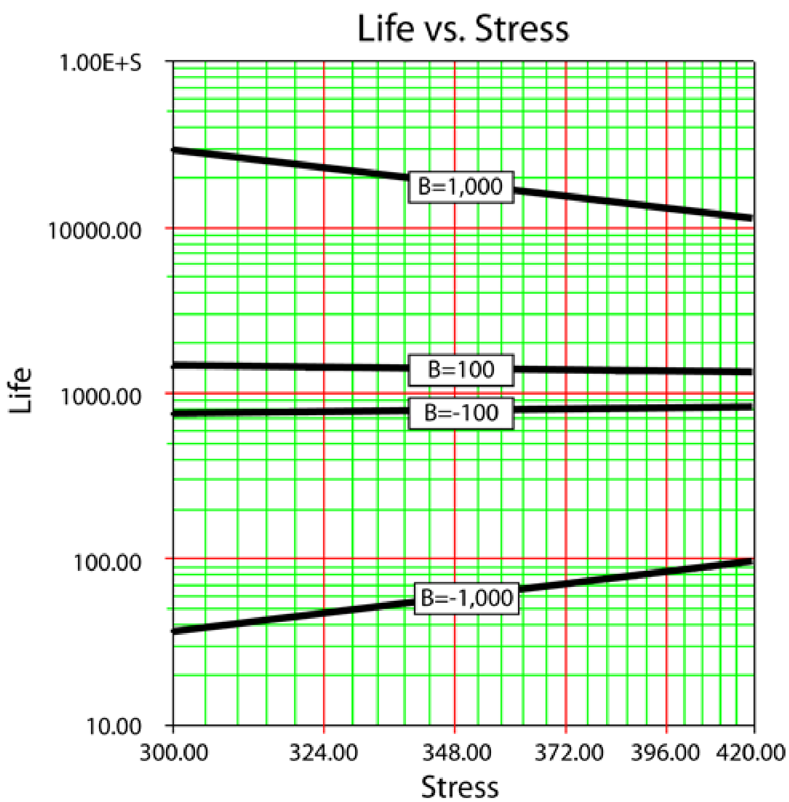


Activation Energy and the Parameter *B*

Depending on the application (and where the stress is exclusively thermal), the parameter B can be replaced by:

$$B = \frac{E_a}{k} = \frac{\text{activation energy}}{\text{Boltzmann's constant}} = \frac{\text{activation energy}}{8.617385 \times 10^{-5}\text{eVK}^{-1}}$$

Note that in this formulation, the activation energy E_a must be known a priori. If the activation energy is known then there is only one model parameter remaining, $C.$ Because in most real life situations this is rarely the case, all subsequent formulations will assume that this activation energy is unknown and treat B as one of the model parameters. B has the same properties as the activation energy. In other words, B is a measure of the effect that the stress (i.e. temperature) has on the life. The larger the value of $B,$ the higher the dependency of the life on the specific stress. Parameter B may also take negative values. In that case, life is increasing with increasing stress. An example of this would be plasma filled bulbs, where low temperature is a higher stress on the bulbs than high temperature.



Acceleration Factor

Most practitioners use the term acceleration factor to refer to the ratio of the life (or acceleration characteristic) between the use level and a higher test stress level or:

$$A_F = \frac{L_{USE}}{L_{Accelerated}}$$

For the Arrhenius model this factor is:

$$A_F = \frac{L_{USE}}{L_{Accelerated}} = \frac{C\,e^{\frac{B}{V_u}}}{C\,e^{\frac{B}{V_A}}} = \frac{e^{\frac{B}{V_u}}}{e^{\frac{B}{V_A}}} = e^{\left(\frac{B}{V_u}-\frac{B}{V_A}\right)}$$

Thus, if B is assumed to be known a priori (using an activation energy), the assumed activation energy alone dictates this acceleration factor!

Arrhenius-Exponential

The *pdf* of the 1-parameter exponential distribution is given by:

$$f(t) = \lambda e^{-\lambda t}$$

It can be easily shown that the mean life for the 1-parameter exponential distribution (presented in detail here) is given by:

$$\lambda = \frac{1}{m}$$

thus:

$$f(t) = \frac{1}{m} e^{-\frac{t}{m}}$$

The Arrhenius-exponential model *pdf* can then be obtained by setting $m = L(V)$:

Therefore:

$$m = L(V) = Ce^{\frac{B}{V}}$$

Substituting for m yields a *pdf* that is both a function of time and stress or:

$$f(t, V) = \frac{1}{Ce^{\frac{B}{V}}} \cdot e^{-\frac{1}{Ce^{\frac{B}{V}}} \cdot t}$$

Arrhenius-Exponential Statistical Properties Summary

Mean or MTTF

The mean, $\overline{T}$, or Mean Time To Failure (MTTF) of the Arrhenius-exponential is given by,

$$\overline{T} = \int_0^\infty t \cdot f(t, V) dt = \int_0^\infty t \cdot \frac{1}{Ce^{\frac{B}{V}}} e^{-\frac{t}{Ce^{\frac{B}{V}}}} dt = Ce^{\frac{B}{V}}$$

Median

The median, $\breve{T}$ of the Arrhenius-exponential model is given by:

$$\breve{T} = 0.693 \cdot Ce^{\frac{B}{V}}$$

Mode

The mode, $\tilde{T},$ of the Arrhenius-exponential model is given by:

$$\tilde{T} = 0$$

Standard Deviation

The standard deviation, σ_T , of the Arrhenius-exponential model is given by:

$$\sigma_T = Ce^{\frac{B}{V}}$$

Arrhenius-Exponential Reliability Function

The Arrhenius-exponential reliability function is given by:

$$R(T,V) = e^{-\frac{T}{Ce^{\frac{B}{V}}}}$$

This function is the complement of the Arrhenius-exponential cumulative distribution function or:

$$R(T,V) = 1 - Q(T,V) = 1 - \int_0^T f(T,V)dT$$

and:

$$R(T,V) = 1 - \int_0^T \frac{1}{Ce^{\frac{B}{V}}} e^{-\frac{T}{Ce^{\frac{B}{V}}}} dT = e^{-\frac{T}{Ce^{\frac{B}{V}}}}$$

Conditional Reliability

The Arrhenius-exponential conditional reliability function is given by:

$$R((t|T),V)=\frac{R(T+t,V)}{R(T,V)}=\frac{e^{-\lambda(T+t)}}{e^{-\lambda T}}=e^{-\frac{t}{Ce^{\frac{B}{V}}}}$$

Reliable Life

For the Arrhenius-exponential model, the reliable life, or the mission duration for a desired reliability goal, t_R, is given by:

$$R(t_R,V)=e^{-\frac{t_R}{Ce^{\frac{B}{V}}}}$$

$$\ln[R(t_R,V)]=-\frac{t_R}{Ce^{\frac{B}{V}}}$$

or:

$$t_R=-Ce^{\frac{B}{V}}\ln[R(t_R,V)]$$

Parameter Estimation

Maximum Likelihood Estimation Method

The log-likelihood function for the exponential distribution is as shown next:

$$\ln(L)=\quad \Lambda=\sum_{i=1}^{F_e}N_i\ln\left[\lambda e^{-\lambda T_i}\right]-\sum_{i=1}^{S}N_i'\lambda T_i'+\sum_{i=1}^{FI}N_i''\ln[R_{Li}''-R_{Ri}'']$$

where:

$$R_{Li}''=e^{-\lambda T_{Li}''}$$

$$R_{Ri}''=e^{-\lambda T_{Ri}''}$$

and:

- F_e is the number of groups of exact times-to-failure data points.
- N_i is the number of times-to-failure in the i^{th} time-to-failure data group.
- λ is the failure rate parameter (unknown).
- T_i is the exact failure time of the i^{th} group.
- S is the number of groups of suspension data points.
- N_i' is the number of suspensions in the i^{th} group of suspension data points.
- T_i' is the time of the i^{th} suspension data group.
- FI is the number of interval data groups.
- N_i'' is the number of intervals in the i^{th} group of data intervals.
- T_{Li}'' is the beginning of the i^{th} interval.
- T_{Ri}'' is the ending of the i^{th} interval.

Substituting the Arrhenius-exponential model into the log-likelihood function yields:

$$\Lambda = \sum_{i=1}^{F_e} N_i \ln\left[\frac{1}{C \cdot e^{\frac{B}{V_i}}} e^{-\frac{1}{C \cdot e^{\frac{B}{V_i}}} T_i}\right] - \sum_{i=1}^{S} N_i' \frac{1}{C \cdot e^{\frac{B}{V_i}}} T_i' + \sum_{i=1}^{FI} N_i'' \ln[R_{Li}'' - R_{Ri}'']$$

where:

$$R_{Li}'' = e^{-\frac{T_{Li}''}{Ce^{\frac{B}{V_i}}}}$$

$$R_{Ri}'' = e^{-\frac{T_{Ri}''}{Ce^{\frac{B}{V_i}}}}$$

The solution (parameter estimates) will be found by solving for the parameters $\widehat{B}, \widehat{C}$ so that $\frac{\partial \Lambda}{\partial B} = 0$ and $\frac{\partial \Lambda}{\partial C} = 0$, where:

$$\frac{\partial \Lambda}{\partial B} = \frac{1}{C}\sum_{i=1}^{F_e} N_i \left(\frac{T_i}{V_i e^{\frac{B}{V_i}}} - \frac{C}{V_i} \right) + \frac{1}{C}\sum_{i=1}^{S} N'_i \frac{T'_i}{V_i e^{\frac{B}{V_i}}} + \sum_{i=1}^{FI} N''_i \frac{T''_{Li} R''_{Li} - T''_{Ri} R''_{Ri}}{(R''_{Li} - R''_{Ri}) C V_i e^{\frac{B}{V_i}}}$$

$$\frac{\partial \Lambda}{\partial C} = \frac{1}{C}\sum_{i=1}^{F_e} N_i \left(\frac{T_i}{C e^{\frac{B}{V_i}}} - 1 \right) + \frac{1}{C^2}\sum_{i=1}^{S} N'_i \frac{T'_i}{e^{\frac{B}{V_i}}} + \sum_{i=1}^{FI} N''_i \frac{T''_{Li} R''_{Li} - T''_{Ri} R''_{Ri}}{(R''_{Li} - R''_{Ri}) C^2 e^{\frac{B}{V_i}}}$$

Arrhenius-Weibull

The *pdf* for the 2-parameter Weibull distribution is given by:

$$f(t) = \frac{\beta}{\eta} \left(\frac{t}{\eta} \right)^{\beta - 1} e^{-\left(\frac{t}{\eta} \right)^{\beta}}$$

The scale parameter (or characteristic life) of the Weibull distribution is η.

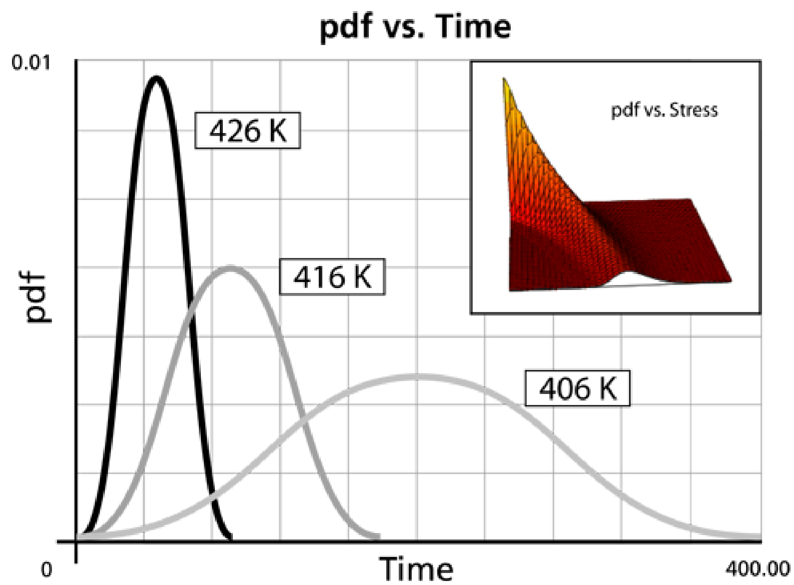
The Arrhenius-Weibull model *pdf* can then be obtained by setting $\eta = L(V)$:

$$\eta = L(V) = C \cdot e^{\frac{B}{V}}$$

and substituting for η in the 2-parameter Weibull distribution equation:

$$f(t, V) = \frac{\beta}{C \cdot e^{\frac{B}{V}}} \left(\frac{t}{C \cdot e^{\frac{B}{V}}} \right)^{\beta - 1} e^{-\left(\frac{t}{C \cdot e^{\frac{B}{V}}} \right)^{\beta}}$$

An illustration of the *pdf* for different stresses is shown in the next figure. As expected, the *pdf* at lower stress levels is more stretched to the right, with a higher scale parameter, while its shape remains the same (the shape parameter is approximately 3). This behavior is observed when the parameter B of the Arrhenius model is positive.



The advantage of using the Weibull distribution as the life distribution lies in its flexibility to assume different shapes. The Weibull distribution is presented in greater detail in The Weibull Distribution.

Arrhenius-Weibull Statistical Properties Summary

Mean or MTTF

The mean, $\overline{T}$ (also called $MTTF$ by some authors), of the Arrhenius-Weibull relationship is given by:

$$\overline{T} = C \cdot e^{\frac{B}{V}} \cdot \Gamma \left(\frac{1}{\beta} + 1 \right)$$

where $\Gamma \left(\frac{1}{\beta} + 1 \right)$ is the gamma function evaluated at the value of $\left(\frac{1}{\beta} + 1 \right)$.

Median

The median, $\breve{T},$ for the Arrhenius-Weibull model is given by:

$$\breve{T} = C \cdot e^{\frac{B}{V}} (\ln 2)^{\frac{1}{\beta}}$$

Mode

The mode, $\tilde{T}$, for the Arrhenius-Weibull model is given by:

$$\tilde{T} = C \cdot e^{\frac{B}{V}} \left(1 - \frac{1}{\beta}\right)^{\frac{1}{\beta}}$$

Standard Deviation

The standard deviation, σ_T, for the Arrhenius-Weibull model is given by:

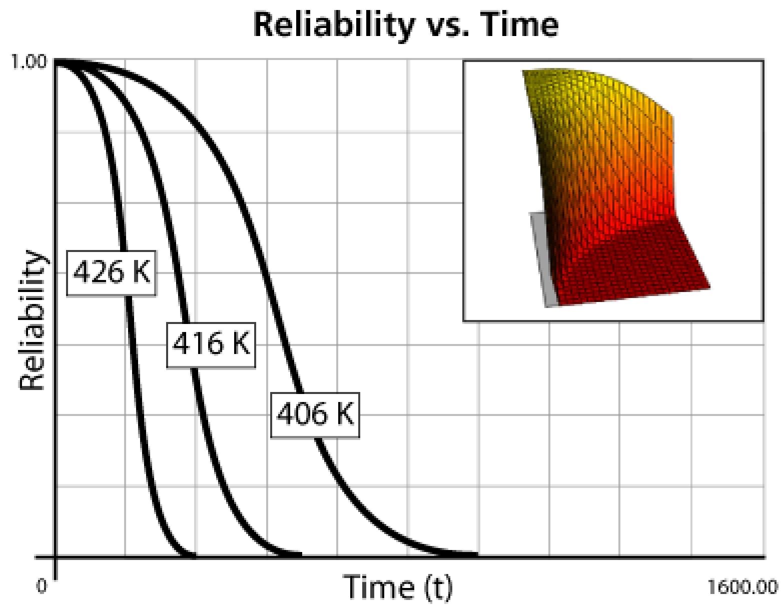
$$\sigma_T = C \cdot e^{\frac{B}{V}} \cdot \sqrt{\Gamma\left(\frac{2}{\beta} + 1\right) - \left(\Gamma\left(\frac{1}{\beta} + 1\right)\right)^2}$$

Arrhenius-Weibull Reliability Function

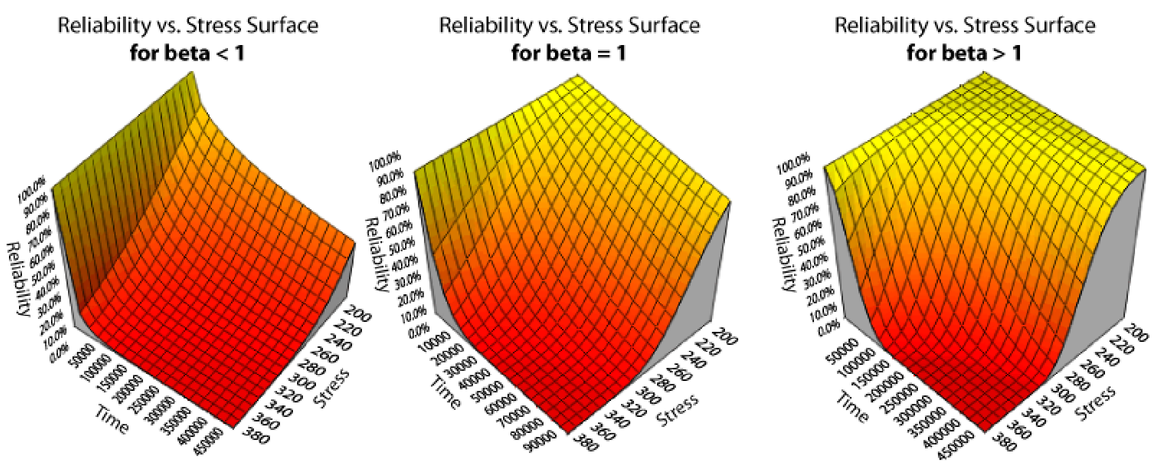
The Arrhenius-Weibull reliability function is given by:

$$R(T, V) = e^{-\left(\frac{T}{C \cdot e^{\frac{B}{V}}}\right)^{\beta}}$$

If the parameter B is positive, then the reliability increases as stress decreases.



The behavior of the reliability function of the Weibull distribution for different values of β was illustrated here. In the case of the Arrhenius-Weibull model, however, the reliability is a function of stress also. A 3D plot such as the ones shown in the next figure is now needed to illustrate the effects of both the stress and β.



Conditional Reliability Function

The Arrhenius-Weibull conditional reliability function at a specified stress level is given by:

$$R((t|T),V)=\frac{R(T+t,V)}{R(T,V)}=\frac{e^{-\left(\frac{T+t}{\eta}\right)^{\beta}}}{e^{-\left(\frac{T}{\eta}\right)^{\beta}}}$$

or:

$$R((t|T),V)=e^{-\left[\left(\frac{T+t}{C\cdot e^{\frac{B}{V}}}\right)^{\beta}-\left(\frac{T}{C\cdot e^{\frac{B}{V}}}\right)^{\beta}\right]}$$

Reliable Life

For the Arrhenius-Weibull relationship, the reliable life, t_R , of a unit for a specified reliability and starting the mission at age zero is given by:

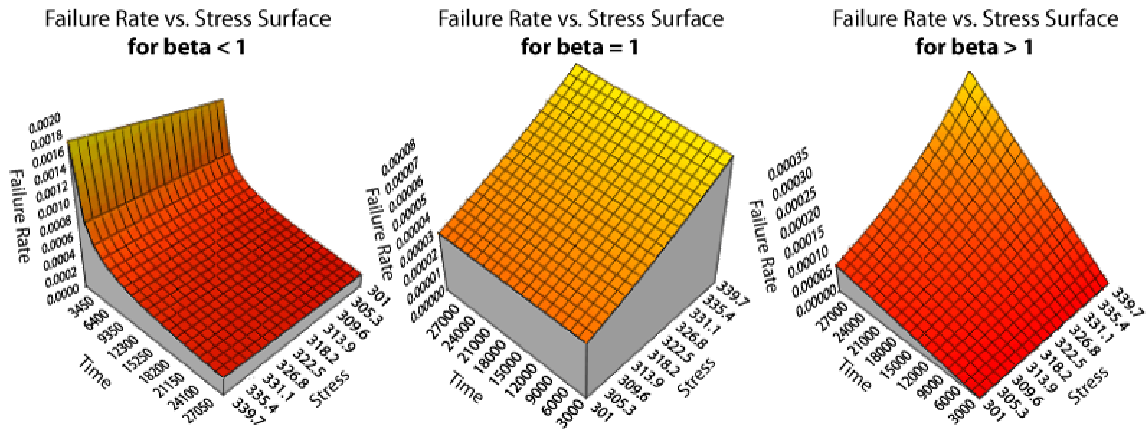
$$t_R=C\cdot e^{\frac{B}{V}}\left\{-\ln[R\left(t_R,V\right)]\right\}^{\frac{1}{\beta}}$$

This is the life for which the unit will function successfully with a reliability of $R(t_R)$. If $R(t_R)=0.50$ then $t_R=\breve{T}$, the median life, or the life by which half of the units will survive.

Arrhenius-Weibull Failure Rate Function

The Arrhenius-Weibull failure rate function, $\lambda(T)$, is given by:

$$\lambda\left(T,V\right)=\frac{f\left(T,V\right)}{R\left(T,V\right)}=\frac{\beta}{C\cdot e^{\frac{B}{V}}}\left(\frac{T}{C\cdot e^{\frac{B}{V}}}\right)^{\beta-1}$$



Parameter Estimation

Maximum Likelihood Estimation Method

The Arrhenius-Weibull log-likelihood function is as follows:

$$\Lambda = \sum_{i=1}^{F_e} N_i \ln\left[\frac{\beta}{C\cdot e^{\frac{B}{V_i}}}\left(\frac{T_i}{C\cdot e^{\frac{B}{V_i}}}\right)^{\beta-1} e^{-\left(\frac{T_i}{C\cdot e^{\frac{B}{V_i}}}\right)^{\beta}}\right] - \sum_{i=1}^{S} N_i'\left(\frac{T_i'}{C\cdot e^{\frac{B}{V_i}}}\right)^{\beta} + \sum_{i=1}^{FI} N_i'' \ln[R''_{Li} - R''_{Ri}]$$

where:

$$R''_{Li} = e^{-\left(\frac{T''_{Li}}{Ce^{\frac{B}{V_i}}}\right)^{\beta}}$$

$$R''_{Ri} = e^{-\left(\frac{T''_{Ri}}{Ce^{\frac{B}{V_i}}}\right)^{\beta}}$$

and:

- F_e is the number of groups of exact times-to-failure data points.
- N_i is the number of times-to-failure data points in the i^{th} time-to-failure data group.

- β is the Weibull shape parameter (unknown, the first of three parameters to be estimated).
- B is the Arrhenius parameter (unknown, the second of three parameters to be estimated).
- C is the second Arrhenius parameter (unknown, the third of three parameters to be estimated).
- V_i is the stress level of the i^{th} group.
- T_i is the exact failure time of the i^{th} group.
- S is the number of groups of suspension data points.
- N_i' is the number of suspensions in the i^{th} group of suspension data points.
- T_i' is the running time of the i^{th} suspension data group.
- FI is the number of interval data groups.
- N_i'' is the number of intervals in the i^{th} group of data intervals.
- T_{Li}'' is the beginning of the i^{th} interval.
- T_{Ri}'' is the ending of the i^{th} interval.

The solution (parameter estimates) will be found by solving for $\widehat{\beta}, \widehat{B}, \widehat{C}$ so that $\frac{\partial \Lambda}{\partial \beta} = 0,$ $\frac{\partial \Lambda}{\partial B} = 0$ and $\frac{\partial \Lambda}{\partial C} = 0$, where:

$$\frac{\partial \Lambda}{\partial \beta} = \frac{1}{\beta}\sum_{i=1}^{F_e} N_i + \sum_{i=1}^{F_e} N_i \ln\left(\frac{T_i}{C \cdot e^{\frac{B}{V_i}}}\right) - \sum_{i=1}^{F_e} N_i \left(\frac{T_i}{C \cdot e^{\frac{B}{V_i}}}\right)^{\beta} \ln\left(\frac{T_i}{C \cdot e^{\frac{B}{V_i}}}\right) - \sum_{i=1}^{S} N_i' \left(\frac{T_i'}{C \cdot e^{\frac{B}{V_i}}}\right)^{\beta} \ln\left(\frac{T_i'}{C \cdot e^{\frac{B}{V_i}}}\right)$$
$$- \sum_{i=1}^{FI} N_i'' \frac{\left(\frac{T_{Li}''}{Ce^{\frac{B}{V_i}}}\right)^{\beta} \ln\left(\frac{T_{Li}''}{Ce^{\frac{B}{V_i}}}\right) R_{Li}'' - \left(\frac{T_{Ri}''}{Ce^{\frac{B}{V_i}}}\right)^{\beta} \ln\left(\frac{T_{Ri}''}{Ce^{\frac{B}{V_i}}}\right) R_{Ri}''}{R_{Li}'' - R_{Ri}''}$$

$$\frac{\partial \Lambda}{\partial B}=-\beta\sum_{i=1}^{F_e}N_i\frac{1}{V_i}+\beta\sum_{i=1}^{F_e}N_i\frac{1}{V_i}\left(\frac{T_i}{\widehat{C}e^{\frac{\widehat{B}}{V_i}}}\right)^{\beta}+\beta\sum_{i=1}^{S}N_i'\frac{1}{V_i}\left(\frac{T_i'}{\widehat{C}e^{\frac{\widehat{B}}{V_i}}}\right)^{\beta}+\sum_{i=1}^{FI}N_i''\frac{\beta}{V_i}\frac{(T_{Li}'')^{\beta}R_{Li}''-(T_{Ri}'')^{\beta}R_{Ri}''}{\left(Ce^{\frac{B}{V_i}}\right)^{\beta}(R_{Li}''-R_{Ri}'')}$$

$$\frac{\partial \Lambda}{\partial C}=-\frac{\beta}{C}\sum_{i=1}^{F_e}N_i+\frac{\beta}{C}\sum_{i=1}^{F_e}N_i\left(\frac{T_i}{C\cdot e^{\frac{B}{V_i}}}\right)^{\beta}+\frac{\beta}{C}\sum_{i=1}^{S}N_i'\left(\frac{T_i'}{C\cdot e^{\frac{B}{V_i}}}\right)^{\beta}+\sum_{i=1}^{FI}N_i''\frac{\beta}{C}\frac{(T_{Li}'')^{\beta}R_{Li}''-(T_{Ri}'')^{\beta}R_{Ri}''}{\left(Ce^{\frac{B}{V_i}}\right)^{\beta}(R_{Li}''-R_{Ri}'')}$$

Arrhenius-Weibull example

Consider the following times-to-failure data at three different stress levels.

Stress	393 K	408 K	423 K
Time Failed (hrs)	3850	3300	2750
	4340	3720	3100
	4760	4080	3400
	5320	4560	3800
	5740	4920	4100
	6160	5280	4400
	6580	5640	4700
	7140	6120	5100
	7980	6840	5700
	8960	7680	6400

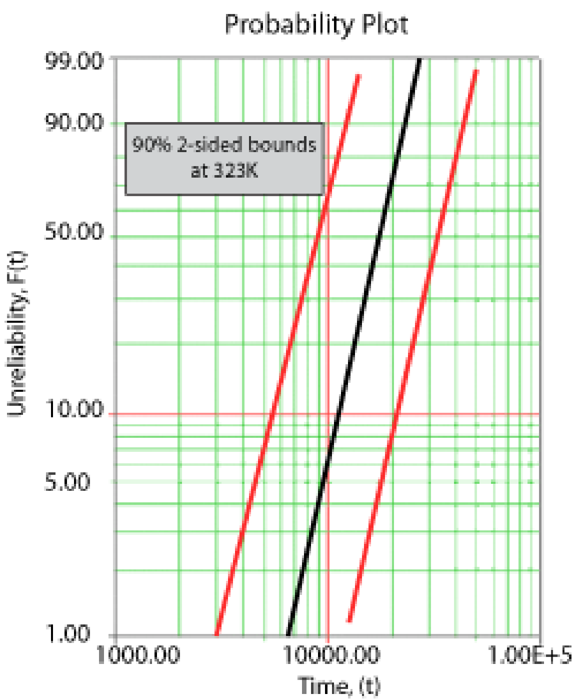
The data set was analyzed jointly and with a complete MLE solution over the entire data set, using ReliaSoft's Weibull++. The analysis yields:

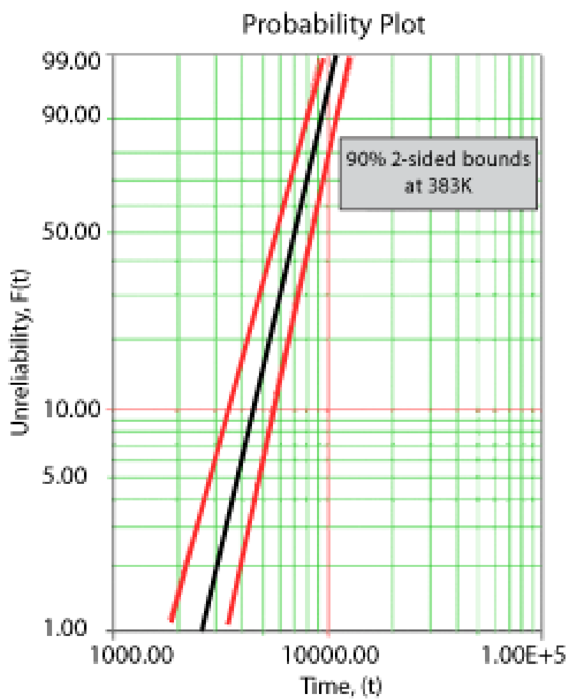
$$\widehat{\beta}=4.2915822$$

$$\widehat{B}=1861.6186657$$

$$\widehat{C}=58.9848692$$

Once the parameters of the model are estimated, extrapolation and other life measures can be directly obtained using the appropriate equations. Using the MLE method, confidence bounds for all estimates can be obtained. Note that in the next figure, the more distant the accelerated stress is from the operating stress, the greater the uncertainty of the extrapolation. The degree of uncertainty is reflected in the confidence bounds. (General theory and calculations for confidence intervals are presented in Appendix A. Specific calculations for confidence bounds on the Arrhenius model are presented in the Arrhenius Confidence Bounds section).





Arrhenius-Lognormal

The *pdf* of the lognormal distribution is given by:

$$f(T) = \frac{1}{T\,\sigma_{T'}\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{T'-\bar{T'}}{\sigma_{T'}}\right)^2}$$

where:

$$T' = \ln(T)$$

and:

- $T =$ times-to-failure.
- $T' =$ mean of the natural logarithms of the times-to-failure.
- $\sigma_{T'} =$ standard deviation of the natural logarithms of the times-to-failure.

The median of the lognormal distribution is given by:

$$\breve{T} = e^{\overline{T}'}$$

The Arrhenius-lognormal model *pdf* can be obtained first by setting $\breve{T} = L(V)$. Therefore:

$$\breve{T} = L(V) = Ce^{\frac{B}{V}}$$

or:

$$e^{\overline{T}'} = Ce^{\frac{B}{V}}$$

Thus:

$$\overline{T}' = \ln(C) + \frac{B}{V}$$

Substituting the above equation into the lognormal *pdf* yields the Arrhenius-lognormal model *pdf* or:

$$f(T,V) = \frac{1}{T\ \sigma_{T'}\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{T'-\ln(C)-\frac{B}{V}}{\sigma_{T'}}\right)^2}$$

Note that in the Arrhenius-lognormal *pdf*, it was assumed that the standard deviation of the natural logarithms of the times-to-failure, $\sigma_{T'}$, is independent of stress. This assumption implies that the shape of the distribution does not change with stress ($\sigma_{T'}$ is the shape parameter of the lognormal distribution).

Arrhenius-Lognormal Statistical Properties Summary

The Mean

The mean life of the Arrhenius-lognormal model (mean of the times-to-failure), $\bar{T}$, is given by:

$$\bar{T} = e^{\bar{T}'+\frac{1}{2}\sigma_{T'}^2} = e^{\ln(C)+\frac{B}{V}+\frac{1}{2}\sigma_{T'}^2}$$

The mean of the natural logarithms of the times-to-failure, $\bar{T}'$, in terms of $\bar{T}$ and σ_T is given by:

$$\bar{T}' = \ln(\bar{T}) - \frac{1}{2}\ln\left(\frac{\sigma_T^2}{\bar{T}^2}+1\right)$$

The Standard Deviation

The standard deviation of the Arrhenius-lognormal model (standard deviation of the times-to-failure), σ_T , is given by:

$$\sigma_T = \sqrt{\left(e^{2\bar{T}'+\sigma_{T'}^2}\right)\left(e^{\sigma_{T'}^2}-1\right)} = \sqrt{\left(e^{2\left(\ln(C)+\frac{B}{V}\right)+\sigma_{T'}^2}\right)\left(e^{\sigma_{T'}^2}-1\right)}$$

The standard deviation of the natural logarithms of the times-to-failure, $\sigma_{T'}$, in terms of $\bar{T}$ and σ_T is given by:

$$\sigma_{T'} = \sqrt{\ln\left(\frac{\sigma_T^2}{\bar{T}^2}+1\right)}$$

The Mode

- The mode of the Arrhenius-lognormal model is given by:

$$\tilde{T} = e^{\overline{T}'-\sigma_{T'}^2} = e^{\ln(C)+\frac{B}{V}-\sigma_{T'}^2}$$

Arrhenius-Lognormal Reliability Function

The reliability for a mission of time T , starting at age 0, for the Arrhenius-lognormal model is determined by:

$$R(T,V) = \int_T^{\infty} f(t,V)dt$$

or:

$$R(T,V)=\int_{T'}^{\infty}\frac{1}{\sigma_{T'}\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{t-\ln(C)-\frac{B}{V}}{\sigma_{T'}}\right)^2}dt$$

There is no closed form solution for the lognormal reliability function. Solutions can be obtained via the use of standard normal tables. Since the application automatically solves for the reliability, we will not discuss manual solution methods.

Reliable Life

For the Arrhenius-lognormal model, the reliable life, or the mission duration for a desired reliability goal, t_R, is estimated by first solving the reliability equation with respect to time, as follows:

$$T'_R=ln(C)+\frac{B}{V}+z\cdot\sigma_{T'}$$

where:

$$z=\Phi^{-1}\left[F\left(T'_R,V\right)\right]$$

and:

$$\Phi(z)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z(T',V)}e^{-\frac{t^2}{2}}dt$$

Since $T'=\ln(T)$ the reliable life, t_R, is given by:

$$t_R=e^{T'_R}$$

Arrhenius-Lognormal Failure Rate

The Arrhenius-lognormal failure rate is given by:

$$\lambda(T,V)=\frac{f(T,V)}{R(T,V)}=\frac{\frac{1}{T\ \sigma_{T'}\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{T'-\ln(C)-\frac{B}{V}}{\sigma_{T'}}\right)^2}}{\int_{T'}^{\infty}\frac{1}{\sigma_{T'}\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{T'-\ln(C)-\frac{B}{V}}{\sigma_{T'}}\right)^2}dt}$$

Parameter Estimation

Maximum Likelihood Estimation Method

The lognormal log-likelihood function for the Arrhenius-lognormal model is as follows:

$$\ln(L)=\quad \Lambda=\sum_{i=1}^{F_e}N_i\ln\left[\frac{1}{\sigma_{T'}T_i}\phi\left(\frac{\ln(T_i)-\ln(C)-\frac{B}{V_i}}{\sigma_{T'}}\right)\right]+\sum_{i=1}^{S}N_i'\ln\left[1-\Phi\left(\frac{\ln(T_i')-\ln(C)-\frac{B}{V_i}}{\sigma_{T'}}\right)\right]+\sum_{i=1}^{FI}N_i''\ln[\Phi(z_{Ri}'')-\Phi(z_{Li}'')]$$

where:

$$z_{Li}''=\frac{\ln T_{Li}''-\ln C-\frac{B}{V_i}}{\sigma_T'}$$

$$z_{Ri}''=\frac{\ln T_{Ri}''-\ln C-\frac{B}{V_i}}{\sigma_T'}$$

and:

- F_e is the number of groups of exact times-to-failure data points.
- N_i is the number of times-to-failure data points in the i^{th} time-to-failure data group.
- $\sigma_{T'}$ is the standard deviation of the natural logarithm of the times-to-failure (unknown, the first of three parameters to be estimated).
- B is the Arrhenius parameter (unknown, the second of three parameters to be estimated).
- C is the second Arrhenius parameter (unknown, the third of three parameters to be estimated).

- V_i is the stress level of the i^{th} group.
- T_i is the exact failure time of the i^{th} group.
- S is the number of groups of suspension data points.
- N'_i is the number of suspensions in the i^{th} group of suspension data points.
- T'_i is the running time of the i^{th} suspension data group.
- FI is the number of interval data groups.
- N''_i is the number of intervals in the i^{th} group of data intervals.
- T''_{Li} is the beginning of the i^{th} interval.
- T''_{Ri} is the ending of the i^{th} interval.

The solution (parameter estimates) will be found by solving for $\widehat{\sigma}_{T'}$, $\widehat{B}, \widehat{C}$ so that $\frac{\partial \Lambda}{\partial \sigma_{T'}} = 0$, $\frac{\partial \Lambda}{\partial B} = 0$ and $\frac{\partial \Lambda}{\partial C} = 0$, where:

$$\frac{\partial \Lambda}{\partial B} = \frac{1}{\sigma_{T'}^2} \sum_{i=1}^{F_e} N_i \frac{1}{V_i} (\ln(T_i) - \ln(C) - \frac{B}{V_i}) + \frac{1}{\sigma_{T'}} \sum_{i=1}^{S} N'_i \frac{1}{V_i} \frac{\phi\left(\frac{\ln(T'_i) - \ln(C) - \frac{B}{V_i}}{\sigma_{T'}}\right)}{1 - \Phi\left(\frac{\ln(T'_i) - \ln(C) - \frac{B}{V_i}}{\sigma_{T'}}\right)} - \sum_{i=1}^{FI} N''_i \frac{\varphi(z''_{Ri}) - \varphi(z''_{Li})}{\sigma'_T V_i (\Phi(z''_{Ri}) - \Phi(z''_{Li}))}$$

$$\frac{\partial \Lambda}{\partial C} = \frac{1}{C \cdot \sigma_{T'}^2} \sum_{i=1}^{F_e} N_i (\ln(T_i) - \ln(C) - \frac{B}{V_i}) + \frac{1}{C \cdot \sigma_{T'}} \sum_{i=1}^{S} N'_i \frac{\phi\left(\frac{\ln(T'_i) - \ln(C) - \frac{B}{V_i}}{\sigma_{T'}}\right)}{1 - \Phi\left(\frac{\ln(T'_i) - \ln(C) - \frac{B}{V_i}}{\sigma_{T'}}\right)} - \sum_{i=1}^{FI} N''_i \frac{\varphi(z''_{Ri}) - \varphi(z''_{Li})}{\sigma'_T C (\Phi(z''_{Ri}) - \Phi(z''_{Li}))}$$

$$\frac{\partial \Lambda}{\partial \sigma_{T'}} = \sum_{i=1}^{F_e} N_i \left(\frac{\left(\ln(T_i) - \ln(C) - \frac{B}{V_i}\right)^2}{\sigma_{T'}^3} - \frac{1}{\sigma_{T'}} \right) + \frac{1}{\sigma_{T'}} \sum_{i=1}^{S} N'_i \frac{\left(\frac{\ln(T'_i) - \ln(C) - \frac{B}{V_i}}{\sigma_{T'}}\right) \phi\left(\frac{\ln(T'_i) - \ln(C) - \frac{B}{V_i}}{\sigma_{T'}}\right)}{1 - \Phi\left(\frac{\ln(T'_i) - \ln(C) - \frac{B}{V_i}}{\sigma_{T'}}\right)} - \sum_{i=1}^{FI} N''_i \frac{z''_{Ri}\varphi(z''_{Ri}) - z''_{Li}\varphi(z''_{Li})}{\sigma'_T (\Phi(z''_{Ri}) - \Phi(z''_{Li}))}$$

and:

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}(x)^2}$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt$$

Arrhenius Confidence Bounds

Approximate Confidence Bounds for the Arrhenius-Exponential

There are different methods for computing confidence bounds. Weibull++ utilizes confidence bounds that are based on the asymptotic theory for maximum likelihood estimates, most commonly referred to as the Fisher matrix bounds.

Confidence Bounds on the Mean Life

The Arrhenius-exponential distribution is given by setting $m = L(V)$ in the exponential *pdf* equation. The upper (m_U) and lower (m_L) bounds on the mean life are then estimated by:

$$m_U = \widehat{m} \cdot e^{\frac{K_\alpha \sqrt{Var(\widehat{m})}}{\widehat{m}}}$$

$$m_L = \widehat{m} \cdot e^{-\frac{K_\alpha \sqrt{Var(\widehat{m})}}{\widehat{m}}}$$

where K_α is defined by:

$$\alpha = \frac{1}{\sqrt{2\pi}} \int_{K_\alpha}^{\infty} e^{-\frac{t^2}{2}} dt = 1 - \Phi(K_\alpha)$$

If δ is the confidence level (i.e., 95%=0.95), then $\alpha = \frac{1-\delta}{2}$ for the two-sided bounds, and $\alpha = 1 - \delta$ for the one-sided bounds. The variance of $\widehat{m}$ is given by:

$$Var(\widehat{m}) = \left(\frac{\partial m}{\partial C}\right)^2 Var(\widehat{C}) + \left(\frac{\partial m}{\partial B}\right)^2 Var(\widehat{B}) + 2\left(\frac{\partial m}{\partial C}\right)\left(\frac{\partial m}{\partial B}\right) Cov(\widehat{B}, \widehat{C})$$

or:

$$Var(\widehat{m})=e^{\frac{2\widehat{B}}{V}}\left[Var(\widehat{C})+\frac{\widehat{C}^2}{V^2}Var(\widehat{B})+\frac{2\widehat{C}}{V}Cov(\widehat{B},\widehat{C})\right]$$

The variances and covariance of B and C are estimated from the local Fisher matrix (evaluated at $\widehat{B}$, $\widehat{C}$) as follows:

$$\begin{bmatrix} Var(\widehat{B}) & Cov(\widehat{B},\widehat{C}) \\ Cov(\widehat{C},\widehat{B}) & Var(\widehat{C}) \end{bmatrix}=\begin{bmatrix} -\frac{\partial^2\Lambda}{\partial B^2} & -\frac{\partial^2\Lambda}{\partial B\partial C} \\ -\frac{\partial^2\Lambda}{\partial C\partial B} & -\frac{\partial^2\Lambda}{\partial C^2} \end{bmatrix}^{-1}$$

Confidence Bounds on Reliability

The bounds on reliability for any given time, T, are estimated by:

$$\begin{aligned} R_U(T)= & \quad e^{-\frac{T}{m_U}} \\ R_L(T)= & \quad e^{-\frac{T}{m_L}} \end{aligned}$$

where m_U and m_L are estimated estimated by:

$$\begin{aligned} m_U &= \widehat{m}\cdot e^{\frac{K_\alpha\sqrt{Var(\widehat{m})}}{\widehat{m}}} \\ m_L &= \widehat{m}\cdot e^{-\frac{K_\alpha\sqrt{Var(\widehat{m})}}{\widehat{m}}} \end{aligned}$$

Confidence Bounds on Time

The bounds on time (ML estimate of time) for a given reliability are estimated by first solving the reliability function with respect to time:

$$\widehat{T}=-\widehat{m}\cdot\ln(R)$$

The corresponding confidence bounds are then estimated from:

$$\begin{aligned} T_U &= -m_U\cdot\ln(R) \\ T_L &= -m_L\cdot\ln(R) \end{aligned}$$

where m_U and m_L are estimated estimated by:

$$m_U = \widehat{m} \cdot e^{\frac{K_\alpha \sqrt{Var(\widehat{m})}}{\widehat{m}}}$$
$$m_L = \widehat{m} \cdot e^{-\frac{K_\alpha \sqrt{Var(\widehat{m})}}{\widehat{m}}}$$

Approximate Confidence Bounds for the Arrhenius-Weibull

Bounds on the Parameters

From the asymptotically normal property of the maximum likelihood estimators, and since $\widehat{\beta}$, and $\widehat{C}$ are positive parameters, $\ln(\widehat{\beta})$, and $\ln(\widehat{C})$ can then be treated as normally distributed. After performing this transformation, the bounds on the parameters can be estimated from:

$$\beta_U = \widehat{\beta} \cdot e^{\frac{K_\alpha \sqrt{Var(\widehat{\beta})}}{\widehat{\beta}}}$$
$$\beta_L = \widehat{\beta} \cdot e^{-\frac{K_\alpha \sqrt{Var(\widehat{\beta})}}{\widehat{\beta}}}$$

also:

$$B_U = \widehat{B} + K_\alpha \sqrt{Var(\widehat{B})}$$
$$B_L = \widehat{B} - K_\alpha \sqrt{Var(\widehat{B})}$$

and:

$$C_U = \widehat{C} \cdot e^{\frac{K_\alpha \sqrt{Var(\widehat{C})}}{\widehat{C}}}$$
$$C_L = \widehat{C} \cdot e^{-\frac{K_\alpha \sqrt{Var(\widehat{C})}}{\widehat{C}}}$$

The variances and covariances of $\beta, B,$ and C are estimated from the local Fisher matrix (evaluated at $\widehat{\beta}, \widehat{B}, \widehat{C})$, as follows:

$$\begin{bmatrix} Var(\widehat{\beta}) & Cov(\widehat{\beta},\widehat{B}) & Cov(\widehat{\beta},\widehat{C}) \\ Cov(\widehat{B},\widehat{\beta}) & Var(\widehat{B}) & Cov(\widehat{B},\widehat{C}) \\ Cov(\widehat{C},\widehat{\beta}) & Cov(\widehat{C},\widehat{B}) & Var(\widehat{C}) \end{bmatrix} = \begin{bmatrix} -\frac{\partial^2\Lambda}{\partial\beta^2} & -\frac{\partial^2\Lambda}{\partial\beta\partial B} & -\frac{\partial^2\Lambda}{\partial\beta\partial C} \\ -\frac{\partial^2\Lambda}{\partial B\partial\beta} & -\frac{\partial^2\Lambda}{\partial B^2} & -\frac{\partial^2\Lambda}{\partial B\partial C} \\ -\frac{\partial^2\Lambda}{\partial C\partial\beta} & -\frac{\partial^2\Lambda}{\partial C\partial B} & -\frac{\partial^2\Lambda}{\partial C^2} \end{bmatrix}^{-1}$$

Confidence Bounds on Reliability

The reliability function for the Arrhenius-Weibull model (ML estimate) is given by:

$$\widehat{R}(T,V) = e^{-\left(\frac{T}{\widehat{C}\cdot e^{\frac{\widehat{B}}{V}}}\right)^{\widehat{\beta}}}$$

or:

$$\widehat{R}(T) = e^{-e^{\ln\left[\left(\frac{T}{\widehat{C}\cdot e^{\frac{\widehat{B}}{V}}}\right)^{\widehat{\beta}}\right]}}$$

Setting:

$$\widehat{u} = \ln\left[\left(\frac{T}{\widehat{C}\cdot e^{\frac{\widehat{B}}{V}}}\right)^{\widehat{\beta}}\right]$$

or:

$$\widehat{u} = \widehat{\beta}\left[\ln(T) - \ln(\widehat{C}) - \frac{\widehat{B}}{V}\right]$$

The reliability function now becomes:

$$\widehat{R}(T,V) = e^{-e^{\widehat{u}}}$$

The next step is to find the upper and lower bounds on $\widehat{u}$:

$$u_U = \widehat{u} + K_\alpha \sqrt{Var(\widehat{u})}$$

$$u_L = \widehat{u} - K_\alpha \sqrt{Var(\widehat{u})}$$

where:

$$Var(\widehat{u}) = \left(\frac{\partial \widehat{u}}{\partial \beta}\right)^2 Var(\widehat{\beta}) + \left(\frac{\partial \widehat{u}}{\partial B}\right)^2 Var(\widehat{B}) + \left(\frac{\partial \widehat{u}}{\partial C}\right)^2 Var(\widehat{C}) + 2\left(\frac{\partial \widehat{u}}{\partial \beta}\right)\left(\frac{\partial \widehat{u}}{\partial B}\right) Cov(\widehat{\beta},\widehat{B}) + 2\left(\frac{\partial \widehat{u}}{\partial \beta}\right)\left(\frac{\partial \widehat{u}}{\partial C}\right) Cov(\widehat{\beta},\widehat{C}) + 2\left(\frac{\partial \widehat{u}}{\partial B}\right)\left(\frac{\partial \widehat{u}}{\partial C}\right) Cov(\widehat{B},\widehat{C})$$

or:

$$Var(\widehat{u}) = \left(\frac{\widehat{u}}{\widehat{\beta}}\right)^2 Var(\widehat{\beta}) + \left(\frac{\widehat{\beta}}{V}\right)^2 Var(\widehat{B}) + \left(\frac{\widehat{\beta}}{\widehat{C}}\right)^2 Var(\widehat{C}) - \frac{2\widehat{u}}{V} Cov(\widehat{\beta},\widehat{B}) - \frac{2\widehat{u}}{\widehat{C}} Cov(\widehat{\beta},\widehat{C}) + \frac{2\widehat{\beta}^2}{V\widehat{C}} Cov(\widehat{B},\widehat{C})$$

The upper and lower bounds on reliability are:

$$R_U(T,V) = e^{-e^{(u_L)}}$$
$$R_L(T,V) = e^{-e^{(u_U)}}$$

Confidence Bounds on Time

The bounds on time for a given reliability are estimated by first solving the reliability function with respect to time:

$$\ln(R) = -\left(\frac{\widehat{T}}{\widehat{C}\cdot e^{\frac{\widehat{B}}{V}}}\right)^{\widehat{\beta}}$$
$$\ln(-\ln(R)) = \widehat{\beta}\left(\ln\widehat{T} - \ln\widehat{C} - \frac{\widehat{B}}{V}\right)$$

or:

$$\widehat{u} = \frac{1}{\widehat{\beta}}\ln(-\ln(R)) + \ln\widehat{C} + \frac{\widehat{B}}{V}$$

where $\widehat{u} = \ln\widehat{T}$.

The upper and lower bounds on u are estimated from:

$$u_U = \widehat{u} + K_\alpha \sqrt{Var(\widehat{u})}$$

$$u_L = \widehat{u} - K_\alpha \sqrt{Var(\widehat{u})}$$

where:

$$Var(\widehat{u}) = \left(\frac{\partial \widehat{u}}{\partial \beta}\right)^2 Var(\widehat{\beta}) + \left(\frac{\partial \widehat{u}}{\partial B}\right)^2 Var(\widehat{B}) + \left(\frac{\partial \widehat{u}}{\partial C}\right)^2 Var(\widehat{C}) + 2\left(\frac{\partial \widehat{u}}{\partial \beta}\right)\left(\frac{\partial \widehat{u}}{\partial B}\right) Cov(\widehat{\beta},\widehat{B}) + 2\left(\frac{\partial \widehat{u}}{\partial \beta}\right)\left(\frac{\partial \widehat{u}}{\partial C}\right) Cov(\widehat{\beta},\widehat{C}) + 2\left(\frac{\partial \widehat{u}}{\partial B}\right)\left(\frac{\partial \widehat{u}}{\partial C}\right) Cov(\widehat{B},\widehat{C})$$

or:

$$Var(\widehat{u}) = \frac{1}{\widehat{\beta}^4}[\ln(-\ln(R))]^2 Var(\widehat{\beta}) + \frac{1}{V^2} Var(\widehat{B}) + \frac{1}{\widehat{C}^2} Var(\widehat{C}) - \frac{2\ln(-\ln(R))}{\widehat{\beta}^2 V} Cov(\widehat{\beta},\widehat{B}) - \frac{2\ln(-\ln(R))}{\widehat{\beta}^2 \widehat{C}} Cov(\widehat{\beta},\widehat{C}) + \frac{2}{V\widehat{C}} Cov(\widehat{B},\widehat{C})$$

The upper and lower bounds on time can then found by:

$$T_U = e^{u_U}$$
$$T_L = e^{u_L}$$

Approximate Confidence Bounds for the Arrhenius-Lognormal

Bounds on the Parameters

The lower and upper bounds on B are estimated from:

$$B_U = \widehat{B} + K_\alpha \sqrt{Var(\widehat{B})} \text{ (Upper bound)}$$
$$B_L = \widehat{B} - K_\alpha \sqrt{Var(\widehat{B})} \text{ (Lower bound)}$$

Since the standard deviation, $\widehat{\sigma}_{T'}$, and the parameter $\widehat{C}$ are positive parameters, $\ln(\widehat{\sigma}_{T'})$ and $\ln(\widehat{C})$ are treated as normally distributed. The bounds are estimated from:

$$C_U = \widehat{C} \cdot e^{\frac{K_\alpha \sqrt{Var(\widehat{C})}}{\widehat{C}}} \text{ (Upper bound)}$$
$$C_L = \frac{\widehat{C}}{e^{\frac{K_\alpha \sqrt{Var(\widehat{C})}}{\widehat{C}}}} \text{ (Lower bound)}$$

and:

$$\sigma_U = \widehat{\sigma}_{T'} \cdot e^{\frac{K_\alpha \sqrt{Var(\widehat{\sigma}_{T'})}}{\widehat{\sigma}_{T'}}} \text{ (Upper bound)}$$
$$\sigma_L = \frac{\widehat{\sigma}_{T'}}{e^{\frac{K_\alpha \sqrt{Var(\widehat{\sigma}_{T'})}}{\widehat{\sigma}_{T'}}}} \text{ (Lower bound)}$$

The variances and covariances of $B, C,$ and $\sigma_{T'}$ are estimated from the local Fisher matrix (evaluated at $\widehat{B}, \widehat{C}, \widehat{\sigma}_{T'}$), as follows:

$$\begin{bmatrix} Var\left(\widehat{\sigma}_{T'}\right) & Cov\left(\widehat{B},\widehat{\sigma}_{T'}\right) & Cov\left(\widehat{C},\widehat{\sigma}_{T'}\right) \\ Cov\left(\widehat{\sigma}_{T'},\widehat{B}\right) & Var\left(\widehat{B}\right) & Cov\left(\widehat{B},\widehat{C}\right) \\ Cov\left(\widehat{\sigma}_{T'},\widehat{C}\right) & Cov\left(\widehat{C},\widehat{B}\right) & Var\left(\widehat{C}\right) \end{bmatrix} = \begin{bmatrix} -\frac{\partial^2 \Lambda}{\partial \sigma_{T'}^2} & -\frac{\partial^2 \Lambda}{\partial \sigma_{T'} \partial B} & -\frac{\partial^2 \Lambda}{\partial \sigma_{T'} \partial C} \\ -\frac{\partial^2 \Lambda}{\partial B \partial \sigma_{T'}} & -\frac{\partial^2 \Lambda}{\partial B^2} & -\frac{\partial^2 \Lambda}{\partial B \partial C} \\ -\frac{\partial^2 \Lambda}{\partial C \partial \sigma_{T'}} & -\frac{\partial^2 \Lambda}{\partial C \partial B} & -\frac{\partial^2 \Lambda}{\partial C^2} \end{bmatrix}^{-1}$$

Bounds on Reliability

The reliability of the lognormal distribution is:

$$R(T', V; B, C, \sigma_{T'}) = \int_{T'}^{\infty} \frac{1}{\widehat{\sigma}_{T'} \sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{t - \ln(\widehat{C}) - \frac{\widehat{B}}{V}}{\widehat{\sigma}_{T'}}\right)^2} dt$$

Let $\hat{z}(t, V; B, C, \sigma_T) = \frac{t - \ln(\widehat{C}) - \frac{\widehat{B}}{V}}{\widehat{\sigma}_{T'}}$, then $\frac{d\hat{z}}{dt} = \frac{1}{\widehat{\sigma}_{T'}}$.

For $t=T'$, $\hat{z}=\frac{T'-\ln(\widehat{C})-\frac{\widehat{B}}{V}}{\widehat{\sigma}_{T'}}$, and for $t=\infty,\hat{z}=\infty.$ The above equation then becomes:

$$R(\hat{z})=\int_{\hat{z}(T')}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^{2}}dz$$

The bounds on z are estimated from:

$$\begin{aligned} z_U &= \hat{z}+K_{\alpha}\sqrt{Var(\hat{z})} \\ z_L &= \hat{z}-K_{\alpha}\sqrt{Var(\hat{z})} \end{aligned}$$

where:

$$\begin{aligned} Var(\hat{z}) = & \left(\frac{\partial\hat{z}}{\partial B}\right)_{\widehat{B}}^{2}Var(\widehat{B})+\left(\frac{\partial\hat{z}}{\partial C}\right)_{\widehat{C}}^{2}Var(\widehat{C})+\left(\frac{\partial\hat{z}}{\partial\sigma_{T'}}\right)_{\widehat{\sigma}_{T'}}^{2}Var(\widehat{\sigma}_{T})+2\left(\frac{\partial\hat{z}}{\partial B}\right)_{\widehat{B}}\left(\frac{\partial\hat{z}}{\partial C}\right)_{\widehat{C}}Cov\left(\widehat{B},\widehat{C}\right) \\ & +2\left(\frac{\partial\hat{z}}{\partial B}\right)_{\widehat{B}}\left(\frac{\partial\hat{z}}{\partial\sigma_{T'}}\right)_{\widehat{\sigma}_{T'}}Cov\left(\widehat{B},\widehat{\sigma}_{T}\right)+2\left(\frac{\partial\hat{z}}{\partial C}\right)_{\widehat{C}}\left(\frac{\partial\hat{z}}{\partial\sigma_{T'}}\right)_{\widehat{\sigma}_{T'}}Cov\left(\widehat{C},\widehat{\sigma}_{T}\right) \end{aligned}$$

or:

$$Var(\hat{z}) = \frac{1}{\widehat{\sigma}_{T'}^{2}}[\frac{1}{V^{2}}Var(\widehat{B})+\frac{1}{C^{2}}Var(\widehat{C})+\hat{z}^{2}Var(\widehat{\sigma}_{T'})+\frac{2}{C\cdot V}Cov\left(\widehat{B},\widehat{C}\right)+\frac{2\hat{z}}{V}Cov\left(\widehat{B},\widehat{\sigma}_{T'}\right)+\frac{2\hat{z}}{C}Cov\left(\widehat{B},\widehat{\sigma}_{T'}\right)]$$

The upper and lower bounds on reliability are:

$$\begin{aligned} R_U &= \int_{z_L}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^{2}}dz\text{ (Upper bound)} \\ R_L &= \int_{z_U}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^{2}}dz\text{ (Lower bound)} \end{aligned}$$

Confidence Bounds on Time

The bounds around time, for a given lognormal percentile (unreliability), are estimated by first solving the reliability equation with respect to time, as follows:

$$T'(V;\widehat{B},\widehat{C},\widehat{\sigma}_{T'})=\ln(\widehat{C})+\frac{\widehat{B}}{V}+z\cdot\widehat{\sigma}_{T'}$$

where:

$$T'(V;\widehat{B},\widehat{C},\widehat{\sigma}_{T'}) = \ln(T)$$
$$z = \Phi^{-1}\left[F(T')\right]$$

and:

$$\Phi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z(T')} e^{-\frac{1}{2}z^2}dz$$

The next step is to calculate the variance of $T'(V;\widehat{B},\widehat{C},\widehat{\sigma}_{T'})$:

$$\begin{aligned}Var(T') = & \left(\frac{\partial T'}{\partial B}\right)^2 Var(\widehat{B}) + \left(\frac{\partial T'}{\partial C}\right)^2 Var(\widehat{C}) + \left(\frac{\partial T'}{\partial \sigma_{T'}}\right)^2 Var(\widehat{\sigma}_{T'}) + 2\left(\frac{\partial T'}{\partial B}\right)\left(\frac{\partial T'}{\partial C}\right) Cov\left(\widehat{B},\widehat{C}\right) \\ & + 2\left(\frac{\partial T'}{\partial B}\right)\left(\frac{\partial T'}{\partial \sigma_{T'}}\right) Cov\left(\widehat{B},\widehat{\sigma}_{T'}\right) + 2\left(\frac{\partial T'}{\partial C}\right)\left(\frac{\partial T'}{\partial \sigma_{T'}}\right) Cov\left(\widehat{C},\widehat{\sigma}_{T'}\right)\end{aligned}$$

or:

$$Var(T') = \frac{1}{V^2}Var(\widehat{B}) + \frac{1}{C^2}Var(\widehat{C}) + \hat{z}^2 Var(\widehat{\sigma}_{T'}) + \frac{2}{B\cdot C}Cov\left(\widehat{B},\widehat{C}\right) + \frac{2\hat{z}}{V}Cov\left(\widehat{B},\widehat{\sigma}_{T'}\right) + \frac{2\hat{z}}{C}Cov\left(\widehat{C},\widehat{\sigma}_{T'}\right)$$

The upper and lower bounds are then found by:

$$T'_U = \quad \ln T_U = T' + K_\alpha\sqrt{Var(T')}$$
$$T'_L = \quad \ln T_L = T' - K_\alpha\sqrt{Var(T')}$$

Solving for T_U and T_L yields:

$$T_U = \quad e^{T'_U} \text{ (Upper bound)}$$
$$T_L = \quad e^{T'_L} \text{ (Lower bound)}$$

Eyring Relationship

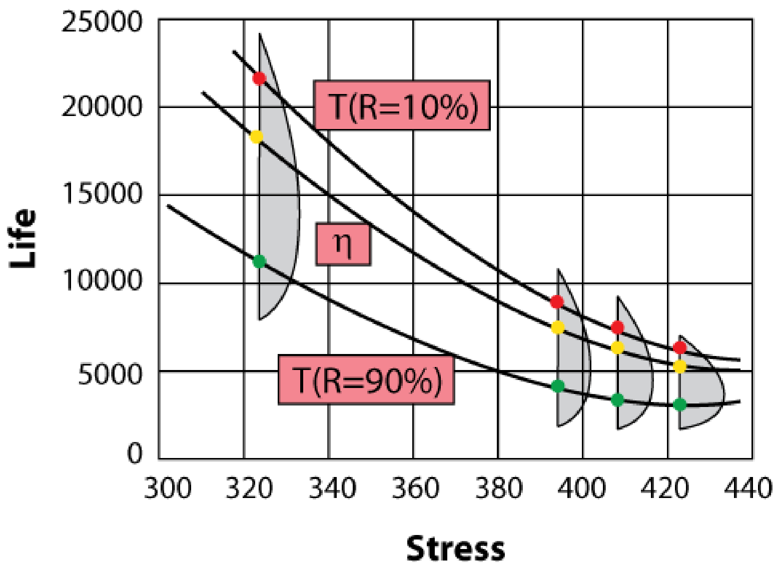
IN THIS CHAPTER

The Eyring relationship was formulated from quantum mechanics principles, as discussed in Glasstone et al. [9], and is most often used when thermal stress (temperature) is the acceleration variable. However, the Eyring relationship is also often used for stress variables other than temperature, such as humidity. The relationship is given by:

$$L(V) = \frac{1}{V}e^{-\left(A - \frac{B}{V}\right)}$$

where:

- L represents a quantifiable life measure, such as mean life, characteristic life, median life, $B(x)$ life, etc.
- V represents the stress level (**temperature values are in absolute units: kelvin or degrees Rankine**).
- A is one of the model parameters to be determined.
- B is another model parameter to be determined.



The Eyring relationship is similar to the Arrhenius relationship. This similarity is more apparent if it is rewritten in the following way:

$$L(V) = \frac{1}{V}e^{-\left(A - \frac{B}{V}\right)} = \frac{e^{-A}}{V}e^{\frac{B}{V}}$$

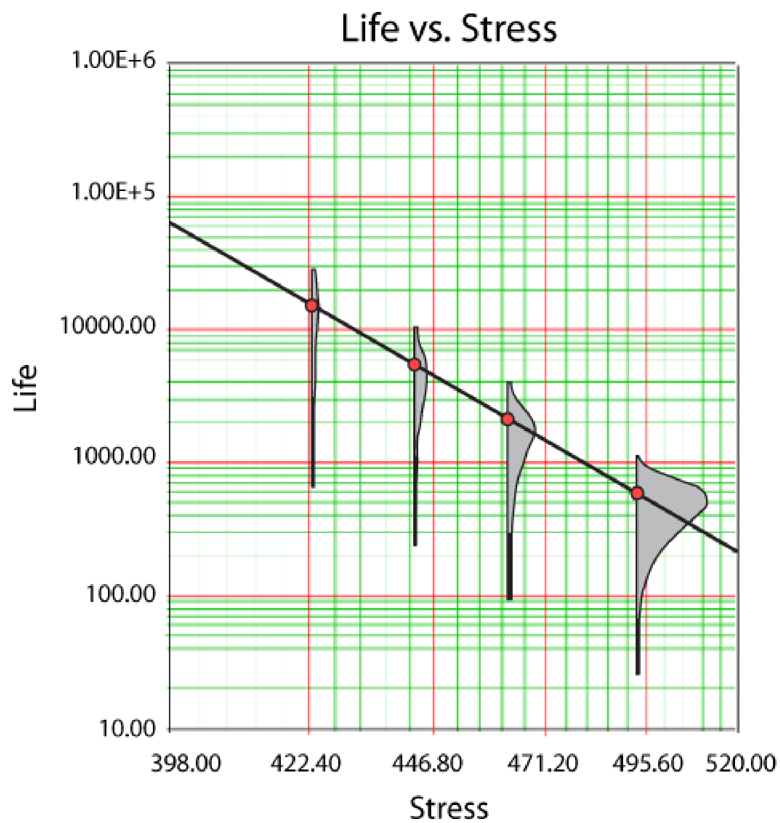
or:

$$L(V) = \frac{1}{V}Const. \cdot e^{\frac{B}{V}}$$

The Arrhenius relationship is given by:

$$L(V) = C \cdot e^{\frac{B}{V}}$$

Comparing the above equation to the Arrhenius relationship, it can be seen that the only difference between the two relationships is the $\frac{1}{V}$ term above. In general, both relationships yield very similar results. Like the Arrhenius, the Eyring relationship is plotted on a log-reciprocal paper.



Acceleration Factor

For the Eyring model the acceleration factor is given by:

$$A_F = \frac{L_{USE}}{L_{Accelerated}} = \frac{\frac{1}{V_u} e^{-\left(A - \frac{B}{V_u}\right)}}{\frac{1}{V_A} e^{-\left(A - \frac{B}{V_A}\right)}} = \frac{e^{\frac{B}{V_u}}}{e^{\frac{B}{V_A}}} = \frac{V_A}{V_u} e^{B\left(\frac{1}{V_u} - \frac{1}{V_A}\right)}$$

Eyring-Exponential

The *pdf* of the 1-parameter exponential distribution is given by:

$$f(t) = \lambda \cdot e^{-\lambda \cdot t}$$

It can be easily shown that the mean life for the 1-parameter exponential distribution (presented in detail here) is given by:

$$\lambda = \frac{1}{m}$$

thus:

$$f(t) = \frac{1}{m} \cdot e^{-\frac{t}{m}}$$

The Eyring-exponential model *pdf* can then be obtained by setting $m = L(V)$:

$$m = L(V) = \frac{1}{V} e^{-\left(A - \frac{B}{V}\right)}$$

and substituting for m in the exponential *pdf* equation:

$$f(t, V) = V \cdot e^{\left(A - \frac{B}{V}\right)} e^{-V \cdot e^{\left(A - \frac{B}{V}\right)} \cdot t}$$

Eyring-Exponential Statistical Properties Summary

Mean or MTTF

The mean, $\overline{T}$, or Mean Time To Failure (MTTF) for the Eyring-exponential is given by:

$$\overline{T} = \int_0^\infty t \cdot f(t, V) dt = \int_0^\infty t \cdot V e^{\left(A - \frac{B}{V}\right)} e^{-tVe^{\left(A - \frac{B}{V}\right)}} dt = \frac{1}{V} e^{-\left(A - \frac{B}{V}\right)}$$

Median

The median, $\breve{T}$, for the Eyring-exponential model is given by:

$$\breve{T} = 0.693\frac{1}{V}e^{-\left(A-\frac{B}{V}\right)}$$

Mode

The mode, $\tilde{T}$, for the Eyring-exponential model is $\tilde{T} = 0.$

Standard Deviation

The standard deviation, σ_T , for the Eyring-exponential model is given by:

$$\sigma_T = \frac{1}{V}e^{-\left(A-\frac{B}{V}\right)}$$

Eyring-Exponential Reliability Function

The Eyring-exponential reliability function is given by:

$$R(T,V) = e^{-T \cdot V \cdot e^{\left(A-\frac{B}{V}\right)}}$$

This function is the complement of the Eyring-exponential cumulative distribution function or:

$$R(T,V) = 1 - Q(T,V) = 1 - \int_0^T f(T,V)dT$$

and:

$$R(T,V) = 1 - \int_0^T Ve^{\left(A-\frac{B}{V}\right)}e^{-T \cdot V \cdot e^{\left(A-\frac{B}{V}\right)}}dT = e^{-T \cdot V \cdot e^{\left(A-\frac{B}{V}\right)}}$$

Conditional Reliability

The conditional reliability function for the Eyring-exponential model is given by:

$$R((t|T),V) = \frac{R(T+t,V)}{R(T,V)} = \frac{e^{-\lambda(T+t)}}{e^{-\lambda T}} = e^{-t \cdot V \cdot e^{\left(A-\frac{B}{V}\right)}}$$

Reliable Life

For the Eyring-exponential model, the reliable life, or the mission duration for a desired reliability goal, t_R, is given by:

$$R(t_R, V) = e^{-t_R \cdot V \cdot e^{\left(A - \frac{B}{V}\right)}}$$

$$\ln[R(t_R, V)] = -t_R \cdot V \cdot e^{\left(A - \frac{B}{V}\right)}$$

or:

$$t_R = -\frac{1}{V} e^{-\left(A - \frac{B}{V}\right)} \ln[R(t_R, V)]$$

Parameter Estimation

Maximum Likelihood Estimation Method

The complete exponential log-likelihood function of the Eyring model is composed of two summation portions:

$$\ln(L) = \quad \Lambda = \sum_{i=1}^{F_e} N_i \ln\left[V_i \cdot e^{\left(A - \frac{B}{V_i}\right)} e^{-V_i \cdot e^{\left(A - \frac{B}{V_i}\right)} \cdot T_i}\right] - \sum_{i=1}^{S} N_i' \cdot V_i \cdot e^{\left(A - \frac{B}{V_i}\right)} \cdot T_i' + \sum_{i=1}^{FI} N_i'' \ln[R_{Li}'' - R_{Ri}'']$$

where:

$$R_{Li}'' = e^{-T_{Li}'' V_i e^{A - \frac{B}{V_i}}}$$

$$R_{Ri}'' = e^{-T_{Ri}'' V_i e^{A - \frac{B}{V_i}}}$$

and:

- F_e is the number of groups of exact times-to-failure data points.
- N_i is the number of times-to-failure in the i^{th} time-to-failure data group.
- V_i is the stress level of the i^{th} group.

- A is the Eyring parameter (unknown, the first of two parameters to be estimated).
- B is the second Eyring parameter (unknown, the second of two parameters to be estimated).
- T_i is the exact failure time of the i^{th} group.
- S is the number of groups of suspension data points.
- N_i' is the number of suspensions in the i^{th} group of suspension data points.
- T_i' is the running time of the i^{th} suspension data group.
- FI is the number of interval data groups.
- N_i'' is the number of intervals in the i^{th} group of data intervals.
- T_{Li}'' is the beginning of the i^{th} interval.
- T_{Ri}'' is the ending of the i^{th} interval.

The solution (parameter estimates) will be found by solving for the parameters $\widehat{A}$ and $\widehat{B}$ so that $\frac{\partial \Lambda}{\partial A}=0$ and $\frac{\partial \Lambda}{\partial B}=0$ where:

$$\frac{\partial \Lambda}{\partial A}= \sum_{i=1}^{F_e} N_i\left(1-V_i\cdot e^{\left(A-\frac{B}{V_i}\right)}T_i\right)-\sum_{i=1}^{S} N_i' V_i\cdot e^{\left(A-\frac{B}{V_i}\right)}T_i' - \sum_{i=1}^{FI} N_i''\frac{\left(T_{Li}''R_{Li}''-T_{Ri}''R_{Ri}''\right)V_i e^{A-\frac{B}{V_i}}}{R_{Li}''-R_{Ri}''}$$

$$\frac{\partial \Lambda}{\partial B}= \sum_{i=1}^{F_e} N_i\left[e^{\left(A-\frac{B}{V_i}\right)}T_i-\frac{1}{V_i}\right]+\sum_{i=1}^{S} N_i'\cdot e^{\left(A-\frac{B}{V_i}\right)}T_i' + \sum_{i=1}^{FI} N_i''\frac{\left(T_{Li}''R_{Li}''-T_{Ri}''R_{Ri}''\right)e^{A-\frac{B}{V_i}}}{R_{Li}''-R_{Ri}''}$$

Eyring-Weibull

The *pdf* for 2-parameter Weibull distribution is given by:

$$f(t)=\frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1}e^{-\left(\frac{t}{\eta}\right)^{\beta}}$$

The scale parameter (or characteristic life) of the Weibull distribution is η. The Eyring-Weibull model *pdf* can then be obtained by setting $\eta=L(V)$:

$$\eta=L(V)=\frac{1}{V}e^{-\left(A-\frac{B}{V}\right)}$$

or:

$$\frac{1}{\eta}=V\cdot e^{\left(A-\frac{B}{V}\right)}$$

Substituting for η into the Weibull *pdf* yields:

$$f(t,V)=\beta\cdot V\cdot e^{\left(A-\frac{B}{V}\right)}\left(t\cdot V\cdot e^{\left(A-\frac{B}{V}\right)}\right)^{\beta-1}e^{-\left(t\cdot V\cdot e^{\left(A-\frac{B}{V}\right)}\right)^{\beta}}$$

Eyring-Weibull Statistical Properties Summary

Mean or MTTF

The mean, $\overline{T}$, or Mean Time To Failure (MTTF) for the Eyring-Weibull model is given by:

$$\overline{T}=\frac{1}{V}e^{-\left(A-\frac{B}{V}\right)}\cdot\Gamma\left(\frac{1}{\beta}+1\right)$$

where $\Gamma\left(\frac{1}{\beta}+1\right)$ is the gamma function evaluated at the value of $\left(\frac{1}{\beta}+1\right)$.

Median

The median, $\breve{T}$ for the Eyring-Weibull model is given by:

$$\breve{T}=\frac{1}{V}e^{-\left(A-\frac{B}{V}\right)}(\ln 2)^{\frac{1}{\beta}}$$

Mode

The mode, $\tilde{T}$, for the Eyring-Weibull model is given by:

$$\tilde{T}=\frac{1}{V}e^{-\left(A-\frac{B}{V}\right)}\left(1-\frac{1}{\beta}\right)^{\frac{1}{\beta}}$$

Standard Deviation

The standard deviation, σ_T, for the Eyring-Weibull model is given by:

$$\sigma_T=\frac{1}{V}e^{-\left(A-\frac{B}{V}\right)}\cdot\sqrt{\Gamma\left(\frac{2}{\beta}+1\right)-\left(\Gamma\left(\frac{1}{\beta}+1\right)\right)^2}$$

Eyring-Weibull Reliability Function

The Eyring-Weibull reliability function is given by:

$$R(T,V)=e^{-\left(V\cdot T\cdot e^{\left(A-\frac{B}{V}\right)}\right)^{\beta}}$$

Conditional Reliability Function

The Eyring-Weibull conditional reliability function at a specified stress level is given by:

$$R((t|T),V)=\frac{R(T+t,V)}{R(T,V)}=\frac{e^{-\left((T+t)\cdot V\cdot e^{\left(A-\frac{B}{V}\right)}\right)^{\beta}}}{e^{-\left(V\cdot T\cdot e^{\left(A-\frac{B}{V}\right)}\right)^{\beta}}}$$

or:

$$R((t|T),V)=e^{-\left[\left((T+t)\cdot V\cdot e^{\left(A-\frac{B}{V}\right)}\right)^{\beta}-\left(V\cdot T\cdot e^{\left(A-\frac{B}{V}\right)}\right)^{\beta}\right]}$$

Reliable Life

For the Eyring-Weibull model, the reliable life, t_R , of a unit for a specified reliability and starting the mission at age zero is given by:

$$t_R=\frac{1}{V}e^{-\left(A-\frac{B}{V}\right)}\left\{-\ln[R\left(T_R,V\right)]\right\}^{\frac{1}{\beta}}$$

Eyring-Weibull Failure Rate Function

The Eyring-Weibull failure rate function, $\lambda(T)$, is given by:

$$\lambda\left(T,V\right)=\frac{f\left(T,V\right)}{R\left(T,V\right)}=\beta\left(T\cdot V\cdot e^{\left(A-\frac{B}{V}\right)}\right)^{\beta-1}$$

Parameter Estimation

Maximum Likelihood Estimation Method

The Eyring-Weibull log-likelihood function is composed of two summation portions:

$$\ln(L)=\quad\Lambda=\sum_{i=1}^{F_e}N_i\ln\left[\beta\cdot V_i\cdot e^{A-\frac{B}{V_i}}\left(T_iV_ie^{A-\frac{B}{V_i}}\right)^{\beta-1}e^{-\left(T_iV_ie^{A-\frac{B}{V_i}}\right)^{\beta}}\right]-\sum_{i=1}^{S}N_i'\left(V_ie^{A-\frac{B}{V_i}}T_i'\right)^{\beta}+\sum_{i=1}^{FI}N_i''\ln[R_{Li}''-R_{Ri}'']$$

where:

$$R_{Li}''=e^{-\left(T_{Li}''V_ie^{A-\frac{B}{V_i}}\right)^{\beta}}$$

$$R_{Ri}''=e^{-\left(T_{Ri}''V_ie^{A-\frac{B}{V_i}}\right)^{\beta}}$$

and:

- F_e is the number of groups of exact times-to-failure data points.
- N_i is the number of times-to-failure data points in the i^{th} time-to-failure data group.
- β is the Weibull shape parameter (unknown, the first of three parameters to be estimated).
- A is the Eyring parameter (unknown, the second of three parameters to be estimated).
- B is the second Eyring parameter (unknown, the third of three parameters to be estimated).
- V_i is the stress level of the i^{th} group.
- T_i is the exact failure time of the i^{th} group.
- S is the number of groups of suspension data points.
- N_i' is the number of suspensions in the i^{th} group of suspension data points.
- T_i' is the running time of the i^{th} suspension data group.
- FI is the number of interval data groups.
- N_i'' is the number of intervals in the i^{th} group of data intervals.
- T_{Li}'' is the beginning of the i^{th} interval.
- T_{Ri}'' is the ending of the i^{th} interval.

The solution (parameter estimates) will be found by solving for the parameters β, A and B so that $\frac{\partial \Lambda}{\partial \beta} = 0$, $\frac{\partial \Lambda}{\partial A} = 0$ and $\frac{\partial \Lambda}{\partial B} = 0$

where:

$$\frac{\partial \Lambda}{\partial A} = \beta\sum_{i=1}^{F_e} N_i - \beta\sum_{i=1}^{F_e} N_i\left(T_i V_i e^{A-\frac{B}{V_i}}\right)^{\beta} - \beta\sum_{i=1}^{S} N_i'\left(T_i' V_i e^{A-\frac{B}{V_i}}\right)^{\beta} - \sum_{i=1}^{FI} N_i'' \frac{\beta V_i^{\beta} e^{A\beta-\frac{B\beta}{V_i}}\left[(T_{Li}'')^{\beta} R_{Li}'' - (T_{Ri}'')^{\beta} R_{Ri}''\right]}{R_{Li}'' - R_{Ri}''}$$

$$\frac{\partial \Lambda}{\partial B}= -\beta\sum_{i=1}^{F_e}N_i\frac{1}{V_i}+\beta\sum_{i=1}^{F_e}N_i\frac{1}{V_i}\left(T_iV_ie^{A-\frac{B}{V_i}}\right)^{\beta}+\beta\sum_{i=1}^{S}N_i'\frac{1}{V_i}\left(T_i'V_ie^{A-\frac{B}{V_i}}\right)^{\beta}+\sum_{i=1}^{FI}N_i''\frac{\beta V_i^{(\beta-1)}e^{A\beta-\frac{B\beta}{V_i}}\left[(T_{Li}'')^{\beta}R_{Li}''-(T_{Ri}'')^{\beta}R_{Ri}''\right]}{R_{Li}''-R_{Ri}''}$$

$$\begin{aligned}\frac{\partial \Lambda}{\partial \beta}=&\frac{1}{\beta}\sum_{i=1}^{F_e}N_i\frac{1}{V_i}+\sum_{i=1}^{F_e}N_i ln\left(T_iV_ie^{A-\frac{B}{V_i}}\right)-\sum_{i=1}^{F_e}N_i\left(T_iV_ie^{A-\frac{B}{V_i}}\right)^{\beta}ln\left(T_iV)ie^{A-\frac{B}{V_i}}\right)\\&-\sum_{i=1}^{S}N_i'\left(T_i'V_Ie^{A-\frac{B}{V_i}}\right)^{\beta}ln\left(T_iV)ie^{A-\frac{B}{V_i}}\right)-\sum_{i=1}^{FI}N_i''V_ie^{A-\frac{B}{V_i}}\frac{R_{Li}''T_{Li}''\left(ln(T_{Li}'V_i)+A-\frac{B}{V_i}\right)-R_{Ri}''T_{Ri}''\left(ln(T_{Ri}''V_i)+A-\frac{B}{V_i}\right)}{R_{L_i}''-F_{Ri}''}\end{aligned}$$

Eyring-Weibull Example

Consider the following times-to-failure data at three different stress levels.

Stress	393 K	408 K	423 K
Time Failed (hrs)	3850	3300	2750
	4340	3720	3100
	4760	4080	3400
	5320	4560	3800
	5740	4920	4100
	6160	5280	4400
	6580	5640	4700
	7140	6120	5100
	7980	6840	5700
	8960	7680	6400

The data set was entered into the life-stress data folio and analyzed using the Eyring-Weibull model, yielding:

$$\widehat{\beta}=4.29186497$$

$$\widehat{A}=-11.08784624$$

$$\widehat{B}=1454.08635742$$

Once the parameters of the model are defined, other life measures can be directly obtained using the appropriate equations. For example, the MTTF can be obtained for the use stress level of 323 K by using:

$$\overline{T} = \frac{1}{V} e^{-\left(A - \frac{B}{V}\right)} \cdot \Gamma\left(\frac{1}{\beta} + 1\right)$$

or:

$$\overline{T} = \frac{1}{323} e^{-\left(-11.08784624 - \frac{1454.08635742}{323}\right)} \cdot \Gamma\left(\frac{1}{4.29186497} + 1\right) = 16,610\ hr$$

Eyring-Lognormal

The *pdf* of the lognormal distribution is given by:

$$f(T) = \frac{1}{T\,\sigma_{T'}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{T' - \overline{T'}}{\sigma_{T'}}\right)^2}$$

where:

$$T' = \ln(T)$$

$$T = \text{times-to-failure}$$

and

- $\overline{T'} =$ mean of the natural logarithms of the times-to-failure.
- $\sigma_{T'} =$ standard deviation of the natural logarithms of the times-to-failure.

The Eyring-lognormal model can be obtained first by setting $\breve{T} = L(V)$:

$$\breve{T} = L(V) = \frac{1}{V} e^{-\left(A - \frac{B}{V}\right)}$$

or:

$$e^{\overline{T}'} = \frac{1}{V} e^{-\left(A - \frac{B}{V}\right)}$$

Thus:

$$\overline{T}' = -\ln(V) - A + \frac{B}{V}$$

Substituting this into the lognormal *pdf* yields the Eyring-lognormal model *pdf*:

$$f(T,V) = \frac{1}{T\, \sigma_{T'} \sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{T' + \ln(V) + A - \frac{B}{V}}{\sigma_{T'}}\right)^2}$$

Eyring-Lognormal Statistical Properties Summary

The Mean

The mean life of the Eyring-lognormal model (mean of the times-to-failure), $\bar{T}$, is given by:

$$\bar{T} = e^{\bar{T}' + \frac{1}{2}\sigma_{T'}^2} = e^{-\ln(V) - A + \frac{B}{V} + \frac{1}{2}\sigma_{T'}^2}$$

The mean of the natural logarithms of the times-to-failure, $\bar{T}'$, in terms of $\bar{T}$ and σ_T is given by:

$$\bar{T}' = \ln(\bar{T}) - \frac{1}{2}\ln\left(\frac{\sigma_T^2}{\bar{T}^2} + 1\right)$$

The Median

The median of the Eyring-lognormal model is given by:

$$\breve{T} = e^{\overline{T}'}$$

The Standard Deviation

The standard deviation of the Eyring-lognormal model (standard deviation of the times-to-failure), σ_T , is given by:

$$\sigma_T = \sqrt{\left(e^{2\bar{T}'+\sigma_{T'}^2}\right)\left(e^{\sigma_{T'}^2}-1\right)} = \sqrt{\left(e^{2\left(-\ln(V)-A+\frac{B}{V}\right)+\sigma_{T'}^2}\right)\left(e^{\sigma_{T'}^2}-1\right)}$$

The standard deviation of the natural logarithms of the times-to-failure, $\sigma_{T'}$, in terms of $\bar{T}$ and σ_T is given by:

$$\sigma_{T'} = \sqrt{\ln\left(\frac{\sigma_T^2}{\bar{T}^2}+1\right)}$$

The Mode

The mode of the Eyring-lognormal model is given by:

$$\tilde{T} = e^{\bar{T}'-\sigma_{T'}^2} = e^{-\ln(V)-A+\frac{B}{V}-\sigma_{T'}^2}$$

Eyring-Lognormal Reliability Function

The reliability for a mission of time T , starting at age 0, for the Eyring-lognormal model is determined by:

$$R(T,V) = \int_T^{\infty} f(t,V)dt$$

or:

$$R(T,V) = \int_{T'}^{\infty} \frac{1}{\sigma_{T'}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{t+\ln(V)+A-\frac{B}{V}}{\sigma_{T'}}\right)^2} dt$$

There is no closed form solution for the lognormal reliability function. Solutions can be obtained via the use of standard normal tables. Since the application automatically solves for the reliability we will not discuss manual solution methods.

Reliable Life

For the Eyring-lognormal model, the reliable life, or the mission duration for a desired reliability goal, t_R, is estimated by first solving the reliability equation with respect to time, as follows:

$$T'_R = -\ln(V) - A + \frac{B}{V} + z \cdot \sigma_{T'}$$

where:

$$z = \Phi^{-1}\left[F\left(T'_R, V\right)\right]$$

and:

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z(T',V)} e^{-\frac{t^2}{2}}\, dt$$

Since $T' = \ln(T)$ the reliable life, t_R, is given by:

$$t_R = e^{T'_R}$$

Eyring-Lognormal Failure Rate

The Eyring-lognormal failure rate is given by:

$$\lambda(T,V) = \frac{f(T,V)}{R(T,V)} = \frac{\frac{1}{T\ \sigma_{T'}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{T'+\ln(V)+A-\frac{B}{V}}{\sigma_{T'}}\right)^2}}{\int_{T'}^{\infty} \frac{1}{\sigma_{T'}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{T'+\ln(V)+A-\frac{B}{V}}{\sigma_{T'}}\right)^2} dt}$$

Parameter Estimation

Maximum Likelihood Estimation Method

The complete Eyring-lognormal log-likelihood function is composed of two summation portions:

$$\ln(L) = \quad \Lambda = \sum_{i=1}^{F_e} N_i \ln\left[\frac{1}{\sigma_{T'} T_i}\phi\left(\frac{\ln(T_i) + \ln(V_i) + A - \frac{B}{V_i}}{\sigma_{T'}}\right)\right] + \sum_{i=1}^{S} N'_i \ln\left[1 - \Phi\left(\frac{\ln(T'_i) + \ln(V_i) + A - \frac{B}{V_i}}{\sigma_{T'}}\right)\right] + \sum_{i=1}^{FI} N''_i \ln[\Phi(z''_{Ri}) - \Phi(z''_{Li})]$$

where:

$$z''_{Li} = \frac{\ln T''_{Li} + \ln V_i + A - \frac{B}{V_i}}{\sigma'_T}$$

$$z''_{Ri} = \frac{\ln T''_{Ri} + \ln V_i + A - \frac{B}{V_i}}{\sigma'_T}$$

and:

- F_e is the number of groups of exact times-to-failure data points.
- N_i is the number of times-to-failure data points in the i^{th} time-to-failure data group.
- $\sigma_{T'}$ is the standard deviation of the natural logarithm of the times-to-failure (unknown, the first of three parameters to be estimated).
- A is the Eyring parameter (unknown, the second of three parameters to be estimated).
- C is the second Eyring parameter (unknown, the third of three parameters to be estimated).
- V_i is the stress level of the i^{th} group.
- T_i is the exact failure time of the i^{th} group.
- S is the number of groups of suspension data points.
- N'_i is the number of suspensions in the i^{th} group of suspension data points.
- T'_i is the running time of the i^{th} suspension data group.
- FI is the number of interval data groups.
- N''_i is the number of intervals in the i^{th} group of data intervals.
- T''_{Li} is the beginning of the i^{th} interval.
- T''_{Ri} is the ending of the i^{th} interval.

The solution (parameter estimates) will be found by solving for $\widehat{\sigma}_{T'}$, $\widehat{A}$, $\widehat{B}$ so that $\frac{\partial \Lambda}{\partial \sigma_{T'}} = 0$, $\frac{\partial \Lambda}{\partial A} = 0$ and $\frac{\partial \Lambda}{\partial B} = 0$:

$$\frac{\partial \Lambda}{\partial A} = -\frac{1}{\sigma_{T'}^2}\sum_{i=1}^{F_e} N_i(\ln(T_i)+\ln(V_i)+A-\frac{B}{V_i}) - \frac{1}{\sigma_{T'}}\sum_{i=1}^{S} N_i' \frac{\phi\left(\frac{\ln(T_i')+\ln(V_i)+A-\frac{B}{V_i}}{\sigma_{T'}}\right)}{1-\Phi\left(\frac{\ln(T_i')+\ln(V_i)+A-\frac{B}{V_i}}{\sigma_{T'}}\right)} + \sum_{i=1}^{FI} N_i'' \frac{\varphi(z''_{Ri})-\varphi(z''_{Li})}{\sigma'_T(\Phi(z''_{Ri})-\Phi(z''_{Li}))}$$

$$\frac{\partial \Lambda}{\partial B} = \frac{1}{\sigma_{T'}^2}\sum_{i=1}^{F_e} N_i \frac{1}{V_i}(\ln(T_i)+\ln(V_i)+A-\frac{B}{V_i}) + \frac{1}{\sigma_{T'}}\sum_{i=1}^{S} N_i' \frac{1}{V_i} \frac{\phi\left(\frac{\ln(T_i')+\ln(V_i)+A-\frac{B}{V_i}}{\sigma_{T'}}\right)}{1-\Phi\left(\frac{\ln(T_i')+\ln(V_i)+A-\frac{B}{V_i}}{\sigma_{T'}}\right)} - \sum_{i=1}^{FI} N_i'' \frac{\varphi(z''_{Ri})-\varphi(z''_{Li})}{\sigma'_T V_i(\Phi(z''_{Ri})-\Phi(z''_{Li}))}$$

$$\frac{\partial \Lambda}{\partial \sigma_{T'}} = \sum_{i=1}^{F_e} N_i \left(\frac{\left(\ln(T_i)+\ln(V_i)+A-\frac{B}{V_i}\right)^2}{\sigma_{T'}^3} - \frac{1}{\sigma_{T'}}\right) + \frac{1}{\sigma_{T'}}\sum_{i=1}^{S} N_i' \frac{\left(\frac{\ln(T_i')+\ln(V_i)+A-\frac{B}{V_i}}{\sigma_{T'}}\right)\phi\left(\frac{\ln(T_i')+\ln(V_i)+A-\frac{B}{V_i}}{\sigma_{T'}}\right)}{1-\Phi\left(\frac{\ln(T_i')+\ln(V_i)+A-\frac{B}{V_i}}{\sigma_{T'}}\right)} - \sum_{i=1}^{FI} N_i'' \frac{z''_{Ri}\varphi(z''_{Ri})-z''_{Li}\varphi(z''_{Li})}{\sigma'_T(\Phi(z''_{Ri})-\Phi(z''_{Li}))}$$

and:

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}(x)^2}$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt$$

Generalized Eyring Relationship

The generalized Eyring relationship is used when temperature and a second non-thermal stress (e.g. voltage) are the accelerated stresses of a test and their interaction is also of interest. This relationship is given by:

$$L(V,U) = \frac{1}{V} e^{A+\frac{B}{V}+CU+D\frac{U}{V}}$$

where:

- V is the temperature (**in absolute units**).
- U is the non-thermal stress (i.e., voltage, vibration, etc.).

A, B, C, D are the parameters to be determined.

The Eyring relationship is a simple case of the generalized Eyring relationship where $C = D = 0$ and $A_{Eyr} = -A_{GEyr}$. Note that the generalized Eyring relationship includes the interaction term of U and V as described by the $D\frac{U}{V}$ term. In other words, this model can estimate the effect of changing one of the factors depending on the level of the other factor.

Acceleration Factor

Most models in actual use do not include any interaction terms, therefore, the acceleration factor can be computed by multiplying the acceleration factors obtained by changing each factor while keeping the other factors constant. In the case of the generalized Eyring relationship, the acceleration factor is derived differently.

The acceleration factor for the generalized Eyring relationship is given by:

$$A_F = \frac{L_{USE}}{L_{Accelerated}} = \frac{\frac{1}{V_U} e^{A + \frac{B}{V_U} + CU_U + D\frac{U_U}{V_U}}}{\frac{1}{T_A} e^{A + \frac{B}{V_A} + CU_A + D\frac{U_A}{V_A}}} = \frac{\frac{1}{V_U} e^{A + \frac{B}{V_U} + CU_U + D\frac{U_U}{V_U}}}{\frac{1}{V_A} e^{A + \frac{B}{V_A} + CU_A + D\frac{U_A}{V_A}}}$$

where:

- L_{USE} is the life at use stress level.
- $L_{Accelerated}$ is the life at the accelerated stress level.
- V_u is the use temperature level.
- V_A is the accelerated temperature level.
- U_A is the accelerated non-thermal level.
- U_u is the use non-thermal level.

Generalized Eyring-Exponential

By setting $m = L(V, U)$, the exponential *pdf* becomes:

$$f(t,V,U)=\left(Ve^{-A-\frac{B}{V}-CU-D\frac{U}{V}}\right)e^{-tVe^{-A-\frac{B}{V}-CU-D\frac{U}{V}}}$$

Generalized Eyring-Exponential Reliability Function

The generalized Eyring exponential model reliability function is given by:

$$R(T,U,V)=e^{-tVe^{-A-\frac{B}{V}-CU-D\frac{U}{V}}}$$

Parameter Estimation

Substituting the generalized Eyring model into the lognormal log-likelihood equation yields:

$$\ln(L)=\quad \Lambda=\sum_{i=1}^{Fe}N_i\ln[\frac{\varphi(z(t))}{\sigma'_T t}]+\sum_{i=1}^{S}N'_i\ln\left(1-\Phi(z(t'_i))\right)+\sum_{i=1}^{FI}N''_i\ln[\Phi(z''_{Ri})-\Phi(z''_{Li})]$$

where:

$$z''_{Ri}=\frac{\ln t''_{Ri}-A-\frac{B}{V''_i}-CU_i-D\frac{U_i}{V''_i}+\ln(V''_i)}{\sigma'_T}$$

$$z''_{Li}=\frac{\ln t''_{Ri}-A-\frac{B}{V''_i}-CU_i-D\frac{U_i}{V''_i}+\ln(V''_i)}{\sigma'_T}$$

and:

- F_e is the number of groups of exact times-to-failure data points.
- N_i is the number of times-to-failure data points in the i^{th} time-to-failure data group.
- A,B,C,D are parameters to be estimated.
- V_i is the temperature level of the i^{th} group.
- U_i is the non-thermal stress level of the i^{th} group.

- T_i is the exact failure time of the i^{th} group.
- S is the number of groups of suspension data points.
- N_i' is the number of suspensions in the i^{th} group of suspension data points.
- T_i' is the running time of the i^{th} suspension data group.
- FI is the number of interval data groups.
- N_i'' is the number of intervals in the i^{th} group of data intervals.
- T_{Li}'' is the beginning of the i^{th} interval.
- T_{Ri}'' is the ending of the i^{th} interval.

The solution (parameter estimates) will be found by solving for the parameters $A, B, C,$ and D so that $\frac{\partial \Lambda}{\partial A} = 0, \frac{\partial \Lambda}{\partial B} = 0, \frac{\partial \Lambda}{\partial D} = 0$ and $\frac{\partial \Lambda}{\partial D} = 0$.

Generalized Eyring-Weibull

By setting $\eta = L(V, U)$ to the Weibull *pdf*, the generalized Eyring Weibull model is given by:

$$f(t, V, U) = \quad \beta \left(Ve^{-A-\frac{B}{V}-CU-D\frac{U}{V}} \right) \left(tVe^{-A-\frac{B}{V}-CU-D\frac{U}{V}} \right)^{\beta-1} e^{-\left(tVe^{-A-\frac{B}{V}-CU-D\frac{U}{V}} \right)^{\beta}}$$

Generalized Eyring-Weibull Reliability Function

The generalized Eyring Weibull reliability function is given by:

$$R(T, V, U) = e^{-\left(tVe^{-A-\frac{B}{V}-CU-D\frac{U}{V}} \right)^{\beta}}$$

Parameter Estimation

Substituting the generalized Eyring model into the Weibull log-likelihood equation yields:

$$\ln(L) = \Lambda = \sum_{i=1}^{Fe} N_i \ln[\beta \left(Ve^{-A-\frac{B}{V}-CU-D\frac{U}{V}} \right) \left(tVe^{-A-\frac{B}{V}-CU-D\frac{U}{V}} \right)^{\beta-1}] - \sum_{i=1}^{Fe} N_i \left(t_i V_i e^{-A-\frac{B}{V_i}-CU_i-D\frac{U_i}{V_i}} \right)^{\beta}$$
$$- \sum_{i=1}^{S} N_i' \left(t_i' V_i' e^{-A-\frac{B}{V_i'}-CU_i'-D\frac{U_i'}{V_i'}} \right)^{\beta} + \sum_{i=1}^{FI} N_i'' \ln[R''_{Li} - R''_{Ri}]$$

where:

$$R''_{Li}(T''_{Li}) = e^{-\left(T''_{Li} V_i'' e^{-A-\frac{B}{V_i''}-CU_i-D\frac{U_i}{V_i''}} \right)^{\beta}}$$

$$R''_{Ri}(T''_{Ri}) = e^{-\left(T''_{Ri} V_i'' e^{-A-\frac{B}{V_i''}-CU_i-D\frac{U_i}{V_i''}} \right)^{\beta}}$$

and:

- F_e is the number of groups of exact times-to-failure data points.
- N_i is the number of times-to-failure data points in the i^{th} time-to-failure data group.
- A, B, C, D are parameters to be estimated.
- V_i is the temperature level of the i^{th} group.
- U_i is the non-thermal stress level of the i^{th} group.
- T_i is the exact failure time of the i^{th} group.
- S is the number of groups of suspension data points.
- N_i' is the number of suspensions in the i^{th} group of suspension data points.
- T_i' is the running time of the i^{th} suspension data group.
- FI is the number of interval data groups.
- N_i'' is the number of intervals in the i^{th} group of data intervals.

- T''_{Li} is the beginning of the i^{th} interval.
- T''_{Ri} is the ending of the i^{th} interval.

The solution (parameter estimates) will be found by solving for the parameters $A, B, C,$ and D so that $\frac{\partial \Lambda}{\partial A}=0, \frac{\partial \Lambda}{\partial B}=0, \frac{\partial \Lambda}{\partial D}=0$ and $\frac{\partial \Lambda}{\partial D}=0$.

Generalized Eyring-Lognormal

By setting $\sigma'_T = L(V,U)$ to the lognormal *pdf*, the generalized Erying lognormal model is given by:

$$f(t,V,U)=\frac{\varphi(z(t))}{\sigma'_T t}$$

where:

$$z(t)=\frac{\ln t - A - \frac{B}{V} - CU - D\frac{U}{V} + \ln(V)}{\sigma'_T}$$

Generalized Eyring-Lognormal Reliability Function

The generalized Erying lognormal reliability function is given by:

$$R(T,V,U)=1-\Phi(z)$$

Parameter Estimation

Substituting the generalized Eyring model into the lognormal log-likelihood equation yields:

$$\ln(L)= \quad \Lambda=\sum_{i=1}^{Fe} N_i \ln[\frac{\varphi(z(t))}{\sigma'_T t}]+\sum_{i=1}^{S} N'_i \ln\left(1-\Phi(z(t'_i))\right)+\sum_{i=1}^{FI} N''_i \ln[\Phi(z''_{Ri})-\Phi(z''_{Li})]$$

where:

$$z''_{Ri}=\frac{\ln t''_{Ri} - A - \frac{B}{V''_i} - CU_i - D\frac{U_i}{V''_i} + \ln(V''_i)}{\sigma'_T}$$

$$z''_{Li} = \frac{\ln t''_{Ri} - A - \frac{B}{V''_i} - CU_i - D\frac{U_i}{V''_i} + \ln(V''_i)}{\sigma'_T}$$

and:

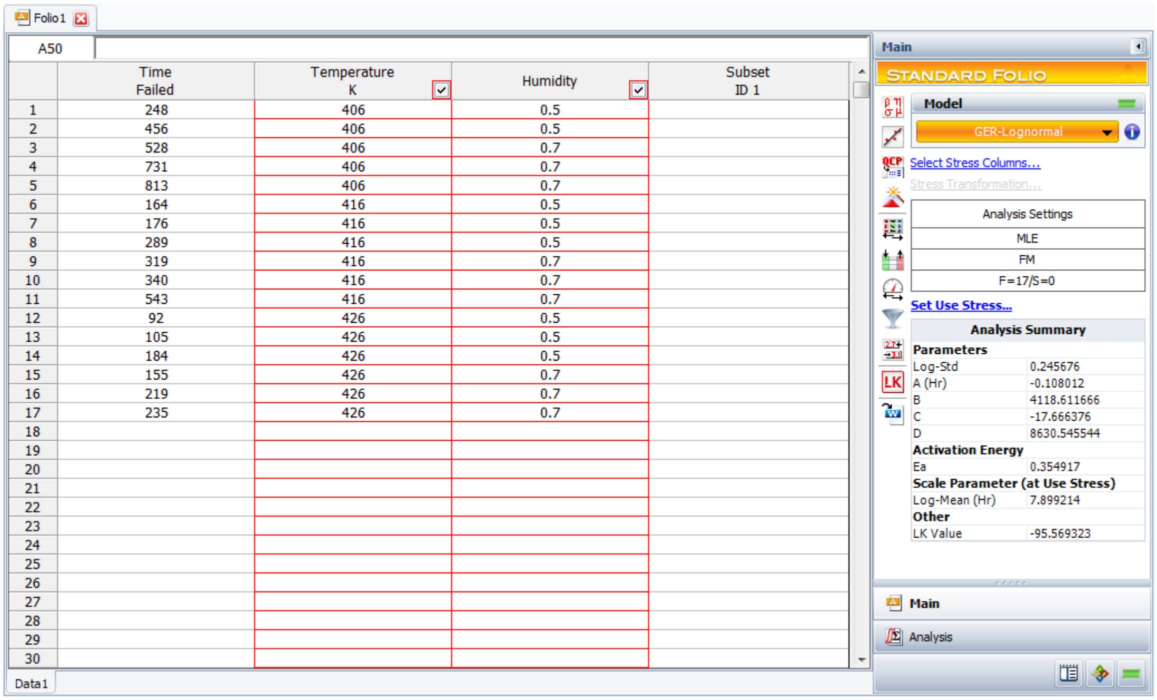
- F_e is the number of groups of exact times-to-failure data points.
- N_i is the number of times-to-failure data points in the i^{th} time-to-failure data group.
- A, B, C, D are parameters to be estimated.
- V_i is the temperature level of the i^{th} group.
- U_i is the non-thermal stress level of the i^{th} group.
- T_i is the exact failure time of the i^{th} group.
- S is the number of groups of suspension data points.
- N'_i is the number of suspensions in the i^{th} group of suspension data points.
- T'_i is the running time of the i^{th} suspension data group.
- FI is the number of interval data groups.
- N''_i is the number of intervals in the i^{th} group of data intervals.
- T''_{Li} is the beginning of the i^{th} interval.
- T''_{Ri} is the ending of the i^{th} interval.

The solution (parameter estimates) will be found by solving for the parameters $A, B, C,$ and D so that $\frac{\partial\Lambda}{\partial A} = 0, \frac{\partial\Lambda}{\partial B} = 0, \frac{\partial\Lambda}{\partial D} = 0$ and $\frac{\partial\Lambda}{\partial D} = 0$.

Generalized Eyring Example

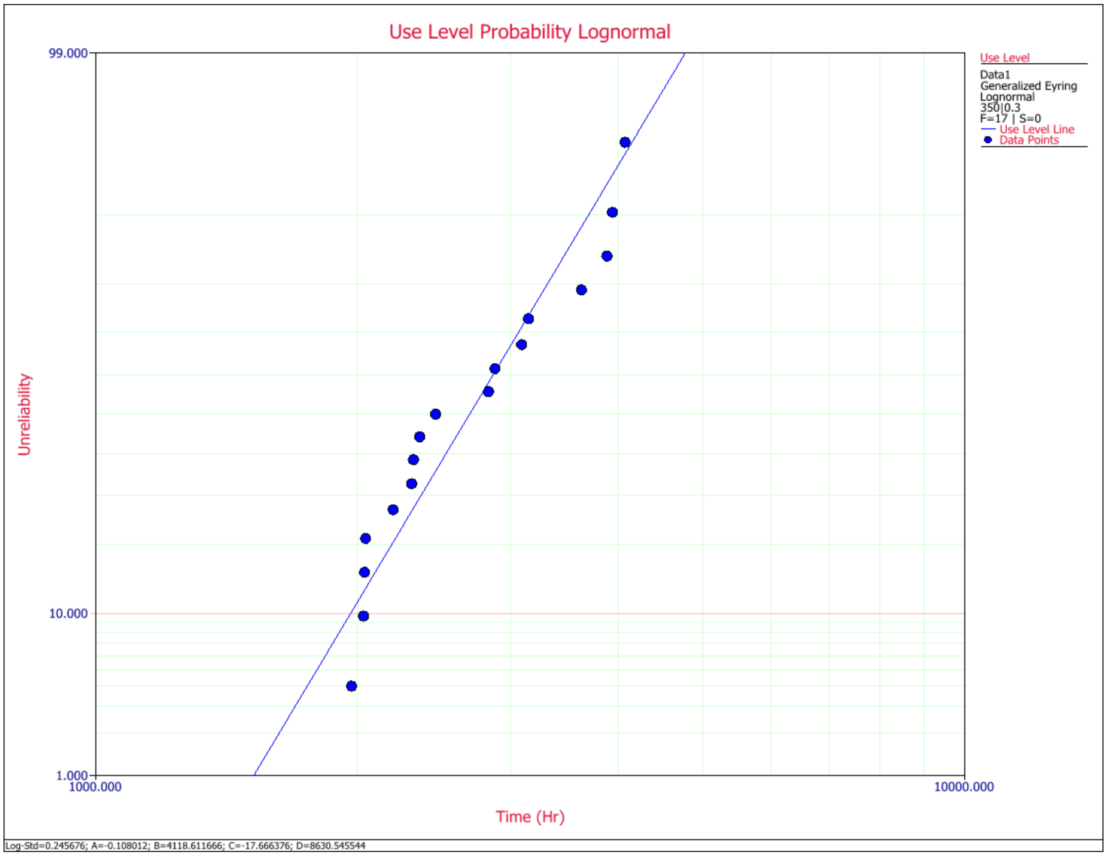
The following data set represents failure times (in hours) obtained from an electronics epoxy packaging accelerated life test performed to understand the synergy between temperature and

and $H = 0.3$. The data set is modeled using the lognormal distribution and the generalized Eyring model.

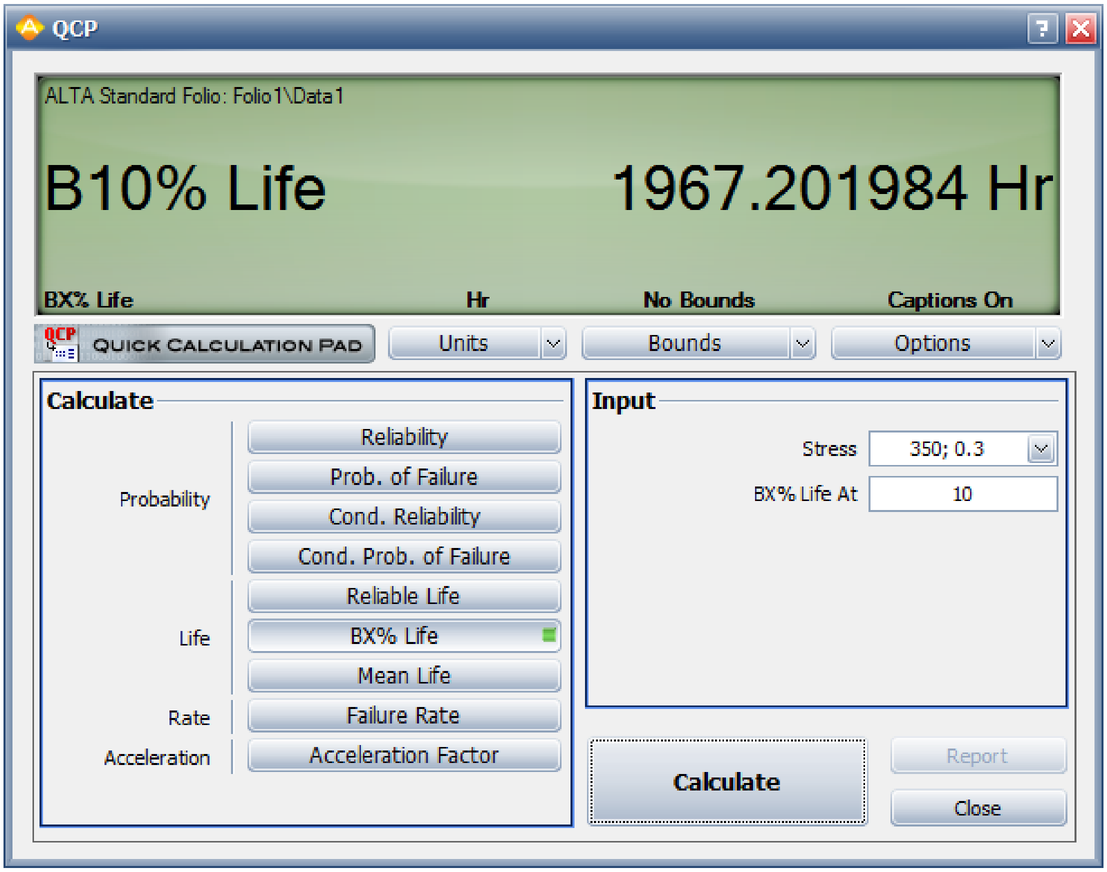


	Time Failed	Temperature K	Humidity	Subset ID 1
1	248	406	0.5	
2	456	406	0.5	
3	528	406	0.7	
4	731	406	0.7	
5	813	406	0.7	
6	164	416	0.5	
7	176	416	0.5	
8	289	416	0.5	
9	319	416	0.7	
10	340	416	0.7	
11	543	416	0.7	
12	92	426	0.5	
13	105	426	0.5	
14	184	426	0.5	
15	155	426	0.7	
16	219	426	0.7	
17	235	426	0.7	

The probability plot at the use conditions is shown next.



The $B10$ information is estimated to be 1967.2 hours, as shown next.



Eyring Confidence Bounds

Approximate Confidence Bounds for the Eyring-Exponential

Confidence Bounds on Mean Life

The mean life for the Eyring relationship is given by setting $m = L(V)$. The upper (m_U) and lower (m_L) bounds on the mean life (ML estimate of the mean life) are estimated by:

$$m_U = \widehat{m} \cdot e^{\frac{K_\alpha \sqrt{Var(\widehat{m})}}{\widehat{m}}}$$

$$m_L = \widehat{m} \cdot e^{-\frac{K_\alpha \sqrt{Var(\widehat{m})}}{\widehat{m}}}$$

where K_α is defined by:

$$\alpha = \frac{1}{\sqrt{2\pi}} \int_{K_\alpha}^{\infty} e^{-\frac{t^2}{2}} dt = 1 - \Phi(K_\alpha)$$

If δ is the confidence level, then $\alpha = \frac{1-\delta}{2}$ for the two-sided bounds, and $\alpha = 1 - \delta$ for the one-sided bounds. The variance of $\widehat{m}$ is given by:

$$Var(\widehat{m}) = \left(\frac{\partial m}{\partial A}\right)^2 Var(\widehat{A}) + \left(\frac{\partial m}{\partial B}\right)^2 Var(\widehat{B}) + 2\left(\frac{\partial m}{\partial A}\right)\left(\frac{\partial m}{\partial B}\right) Cov(\widehat{A}, \widehat{B})$$

or:

$$Var(\widehat{m}) = \frac{1}{V^2} e^{-2\left(\widehat{A} - \frac{\widehat{B}}{V}\right)} \left[Var(\widehat{A}) + \frac{1}{V^2} Var(\widehat{B}) - \frac{1}{V} Cov(\widehat{A}, \widehat{B})\right]$$

The variances and covariance of A and B are estimated from the local Fisher matrix (evaluated at $\widehat{A}$, $\widehat{B}$) as follows:

$$\begin{bmatrix} Var(\widehat{A}) & Cov(\widehat{A}, \widehat{B}) \\ Cov(\widehat{B}, \widehat{A}) & Var(\widehat{B}) \end{bmatrix} = \begin{bmatrix} -\frac{\partial^2 \Lambda}{\partial A^2} & -\frac{\partial^2 \Lambda}{\partial A \partial B} \\ -\frac{\partial^2 \Lambda}{\partial B \partial A} & -\frac{\partial^2 \Lambda}{\partial B^2} \end{bmatrix}^{-1}$$

Confidence Bounds on Reliability

The bounds on reliability at a given time, T, are estimated by:

$$R_U = e^{-\frac{T}{m_U}}$$

$$R_L = e^{-\frac{T}{m_L}}$$

Confidence Bounds on Time

The bounds on time (ML estimate of time) for a given reliability are estimated by first solving the reliability function with respect to time:

$$\widehat{T} = -\widehat{m} \cdot \ln(R)$$

The corresponding confidence bounds are estimated from:

$$T_U = -m_U \cdot \ln(R)$$

$$T_L = -m_L \cdot \ln(R)$$

Approximate Confidence Bounds for the Eyring-Weibull

Bounds on the Parameters

From the asymptotically normal property of the maximum likelihood estimators, and since $\widehat{\beta}$ is a positive parameter, $\ln(\widehat{\beta})$ can then be treated as normally distributed. After performing this transformation, the bounds on the parameters are estimated from:

$$\beta_U = \widehat{\beta} \cdot e^{\frac{K_\alpha \sqrt{Var(\widehat{\beta})}}{\widehat{\beta}}}$$

$$\beta_L = \widehat{\beta} \cdot e^{-\frac{K_\alpha \sqrt{Var(\widehat{\beta})}}{\widehat{\beta}}}$$

also:

$$A_U = \widehat{A} + K_\alpha \sqrt{Var(\widehat{A})}$$

$$A_L = \widehat{A} - K_\alpha \sqrt{Var(\widehat{A})}$$

and:

$$B_U = \widehat{B} + K_\alpha \sqrt{Var(\widehat{B})}$$

$$B_L = \widehat{B} - K_\alpha \sqrt{Var(\widehat{B})}$$

The variances and covariances of $\beta, A,$ and B are estimated from the Fisher matrix (evaluated at $\widehat{\beta}, \widehat{A}, \widehat{B})$ as follows:

$$\begin{bmatrix} Var(\widehat{\beta}) & Cov(\widehat{\beta},\widehat{A}) & Cov(\widehat{\beta},\widehat{B}) \\ Cov(\widehat{A},\widehat{\beta}) & Var(\widehat{A}) & Cov(\widehat{A},\widehat{B}) \\ Cov(\widehat{B},\widehat{\beta}) & Cov(\widehat{B},\widehat{A}) & Var(\widehat{B}) \end{bmatrix} = \begin{bmatrix} -\frac{\partial^2\Lambda}{\partial\beta^2} & -\frac{\partial^2\Lambda}{\partial\beta\partial A} & -\frac{\partial^2\Lambda}{\partial\beta\partial B} \\ -\frac{\partial^2\Lambda}{\partial A\partial\beta} & -\frac{\partial^2\Lambda}{\partial A^2} & -\frac{\partial^2\Lambda}{\partial A\partial B} \\ -\frac{\partial^2\Lambda}{\partial B\partial\beta} & -\frac{\partial^2\Lambda}{\partial B\partial A} & -\frac{\partial^2\Lambda}{\partial B^2} \end{bmatrix}^{-1}$$

Confidence Bounds on Reliability

The reliability function for the Eyring-Weibull model (ML estimate) is given by:

$$\widehat{R}(T,V) = e^{-\left(T\cdot V\cdot e^{\left(\widehat{A}-\frac{\widehat{B}}{V}\right)}\right)^{\widehat{\beta}}}$$

or:

$$\widehat{R}(T,V) = e^{-e^{\ln\left[\left(T\cdot V\cdot e^{\left(\widehat{A}-\frac{\widehat{B}}{V}\right)}\right)^{\widehat{\beta}}\right]}}$$

Setting:

$$\widehat{u} = \ln\left[\left(T\cdot V\cdot e^{\left(\widehat{A}-\frac{\widehat{B}}{V}\right)}\right)^{\widehat{\beta}}\right]$$

or:

$$\widehat{u} = \widehat{\beta}\left[\ln(T) + \ln(V) + \widehat{A} - \frac{\widehat{B}}{V}\right]$$

The reliability function now becomes:

$$\widehat{R}(T,V) = e^{-e^{\hat{u}}}$$

The next step is to find the upper and lower bounds on $\widehat{u}$:

$$u_U = \widehat{u} + K_\alpha \sqrt{Var(\widehat{u})}$$

$$u_L = \widehat{u} - K_\alpha \sqrt{Var(\widehat{u})}$$

where:

$$Var(\widehat{u}) = \left(\frac{\partial \widehat{u}}{\partial \beta}\right)^2 Var(\widehat{\beta}) + \left(\frac{\partial \widehat{u}}{\partial A}\right)^2 Var(\widehat{A}) + \left(\frac{\partial \widehat{u}}{\partial B}\right)^2 Var(\widehat{B}) + 2\left(\frac{\partial \widehat{u}}{\partial \beta}\right)\left(\frac{\partial \widehat{u}}{\partial A}\right) Cov(\widehat{\beta}, \widehat{A}) + 2\left(\frac{\partial \widehat{u}}{\partial \beta}\right)\left(\frac{\partial \widehat{u}}{\partial B}\right) Cov(\widehat{\beta}, \widehat{B}) + 2\left(\frac{\partial \widehat{u}}{\partial A}\right)\left(\frac{\partial \widehat{u}}{\partial B}\right) Cov(\widehat{A}, \widehat{B})$$

or:

$$Var(\widehat{u}) = \left(\frac{\widehat{u}}{\widehat{\beta}}\right)^2 Var(\widehat{\beta}) + \widehat{\beta}^2 Var(\widehat{A}) + \left(\frac{\widehat{\beta}}{V}\right)^2 Var(\widehat{B}) + 2\widehat{u} \cdot Cov(\widehat{\beta}, \widehat{A}) - \frac{2\widehat{u}}{V} Cov(\widehat{\beta}, \widehat{B}) - \frac{2\widehat{\beta}^2}{V} Cov(\widehat{A}, \widehat{B})$$

The upper and lower bounds on reliability are:

$$R_U = e^{-e^{(u_L)}}$$
$$R_L = e^{-e^{(u_U)}}$$

Confidence Bounds on Time

The bounds on time (ML estimate of time) for a given reliability are estimated by first solving the reliability function with respect to time:

$$\ln(R) = -\left(\widehat{T} \cdot V \cdot e^{\left(\widehat{A} - \frac{\widehat{B}}{V}\right)}\right)^{\widehat{\beta}}$$

$$\ln(-\ln(R)) = \widehat{\beta}\left(\ln \widehat{T} + \ln V + \widehat{A} - \frac{\widehat{B}}{V}\right)$$

or:

$$\widehat{u} = \frac{1}{\widehat{\beta}} \ln(-\ln(R)) - \ln V - \widehat{A} + \frac{\widehat{B}}{V}$$

where $\widehat{u}=ln(\widehat{T})$. The upper and lower bounds on $\widehat{u}$ are then estimated from:

$$u_U=\widehat{u}+K_{\alpha}\sqrt{Var(\widehat{u})}$$

$$u_L=\widehat{u}-K_{\alpha}\sqrt{Var(\widehat{u})}$$

where:

$$\begin{aligned}Var(\widehat{u})=&\left(\frac{\partial\widehat{u}}{\partial\beta}\right)^2Var(\widehat{\beta})+\left(\frac{\partial\widehat{u}}{\partial A}\right)^2Var(\widehat{A})+\left(\frac{\partial\widehat{u}}{\partial B}\right)^2Var(\widehat{B})+2\left(\frac{\partial\widehat{u}}{\partial\beta}\right)\left(\frac{\partial\widehat{u}}{\partial A}\right)Cov(\widehat{\beta},\widehat{A})\\&+2\left(\frac{\partial\widehat{u}}{\partial\beta}\right)\left(\frac{\partial\widehat{u}}{\partial B}\right)Cov(\widehat{\beta},\widehat{B})+2\left(\frac{\partial\widehat{u}}{\partial A}\right)\left(\frac{\partial\widehat{u}}{\partial B}\right)Cov(\widehat{A},\widehat{B})\end{aligned}$$

or:

$$Var(\widehat{u})=\frac{1}{\widehat{\beta}^4}[\ln(-\ln(R))]^2Var(\widehat{\beta})+Var(\widehat{A})+\frac{1}{V^2}Var(\widehat{B})+\frac{2\ln(-\ln(R))}{\widehat{\beta}^2}Cov(\widehat{\beta},\widehat{A})-\frac{2\ln(-\ln(R))}{\widehat{\beta}^2V}Cov(\widehat{\beta},\widehat{B})-\frac{2}{V}Cov(\widehat{A},\widehat{B})$$

The upper and lower bounds on time are then found by:

$$T_U=e^{u_U}$$
$$T_L=e^{u_L}$$

Approximate Confidence Bounds for the Eyring-Lognormal

Bounds on the Parameters

The lower and upper bounds on A and B are estimated from:

$$A_U=\widehat{A}+K_{\alpha}\sqrt{Var(\widehat{A})}\text{ (Upper bound)}$$
$$A_L=\widehat{A}-K_{\alpha}\sqrt{Var(\widehat{A})}\text{ (Lower bound)}$$

and:

$$B_U=\widehat{B}+K_{\alpha}\sqrt{Var(\widehat{B})}\text{ (Upper bound)}$$
$$B_L=\widehat{B}-K_{\alpha}\sqrt{Var(\widehat{B})}\text{ (Lower bound)}$$

Since the standard deviation, $\widehat{\sigma}_{T'}$, is a positive parameter, $\ln(\widehat{\sigma}_{T'})$ is treated as normally distributed, and the bounds are estimated from:

$$\sigma_U = \widehat{\sigma}_{T'} \cdot e^{\frac{K_\alpha \sqrt{Var(\widehat{\sigma}_{T'})}}{\widehat{\sigma}_{T'}}} \text{ (Upper bound)}$$

$$\sigma_L = \frac{\widehat{\sigma}_{T'}}{e^{\frac{K_\alpha \sqrt{Var(\widehat{\sigma}_{T'})}}{\widehat{\sigma}_{T'}}}} \text{ (Lower bound)}$$

The variances and covariances of $A, B,$ and $\sigma_{T'}$ are estimated from the local Fisher matrix (evaluated at $\widehat{A}, \widehat{B}, \widehat{\sigma}_{T'}$) as follows:

$$\begin{pmatrix} Var(\widehat{\sigma}_{T'}) & Cov\left(\widehat{A}, \widehat{\sigma}_{T'}\right) & Cov\left(\widehat{B}, \widehat{\sigma}_{T'}\right) \\ Cov\left(\widehat{\sigma}_{T'}, \widehat{A}\right) & Var\left(\widehat{A}\right) & Cov\left(\widehat{A}, \widehat{B}\right) \\ Cov\left(\widehat{\sigma}_{T'}, \widehat{B}\right) & Cov\left(\widehat{B}, \widehat{A}\right) & Var\left(\widehat{B}\right) \end{pmatrix} = [F]^{-1}$$

where:

$$F = \begin{pmatrix} -\frac{\partial^2 \Lambda}{\partial \sigma_{T'}^2} & -\frac{\partial^2 \Lambda}{\partial \sigma_{T'} \partial A} & -\frac{\partial^2 \Lambda}{\partial \sigma_{T'} \partial B} \\ -\frac{\partial^2 \Lambda}{\partial A \partial \sigma_{T'}} & -\frac{\partial^2 \Lambda}{\partial A^2} & -\frac{\partial^2 \Lambda}{\partial A \partial B} \\ -\frac{\partial^2 \Lambda}{\partial B \partial \sigma_{T'}} & -\frac{\partial^2 \Lambda}{\partial B \partial A} & -\frac{\partial^2 \Lambda}{\partial B^2} \end{pmatrix}$$

Bounds on Reliability

The reliability of the lognormal distribution is given by:

$$R(T', V; A, B, \sigma_{T'}) = \int_{T'}^{\infty} \frac{1}{\widehat{\sigma}_{T'} \sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{t + \ln(V) + \widehat{A} - \frac{\widehat{B}}{V}}{\widehat{\sigma}_{T'}}\right)^2} dt$$

Let $\hat{z}(t, V; A, B, \sigma_T) = \frac{t + \ln(V) + \widehat{A} - \frac{\widehat{B}}{V}}{\widehat{\sigma}_{T'}}$, then $\frac{d\hat{z}}{dt} = \frac{1}{\widehat{\sigma}_{T'}}$.

For $t=T'$, $\hat{z}=\frac{T'+\ln(V)+\widehat{A}-\frac{\widehat{B}}{V}}{\widehat{\sigma}_{T'}}$, and for $t=\infty$, $\hat{z}=\infty.$ The above equation then becomes:

$$R(\hat{z})=\int_{\hat{z}(T',V)}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz$$

The bounds on z are estimated from:

$$\begin{aligned} z_U &= \hat{z}+K_\alpha\sqrt{Var(\hat{z})} \\ z_L &= \hat{z}-K_\alpha\sqrt{Var(\hat{z})} \end{aligned}$$

where:

$$Var(\hat{z})=\left(\frac{\partial\hat{z}}{\partial A}\right)_{\widehat{A}}^2 Var(\widehat{A})+\left(\frac{\partial\hat{z}}{\partial B}\right)_{\widehat{B}}^2 Var(\widehat{B})+\left(\frac{\partial\hat{z}}{\partial\sigma_{T'}}\right)_{\widehat{\sigma}_{T'}}^2 Var(\widehat{\sigma}_T)+2\left(\frac{\partial\hat{z}}{\partial A}\right)_{\widehat{A}}\left(\frac{\partial\hat{z}}{\partial B}\right)_{\widehat{B}}Cov\left(\widehat{A},\widehat{B}\right)$$
$$+2\left(\frac{\partial\hat{z}}{\partial A}\right)_{\widehat{A}}\left(\frac{\partial\hat{z}}{\partial\sigma_{T'}}\right)_{\widehat{\sigma}_{T'}}Cov\left(\widehat{A},\widehat{\sigma}_T\right)+2\left(\frac{\partial\hat{z}}{\partial B}\right)_{\widehat{B}}\left(\frac{\partial\hat{z}}{\partial\sigma_{T'}}\right)_{\widehat{\sigma}_{T'}}Cov\left(\widehat{B},\widehat{\sigma}_T\right)$$

or:

$$Var(\hat{z})=\frac{1}{\widehat{\sigma}_{T'}^2}[Var(\widehat{A})+\frac{1}{V^2}Var(\widehat{B})+\hat{z}^2 Var(\widehat{\sigma}_{T'})-\frac{2}{V}Cov\left(\widehat{A},\widehat{B}\right)-2\hat{z}Cov\left(\widehat{A},\widehat{\sigma}_{T'}\right)+\frac{2\hat{z}}{V}Cov\left(\widehat{B},\widehat{\sigma}_{T'}\right)]$$

The upper and lower bounds on reliability are:

$$\begin{aligned} R_U &= \int_{z_L}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz\text{ (Upper bound)} \\ R_L &= \int_{z_U}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz\text{ (Lower bound)} \end{aligned}$$

Confidence Bounds on Time

The bounds around time for a given lognormal percentile (unreliability) are estimated by first solving the reliability equation with respect to time as follows:

$$T'(V;\widehat{A},\widehat{B},\widehat{\sigma}_{T'})=-\ln(V)-\widehat{A}+\frac{\widehat{B}}{V}+z\cdot\widehat{\sigma}_{T'}$$

where:

$$\begin{aligned} T'(V;\widehat{A},\widehat{B},\widehat{\sigma}_{T'}) = & \ \ln(T) \\ z = & \ \Phi^{-1}\left[F(T')\right] \end{aligned}$$

and:

$$\Phi(z)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z(T')} e^{-\frac{1}{2}z^2}dz$$

The next step is to calculate the variance of $T'(V;\widehat{A},\widehat{B},\widehat{\sigma}_{T'})$:

$$\begin{aligned} Var(T') = & \left(\frac{\partial T'}{\partial A}\right)^2 Var(\widehat{A})+\left(\frac{\partial T'}{\partial B}\right)^2 Var(\widehat{B})+\left(\frac{\partial T'}{\partial \sigma_{T'}}\right)^2 Var(\widehat{\sigma}_{T'})+2\left(\frac{\partial T'}{\partial A}\right)\left(\frac{\partial T'}{\partial B}\right)Cov\left(\widehat{A},\widehat{B}\right) \\ & +2\left(\frac{\partial T'}{\partial A}\right)\left(\frac{\partial T'}{\partial \sigma_{T'}}\right)Cov\left(\widehat{A},\widehat{\sigma}_{T'}\right)+2\left(\frac{\partial T'}{\partial B}\right)\left(\frac{\partial T'}{\partial \sigma_{T'}}\right)Cov\left(\widehat{B},\widehat{\sigma}_{T'}\right) \end{aligned}$$

or:

$$Var(T')=Var(\widehat{A})+\frac{1}{V}Var(\widehat{B})+\hat{z}^2 Var(\widehat{\sigma}_{T'})-\frac{2}{V}Cov\left(\widehat{A},\widehat{B}\right)-2\hat{z}Cov\left(\widehat{A},\widehat{\sigma}_{T'}\right)+\frac{2\hat{z}}{V}Cov\left(\widehat{B},\widehat{\sigma}_{T'}\right)$$

The upper and lower bounds are then found by:

$$\begin{aligned} T'_U = & \quad \ln T_U = T' + K_\alpha\sqrt{Var(T')} \\ T'_L = & \quad \ln T_L = T' - K_\alpha\sqrt{Var(T')} \end{aligned}$$

Solving for T_U and T_L yields:

$$\begin{aligned} T_U = & \quad e^{T'_U} \text{ (Upper bound)} \\ T_L = & \quad e^{T'_L} \text{ (Lower bound)} \end{aligned}$$

Inverse Power Law Relationship

IN THIS CHAPTER

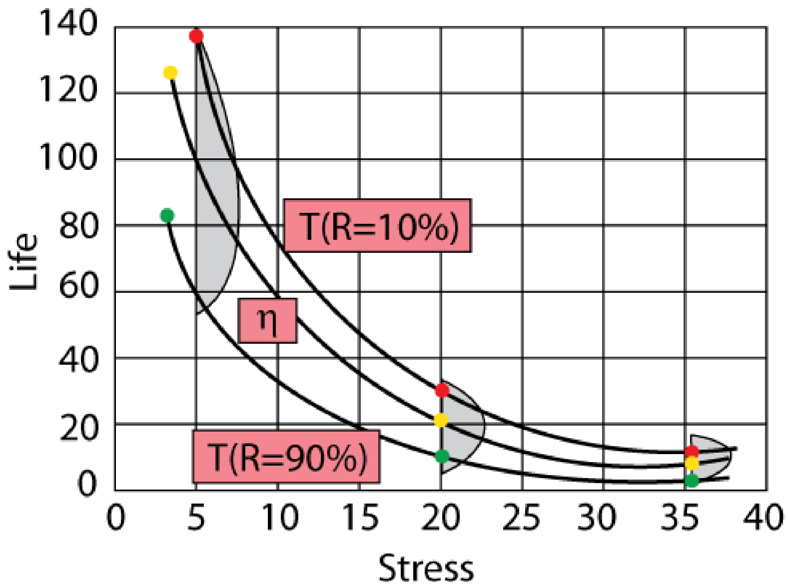
The inverse power law (IPL) model (or relationship) is commonly used for non-thermal accelerated stresses and is given by:

$$L(V) = \frac{1}{KV^n}$$

where:

- L represents a quantifiable life measure, such as mean life, characteristic life, median life, $B(x)$ life, etc.

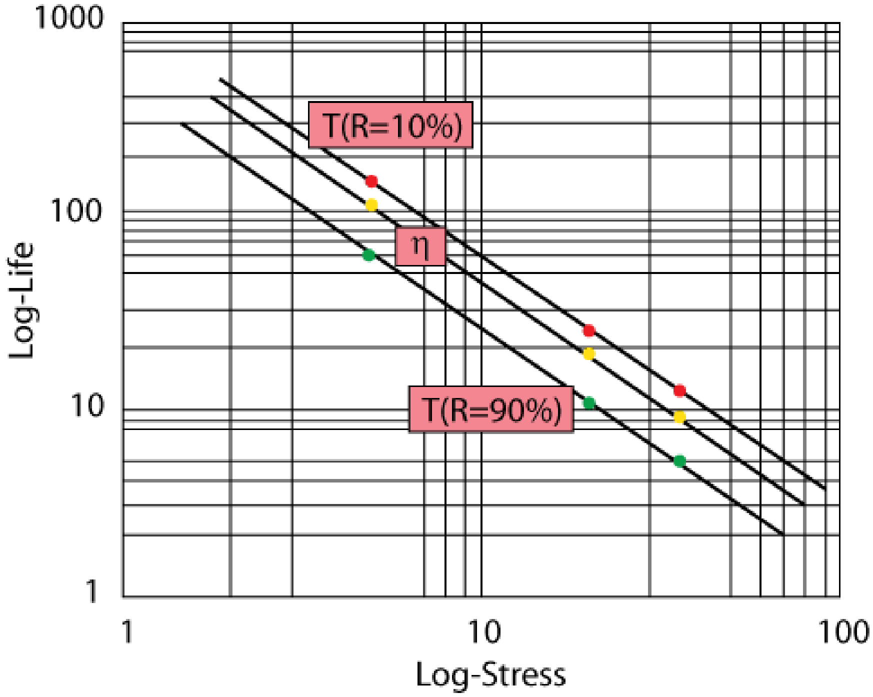
- V represents the stress level.
- K is one of the model parameters to be determined, $(K > 0)$.
- n is another model parameter to be determined.



The inverse power law appears as a straight line when plotted on a log-log paper. The equation of the line is given by:

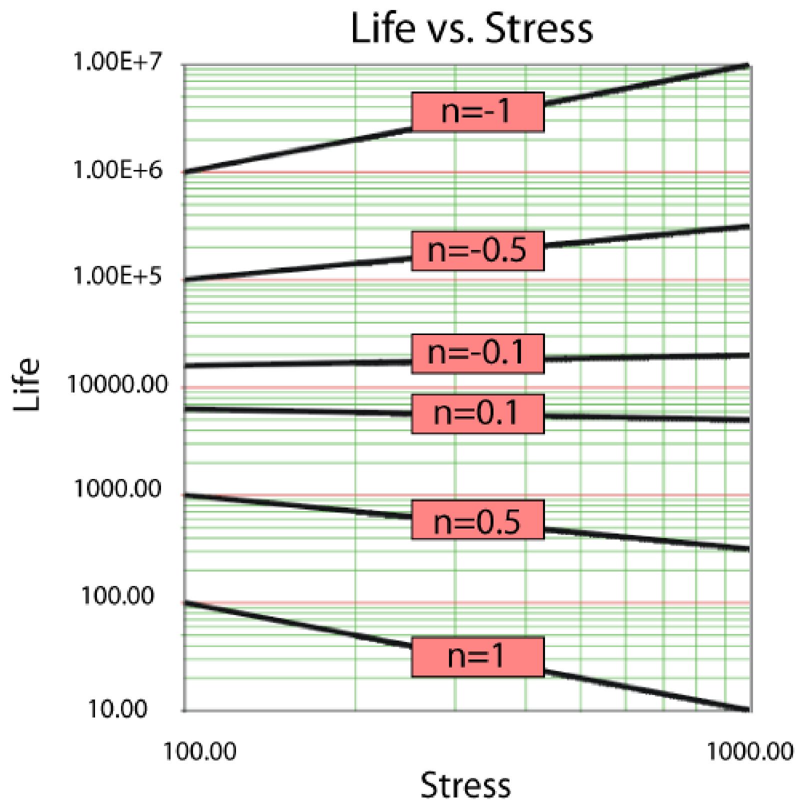
$$\ln(L) = -\ln(K) - n\ln(V)$$

Plotting methods are widely used in estimating the parameters of the inverse power law relationship since obtaining K and n is as simple as finding the slope and the intercept in the above equation.



A Look at the Parameter n

The parameter n in the inverse power relationship is a measure of the effect of the stress on the life. As the absolute value of n increases, the greater the effect of the stress. Negative values of n indicate an increasing life with increasing stress. An absolute value of n approaching zero indicates small effect of the stress on the life, with no effect (constant life with stress) when $n = 0$.



Acceleration Factor

For the IPL relationship the acceleration factor is given by:

$$A_F = \frac{L_{USE}}{L_{Accelerated}} = \frac{\frac{1}{KV_u^n}}{\frac{1}{KV_A^n}} = \left(\frac{V_A}{V_u}\right)^n$$

where:

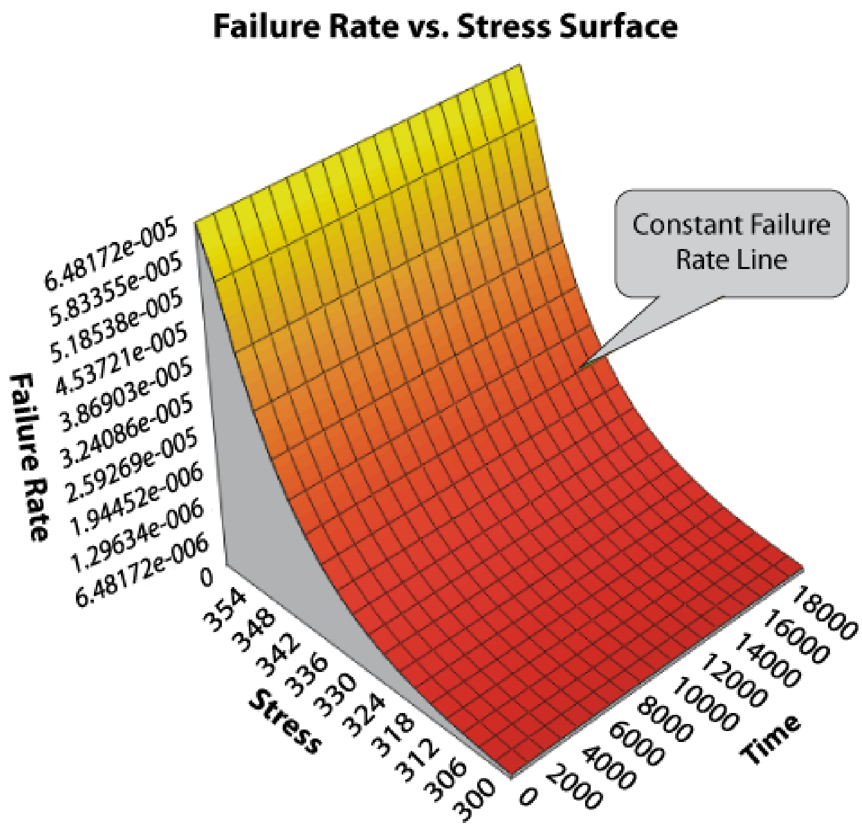
- L_{USE} is the life at use stress level.
- $L_{Accelerated}$ is the life at the accelerated stress level.
- V_u is the use stress level.
- V_A is the accelerated stress level.

IPL-Exponential

The IPL-exponential model can be derived by setting $m = L(V)$ in the exponential *pdf*, yielding the following IPL-exponential *pdf*:

$$f(t, V) = KV^n e^{-KV^n t}$$

Note that this is a 2-parameter model. The failure rate (the parameter of the exponential distribution) of the model is simply $\lambda = KV^n$, and is only a function of stress.



IPL-Exponential Statistical Properties Summary

Mean or MTTF

The mean, $\overline{T}$, or Mean Time To Failure (MTTF) for the IPL-exponential relationship is given by:

$$\overline{T} = \int_0^\infty t \cdot f(t,V)dt = \int_0^\infty t \cdot KV^n e^{-KV^n t}dt = \frac{1}{KV^n}$$

Note that the MTTF is a function of stress only and is simply equal to the IPL relationship (which is the original assumption), when using the exponential distribution.

Median

The median, $\breve{T}$, for the IPL-exponential model is given by:

$$\breve{T} = 0.693\frac{1}{KV^n}$$

Mode

The mode, $\tilde{T}$, for the IPL-exponential model is given by:

$$\tilde{T} = 0$$

Standard Deviation

The standard deviation, σ_T , for the IPL-exponential model is given by:

$$\sigma_T = \frac{1}{KV^n}$$

IPL-Exponential Reliability Function

The IPL-exponential reliability function is given by:

$$R(T,V) = e^{-TKV^n}$$

This function is the complement of the IPL-exponential cumulative distribution function:

$$R(T,V) = 1 - Q(T,V) = 1 - \int_0^T f(T,V)dT$$

or:

$$R(T,V)=1-\int_0^T KV^n e^{-KV^nT}dT=e^{-KV^nT}$$

Conditional Reliability

The conditional reliability function for the IPL-exponential model is given by:

$$R((t|T),V)=\frac{R(T+t,V)}{R(T,V)}=\frac{e^{-\lambda(T+t)}}{e^{-\lambda T}}=e^{-KV^nt}$$

Reliable Life

For the IPL-exponential model, the reliable life or the mission duration for a desired reliability goal, t_R, is given by:

$$R(t_R,V)=e^{-KV^nt_R}$$

$$\ln[R(t_R,V)]=-KV^nt_R$$

or:

$$t_R=-\frac{1}{KV^n}\ln[R(t_R,V)]$$

Parameter Estimation

Maximum Likelihood Parameter Estimation

Substituting the inverse power law relationship into the exponential log-likelihood equation yields:

$$\ln(L)=\Lambda=\sum_{i=1}^{F_e}N_i\ln\left[KV_i^ne^{-KV_i^nT_i}\right]-\sum_{i=1}^{S}N_i'KV_i^nT_i'+\sum_{i=1}^{FI}N_i''\ln[R_{Li}''-R_{Ri}'']$$

where:

$$R_{Li}''=e^{-T_{Li}''KV_i^n}$$

$$R''_{Ri} = e^{-T''_{Ri} K V_i^n}$$

and:

- F_e is the number of groups of exact times-to-failure data points.
- N_i is the number of times-to-failure in the i^{th} time-to-failure data group.
- V_i is the stress level of the i^{th} group.
- K is the IPL parameter (unknown, the first of two parameters to be estimated).
- n is the second IPL parameter (unknown, the second of two parameters to be estimated).
- T_i is the exact failure time of the i^{th} group.
- S is the number of groups of suspension data points.
- N'_i is the number of suspensions in the i^{th} group of suspension data points.
- T'_i is the running time of the i^{th} suspension data group.
- FI is the number of interval data groups.
- N''_i is the number of intervals in the i^{th} group of data intervals.
- T''_{Li} is the beginning of the i^{th} interval.
- T''_{Ri} is the ending of the i^{th} interval.

The solution (parameter estimates) will be found by solving for the parameters $\widehat{K}, \widehat{n}$ so that $\frac{\partial \Lambda}{\partial K} = 0$ and $\frac{\partial \Lambda}{\partial n} = 0$, where:

$$\frac{\partial \Lambda}{\partial K} = \frac{1}{K}\sum_{i=1}^{F_e} N_i - \sum_{i=1}^{F_e} N_i V_i^n T_i - \sum_{i=1}^{S} N'_i V_i^n T'_i - \sum_{i=1}^{FI} N''_i \frac{\left(T''_{Li} R''_{Li} - T''_{Ri} R''_{Ri}\right) V_i^n}{R''_{Li} - R''_{Ri}}$$

$$\frac{\partial \Lambda}{\partial n}=\sum_{i=1}^{F_e} N_i \ln(V_i)-K\sum_{i=1}^{F_e} N_i V_i^n \ln(V_i) T_i-K\sum_{i=1}^{S} N_i' V_i^n \ln(V_i) T_i' - \sum_{i=1}^{FI} N_i'' \frac{K V_i^n \ln(V_i)\left(T_{Li}'' R_{Li}''-T_{Ri}'' R_{Ri}''\right)}{R_{Li}''-R_{Ri}''}$$

IPL-Weibull

The IPL-Weibull model can be derived by setting $\eta = L(V)$ in the Weibull *pdf*, yielding the following IPL-Weibull *pdf*:

$$f(t,V)=\beta K V^n (K V^n t)^{\beta-1} e^{-(K V^n t)^{\beta}}$$

This is a three parameter model. Therefore it is more flexible but it also requires more laborious techniques for parameter estimation. The IPL-Weibull model yields the IPL-exponential model for $\beta = 1.$

IPL-Weibull Statistical Properties Summary

Mean or MTTF

The mean, $\overline{T}$ (also called $MTTF$), of the IPL-Weibull model is given by:

$$\overline{T}=\frac{1}{KV^n}\cdot\Gamma\left(\frac{1}{\beta}+1\right)$$

where $\Gamma\left(\frac{1}{\beta}+1\right)$ is the gamma function evaluated at the value of $\left(\frac{1}{\beta}+1\right)$.

Median

The median, $\breve{T},$ of the IPL-Weibull model is given by:

$$\breve{T}=\frac{1}{KV^n}(\ln 2)^{\frac{1}{\beta}}$$

Mode

The mode, $\tilde{T},$ of the IPL-Weibull model is given by:

$$\tilde{T} = \frac{1}{KV^n}\left(1 - \frac{1}{\beta}\right)^{\frac{1}{\beta}}$$

Standard Deviation

The standard deviation, σ_T, of the IPL-Weibull model is given by:

$$\sigma_T = \frac{1}{KV^n} \cdot \sqrt{\Gamma\left(\frac{2}{\beta} + 1\right) - \left(\Gamma\left(\frac{1}{\beta} + 1\right)\right)^2}$$

IPL-Weibull Reliability Function

The IPL-Weibull reliability function is given by:

$$R(T, V) = e^{-(KV^n T)^\beta}$$

Conditional Reliability Function

The IPL-Weibull conditional reliability function at a specified stress level is given by:

$$R((t|T), V) = \frac{R(T + t, V)}{R(T, V)} = \frac{e^{-[KV^n(T+t)]^\beta}}{e^{-(KV^n T)^\beta}}$$

or:

$$R((t|T), V) = e^{-\left[(KV^n(T+t))^\beta - (KV^n T)^\beta\right]}$$

Reliable Life

For the IPL-Weibull model, the reliable life, T_R, of a unit for a specified reliability and starting the mission at age zero is given by:

$$T_R = \frac{1}{KV^n}\{-\ln[R(T_R, V)]\}^{\frac{1}{\beta}}$$

IPL-Weibull Failure Rate Function

The IPL-Weibull failure rate function, $\lambda(T)$, is given by:

$$\lambda(T,V)=\frac{f(T,V)}{R(T,V)}=\beta K V^{n}\left(KV^{n}T\right)^{\beta-1}$$

Parameter Estimation

Maximum Likelihood Estimation Method

Substituting the inverse power law relationship into the Weibull log-likelihood function yields:

$$\Lambda=\sum_{i=1}^{F_e}N_i\ln\left[\beta K V_i^{n}\left(KV_i^{n}T_i\right)^{\beta-1}e^{-\left(KV_i^{n}T_i\right)^{\beta}}\right]-\sum_{i=1}^{S}N_i'\left(KV_i^{n}T_i'\right)^{\beta}+\sum_{i=1}^{FI}N_i''\ln[R_{Li}''-R_{Ri}'']$$

where:

$$R_{Li}''=e^{-\left(KV_i^{n}T_{Li}''\right)^{\beta}}$$

$$R_{Ri}''=e^{-\left(KV_i^{n}T_{Ri}''\right)^{\beta}}$$

and:

- F_e is the number of groups of exact times-to-failure data points.
- N_i is the number of times-to-failure data points in the i^{th} time-to-failure data group.
- β is the Weibull shape parameter (unknown, the first of three parameters to be estimated).
- K is the IPL parameter (unknown, the second of three parameters to be estimated).
- n is the second IPL parameter (unknown, the third of three parameters to be estimated).
- V_i is the stress level of the i^{th} group.
- T_i is the exact failure time of the i^{th} group.
- S is the number of groups of suspension data points.
- N_i' is the number of suspensions in the i^{th} group of suspension data points.
- T_i' is the running time of the i^{th} suspension data group.
- FI is the number of interval data groups.

- N''_i is the number of intervals in the i^{th} group of data intervals.
- T''_{Li} is the beginning of the i^{th} interval.
- T''_{Ri} is the ending of the i^{th} interval.

The solution (parameter estimates) will be found by solving for β, K, n so that $\frac{\partial \Lambda}{\partial \beta} = 0$, $\frac{\partial \Lambda}{\partial K} = 0$ and $\frac{\partial \Lambda}{\partial n} = 0$, where:

$$\begin{aligned}\frac{\partial \Lambda}{\partial \beta} = {} & \frac{1}{\beta}\sum_{i=1}^{F_e} N_i + \sum_{i=1}^{F_e} N_i \ln\left(KV_i^n T_i\right) - \sum_{i=1}^{F_e} N_i \left(KV_i^n T_i\right)^{\beta} \ln\left(KV_i^n T_i\right) - \sum_{i=1}^{S} N'_i \left(KV_i^n T'_i\right)^{\beta} \ln\left(KV_i^n T'_i\right) \\ & - \sum_{i=1}^{FI} N''_i \frac{\left(KV_i^n\right)^{\beta}\left[R''_{Li} T''^{\beta}_{Li} \left(\ln(KV_i^n T''_{Li})\right) - R''_{Ri} T''^{\beta}_{Ri} \left(\ln(KV_i^n T''_{Ri})\right)\right]}{R''_{Li} - R''_{Ri}}\end{aligned}$$

$$\frac{\partial \Lambda}{\partial K} = \frac{\beta}{K}\sum_{i=1}^{F_e} N_i - \frac{\beta}{K}\sum_{i=1}^{F_e} N_i \left(KV_i^n T_i\right)^{\beta} - \frac{\beta}{K}\sum_{i=1}^{S} N'_i \left(KV_i^n T'_i\right)^{\beta} - \beta\sum_{i=1}^{FI} N''_i \frac{K^{\beta-1} V_i^{n\beta}\left[T''^{\beta}_{Li} R''_{Li} - T''^{\beta}_{Ri} R''_{Ri}\right]}{R''_{Li} - R''_{Ri}}$$

$$\frac{\partial \Lambda}{\partial n} = \beta\sum_{i=1}^{F_e} N_i \ln(V_i) - \beta\sum_{i=1}^{F_e} N_i \ln(V_i)\left(KV_i^n T_i\right)^{\beta} - \beta\sum_{i=1}^{S} N'_i \ln(V_i)\left(KV_i^n T_i\right)^{\beta} - \sum_{i=1}^{FI} N''_i \frac{nK^{\beta} V_i^{\beta(n-1)}\left[T''^{\beta}_{Li} R''_{Li} - T''^{\beta}_{Ri} R''_{Ri}\right]}{R''_{Li} - R''_{Ri}}$$

IPL-Weibull Example

Consider the following times-to-failure data at two different stress levels.

Stress	20 V	36 V	36 V
Time Failed (hrs)		2.3	
		3.2	7.5
	11.7	3.7	8.6
	16.2	4.7	9.0
	18.3	4.9	10.2
	23.8	5.7	10.9
		6.0	12.0
		6.8	14.0

The data set was analyzed jointly in a life-stress data folio using the IPL-Weibull model, with a complete MLE solution over the entire data set. The analysis yields:

$$\widehat{\beta} = 2.616464$$

$$\widehat{K} = 0.001022$$

$$\widehat{n} = 1.327292$$

IPL-Lognormal

The *pdf* for the Inverse Power Law relationship and the lognormal distribution is given next.

The *pdf* of the lognormal distribution is given by:

$$f(T) = \frac{1}{T\ \sigma_{T'}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{T'-\overline{T'}}{\sigma_{T'}}\right)^2}$$

where:

$$T' = ln(T)$$

and:

T = times-to-failure.

$\overline{T}'$ = mean of the natural logarithms of the times-to-failure.

$\sigma_{T'}$ = standard deviation of the natural logarithms of the times-to-failure.

The median of the lognormal distribution is given by:

$$\breve{T} = e^{\overline{T}'}$$

The IPL-lognormal model *pdf* can be obtained first by setting $\breve{T} = L(V)$ in the lognormal *pdf*. Therefore:

$$\breve{T} = L(V) = \frac{1}{K \cdot V^n}$$

or:

$$e^{\overline{T'}} = \frac{1}{K \cdot V^n}$$

Thus:

$$\bar{T}' = -ln(K) - nln(V)$$

So the IPL-lognormal model *pdf* is:

$$f(T,V) = \frac{1}{T\ \sigma_{T'}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{T'+ln(K)+nln(V)}{\sigma_{T'}}\right)^2}$$

IPL-Lognormal Statistical Properties Summary

The Mean

The mean life of the IPL-lognormal model (mean of the times-to-failure), $\bar{T}$, is given by:

$$\bar{T} = e^{\bar{T}'+\frac{1}{2}\sigma_{T'}^2} = e^{-ln(K)-nln(V)+\frac{1}{2}\sigma_{T'}^2}$$

The mean of the natural logarithms of the times-to-failure, $\bar{T}'$, in terms of $\bar{T}$ and σ_T is given by:

$$\bar{T}' = \ln(\bar{T}) - \frac{1}{2}\ln\left(\frac{\sigma_T^2}{\bar{T}^2}+1\right)$$

The Standard Deviation

The standard deviation of the IPL-lognormal model (standard deviation of the times-to-failure), σ_T, is given by:

$$\sigma_T = \sqrt{\left(e^{2\bar{T}'+\sigma_{T'}^2}\right)\left(e^{\sigma_{T'}^2}-1\right)} = \sqrt{\left(e^{2(-\ln(K)-n\ln(V))+\sigma_{T'}^2}\right)\left(e^{\sigma_{T'}^2}-1\right)}$$

The standard deviation of the natural logarithms of the times-to-failure, $\sigma_{T'}$, in terms of $\bar{T}$ and σ_T is given by:

$$\sigma_{T'} = \sqrt{\ln\left(\frac{\sigma_T^2}{\bar{T}^2}+1\right)}$$

The Mode

The mode of the IPL-lognormal model is given by:

$$\tilde{T} = e^{\bar{T}' - \sigma_{T'}^2} = e^{-\ln(K) - n\ln(V) - \sigma_{T'}^2}$$

IPL-Lognormal Reliability

The reliability for a mission of time T, starting at age 0, for the IPL-lognormal model is determined by:

$$R(T, V) = \int_T^{\infty} f(t, V)dt$$

or:

$$R(T, V) = \int_{T'}^{\infty} \frac{1}{\sigma_{T'}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{t + \ln(K) + n\ln(V)}{\sigma_{T'}}\right)^2} dt$$

Reliable Life

The reliable life, or the mission duration for a desired reliability goal, t_R, is estimated by first solving the reliability equation with respect to time, as follows:

$$T'_R = -\ln(K) - n\ln(V) + z \cdot \sigma_{T'}$$

where:

$$z = \Phi^{-1}\left[F\left(T'_R, V\right)\right]$$

and:

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z(T', V)} e^{-\frac{t^2}{2}} dt$$

Since $T' = \ln(T)$ the reliable life, t_R, , is given by:

$$t_R = e^{T'_R}$$

Lognormal Failure Rate

The lognormal failure rate is given by:

$$\lambda(T,V)=\frac{f(T,V)}{R(T,V)}=\frac{\frac{1}{T\,\sigma_{T'}\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{T'+\ln(K)+n\ln(V)}{\sigma_{T'}}\right)^2}}{\int_{T'}^{\infty}\frac{1}{\sigma_{T'}\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{T'+\ln(K)+n\ln(V)}{\sigma_{T'}}\right)^2}dt}$$

Parameter Estimation

Maximum Likelihood Estimation Method

The complete IPL-lognormal log-likelihood function is:

$$\ln(L)=\Lambda=\sum_{i=1}^{F_e}N_i\ln\left[\frac{1}{\sigma_{T'}T_i}\varphi\left(\frac{\ln(T_i)+\ln(K)+n\ln(V_i)}{\sigma_{T'}}\right)\right]+\sum_{i=1}^{S}N_i'\ln\left[1-\Phi\left(\frac{\ln(T_i')+\ln(K)+n\ln(V_i)}{\sigma_{T'}}\right)\right]+\sum_{i=1}^{FI}N_i''\ln[\Phi(z_{Ri}'')-\Phi(z_{Li}'')]$$

where:

$$z_{Li}''=\frac{\ln T_{Li}''+\ln K+n\ln V_i}{\sigma_T'}$$

$$z_{Ri}''=\frac{\ln T_{Ri}''+\ln K+n\ln V_i}{\sigma_T'}$$

and:

- F_e is the number of groups of exact times-to-failure data points.
- N_i is the number of times-to-failure data points in the i^{th} time-to-failure data group.
- $s_{T'}$ is the standard deviation of the natural logarithm of the times-to-failure (unknown, the first of three parameters to be estimated).
- K is the IPL parameter (unknown, the second of three parameters to be estimated).
- n is the second IPL parameter (unknown, the third of three parameters to be estimated).
- Vi is the stress level of the i^{th} group.
- Ti is the exact failure time of the i^{th} group.
- S is the number of groups of suspension data points.

- N_i' is the number of suspensions in the i^{th} group of suspension data points.
- T_i' is the running time of the i^{th} suspension data group.
- FI is the number of interval data groups.
- N_i'' is the number of intervals in the i^{th} group of data intervals.
- T_{Li}'' is the beginning of the i^{th} interval.
- T_{Ri}'' is the ending of the i^{th} interval.

The solution (parameter estimates) will be found by solving for $\hat{\sigma}_{T'}$, $\hat{K}$, $\hat{n}$ so that $\frac{\partial \Lambda}{\partial \sigma_{T'}} = 0$, , $\frac{\partial \Lambda}{\partial K} = 0$ and $\frac{\partial \Lambda}{\partial n} = 0$: :

$$\frac{\partial \Lambda}{\partial K} = -\frac{1}{K \cdot \sigma_{T'}^2} \sum_{i=1}^{F_e} N_i (\ln(T_i) + \ln(K) + n \ln(V_i)) - \frac{1}{K \cdot \sigma_{T'}} \sum_{i=1}^{S} N_i' \frac{\varphi\left(\frac{\ln(T_i') + \ln(K) + n \ln(V_i)}{\sigma_{T'}}\right)}{1 - \Phi\left(\frac{\ln(T_i') + \ln(K) + n \ln(V_i)}{\sigma_{T'}}\right)} + \sum_{i=1}^{FI} N_i'' \frac{\phi(z_{Ri}'') - \phi(z_{Li}'')}{K \sigma_T' (\Phi(z_{Ri}'') - \Phi(z_{Li}''))}$$

$$\frac{\partial \Lambda}{\partial n} = -\frac{1}{\sigma_{T'}^2} \sum_{i=1}^{F_e} N_i \ln(V_i) \left[\ln(T_i) + \ln(K) + n \ln(V_i)\right] - \frac{1}{\sigma_{T'}} \sum_{i=1}^{S} N_i' \ln(V_i) \frac{\varphi\left(\frac{\ln(T_i') + \ln(K) + n \ln(V_i)}{\sigma_{T'}}\right)}{1 - \Phi\left(\frac{\ln(T_i') + \ln(K) + n \ln(V_i)}{\sigma_{T'}}\right)} + \sum_{i=1}^{FI} N_i'' \frac{\ln V_i \left(\phi(z_{Ri}'') - \phi(z_{Li}'')\right)}{\sigma_T' (\Phi(z_{Ri}'') - \Phi(z_{Li}''))}$$

$$\frac{\partial \Lambda}{\partial \sigma_{T'}} = \sum_{i=1}^{F_e} N_i \left(\frac{(\ln(T_i) + \ln(K) + n \ln(V_i))^2}{\sigma_{T'}^3} - \frac{1}{\sigma_{T'}} \right) + \frac{1}{\sigma_{T'}} \sum_{i=1}^{S} N_i' \frac{\left(\frac{\ln(T_i') + \ln(K) + n \ln(V_i)}{\sigma_{T'}}\right) \varphi\left(\frac{\ln(T_i') + \ln(K) + n \ln(V_i)}{\sigma_{T'}}\right)}{1 - \Phi\left(\frac{\ln(T_i') + \ln(K) + n \ln(V_i)}{\sigma_{T'}}\right)} - \sum_{i=1}^{FI} N_i'' \frac{z_{Ri}'' \phi(z_{Ri}'') - z_{Li}'' \phi(z_{Li}'')}{\sigma_T' (\Phi(z_{Ri}'') - \Phi(z_{Li}''))}$$

and:

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}(x)^2}$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt$$

IPL and the Coffin-Manson Relationship

In accelerated life testing analysis, thermal cycling is commonly treated as a low-cycle fatigue problem, using the inverse power law relationship. Coffin and Manson suggested that the number of cycles-to-failure of a metal subjected to thermal cycling is given by Nelson [28]:

$$N = \frac{C}{(\Delta T)^{\gamma}}$$

where:

- N is the number of cycles to failure.
- C is a constant, characteristic of the metal.
- γ is another constant, also characteristic of the metal.
- ΔT is the range of the thermal cycle.

This relationship is basically the inverse power law relationship, where the stress $V,$ is substituted by the range ΔV . This is an attempt to simplify the analysis of a time-varying stress test by using a constant stress model. It is a very commonly used methodology for thermal cycling and mechanical fatigue tests. However, by performing such a simplification, the following assumptions and shortcomings are inevitable. First, the acceleration effects due to the stress rate of change are ignored. In other words, it is assumed that the failures are accelerated by the stress difference and not by how rapidly this difference occurs. Secondly, the acceleration effects due to stress relaxation and creep are ignored.

Example

In this example the use of the Coffin-Manson relationship will be illustrated. This is a very simple example which can be repeated at any time. The reader is encouraged to perform this test.

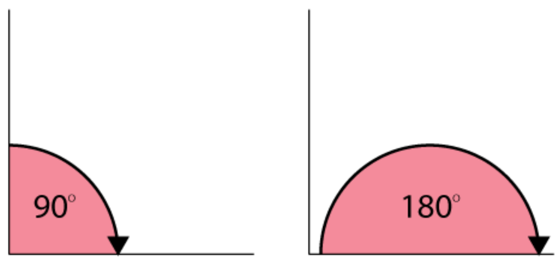
Product: ACME Paper Clip Model 4456

Reliability Target: 99% at a 90% confidence after 30 cycles of 45°

After consulting with our paper-clip engineers, the acceleration stress was determined to be the angle to which the clips are bent. Two bend stresses of 90° and 180° were used. A sample of six paper clips was tested to failure at both 90° and 180° bends with the following data obtained.

Degrees	90°	180°
Cycles-to-Failure	16	4
	17	4
	18	5
	21	6
	22	6
	23	8

The test was performed as shown in the next figures (a side-view of the paper-clip is shown).

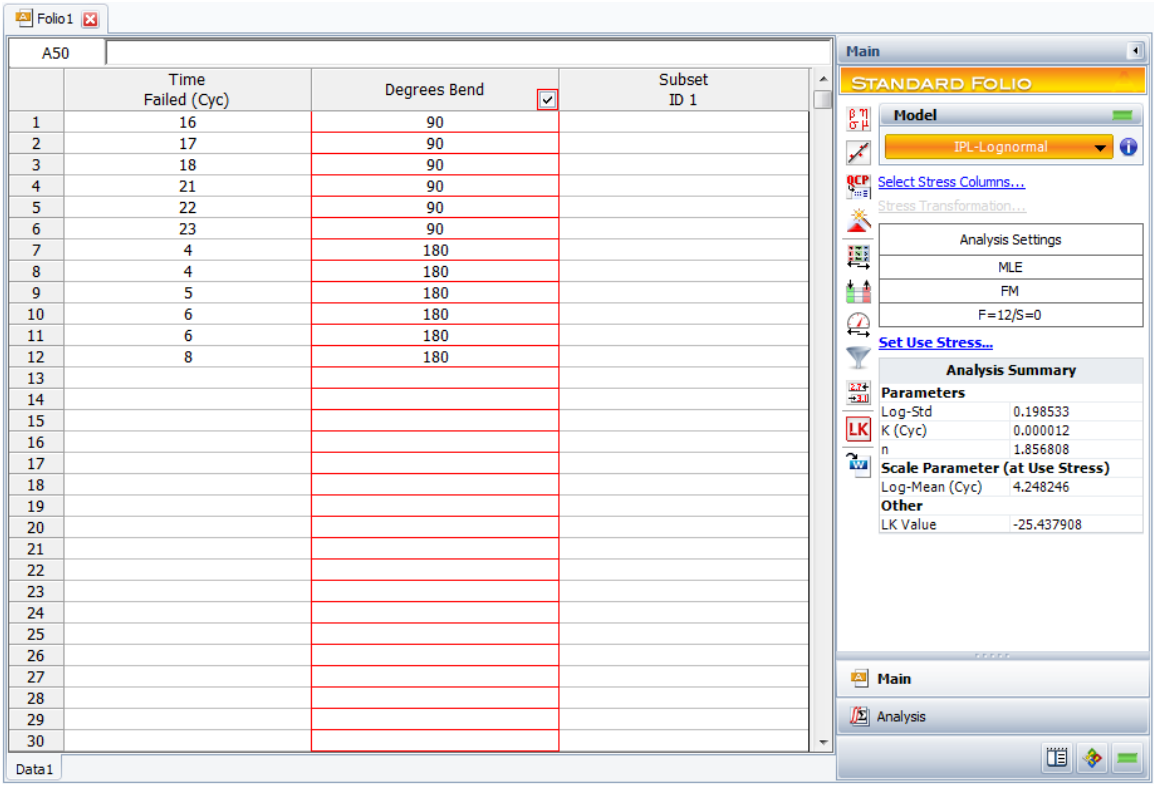


Using the IPL-lognormal model, determine whether the reliability target was met.

Solution

By using the IPL relationship to analyze the data, we are actually using a constant stress model to analyze a cycling process. Caution must be exercised when performing the test. The rate of change in the angle must be constant and equal for both the 90° and 180° bends and constant and equal to the rate of change in the angle for the use life of 45° bend. Rate effects are influencing the life of the paper clip. By keeping the rate constant and equal at all stress levels, we can then eliminate these rate effects from our analysis. Otherwise the analysis will not be valid.

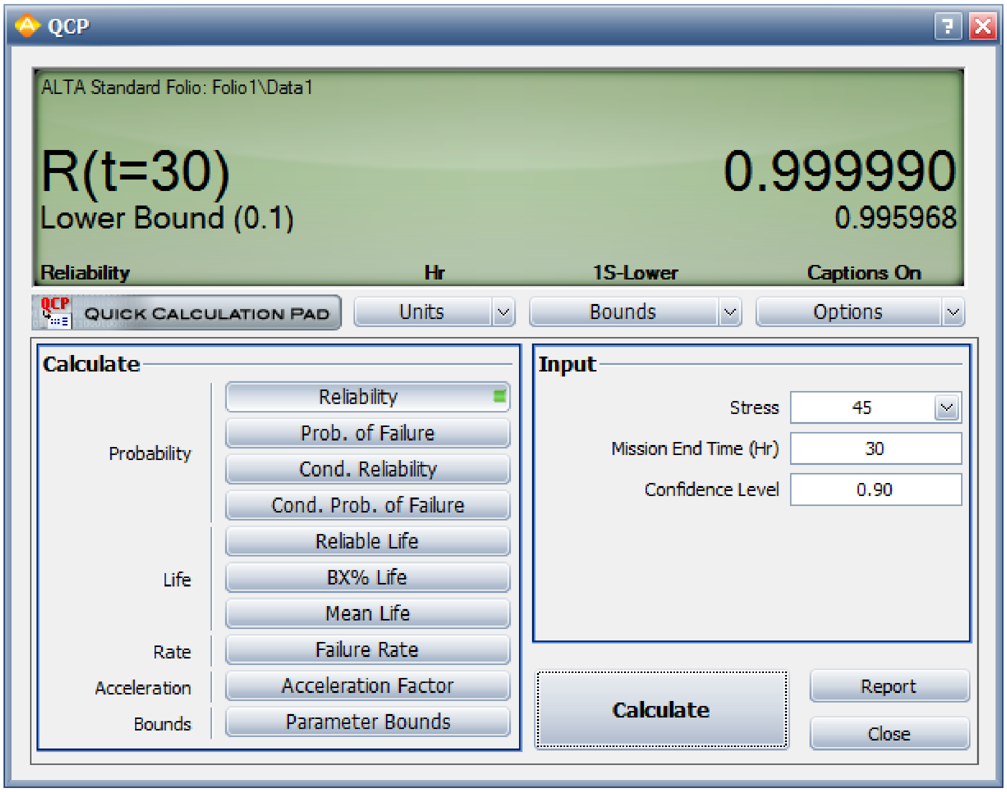
The data were entered and analyzed using ReliaSoft's Weibull++.



The parameters of the IPL-lognormal model were estimated to be:

$$\begin{aligned} \sigma = & \quad 0.198533 \\ K = & \quad 0.000012 \\ n = & \quad 1.856808 \end{aligned}$$

Using the QCP, the 90% lower 1-sided confidence bound on reliability after 30 cycles for a 45° bend was estimated to be 99.6%, as shown below.



This meets the target reliability of 99%.

IPL Confidence Bounds

Approximate Confidence Bounds on IPL-Exponential

Confidence Bounds on the Mean Life

From the inverse power law relationship the mean life for the exponential distribution is given by setting $m = L(V)$. The upper (m_U) and lower (m_L) bounds on the mean life (ML estimate of the mean life) are estimated by:

$$m_U = \widehat{m} \cdot e^{\frac{K_\alpha \sqrt{Var(\widehat{m})}}{\widehat{m}}}$$

$$m_L = \widehat{m} \cdot e^{-\frac{K_\alpha \sqrt{Var(\widehat{m})}}{\widehat{m}}}$$

where K_α is defined by:

$$\alpha = \frac{1}{\sqrt{2\pi}} \int_{K_\alpha}^{\infty} e^{-\frac{t^2}{2}} dt = 1 - \Phi(K_\alpha)$$

If δ is the confidence level, then $\alpha = \frac{1-\delta}{2}$ for the two-sided bounds, and $\alpha = 1 - \delta$ for the one-sided bounds. The variance of $\widehat{m}$ is given by:

$$Var(\widehat{m}) = \left(\frac{\partial m}{\partial K}\right)^2 Var(\widehat{K}) + \left(\frac{\partial m}{\partial n}\right)^2 Var(\widehat{n}) + 2\left(\frac{\partial m}{\partial K}\right)\left(\frac{\partial m}{\partial n}\right) Cov(\widehat{K}, \widehat{n})$$

or:

$$Var(\widehat{m}) = \frac{1}{\widehat{K}^2 V^{2\widehat{n}}} \left[\frac{1}{\widehat{K}^2} Var(\widehat{K}) + [\ln(V)]^2 Var(\widehat{n}) + \frac{2\ln(V)}{\widehat{K}} Cov(\widehat{K}, \widehat{n})\right]$$

The variances and covariance of K and n are estimated from the Fisher matrix (evaluated at $\widehat{K}, \widehat{n})$ as follows:

$$\begin{bmatrix} Var(\widehat{K}) & Cov(\widehat{K}, \widehat{n}) \\ Cov(\widehat{n}, \widehat{K}) & Var(\widehat{n}) \end{bmatrix} = \begin{bmatrix} -\frac{\partial^2 \Lambda}{\partial K^2} & -\frac{\partial^2 \Lambda}{\partial K \partial n} \\ -\frac{\partial^2 \Lambda}{\partial n \partial K} & -\frac{\partial^2 \Lambda}{\partial n^2} \end{bmatrix}^{-1}$$

Confidence Bounds on Reliability

The bounds on reliability at a given time, T, are estimated by:

$$R_U = e^{-\frac{T}{m_U}}$$
$$R_L = e^{-\frac{T}{m_L}}$$

Confidence Bounds on Time

The bounds on time (ML estimate of time) for a given reliability are estimated by first solving the reliability function with respect to time:

$$\widehat{T}=-\widehat{m}\cdot \ln(R)$$

The corresponding confidence bounds are estimated from:

$$\begin{aligned} T_U = & -m_U\cdot \ln(R) \\ T_L = & -m_L\cdot \ln(R) \end{aligned}$$

Approximate Confidence Bounds on IPL-Weibull

Bounds on the Parameters

Using the same approach as previously discussed ($\widehat{\beta}$ and $\widehat{K}$ positive parameters):

$$\begin{aligned} \beta_U = & \widehat{\beta}\cdot e^{\frac{K_\alpha\sqrt{Var(\widehat{\beta})}}{\widehat{\beta}}} \\ \beta_L = & \widehat{\beta}\cdot e^{-\frac{K_\alpha\sqrt{Var(\widehat{\beta})}}{\widehat{\beta}}} \end{aligned}$$

$$\begin{aligned} K_U = & \widehat{K}\cdot e^{\frac{K_\alpha\sqrt{Var(\widehat{K})}}{\widehat{K}}} \\ K_L = & \widehat{K}\cdot e^{-\frac{K_\alpha\sqrt{Var(\widehat{K})}}{\widehat{K}}} \end{aligned}$$

and:

$$\begin{aligned} n_U = & \widehat{n}+K_\alpha\sqrt{Var(\widehat{n})} \\ n_L = & \widehat{n}-K_\alpha\sqrt{Var(\widehat{n})} \end{aligned}$$

The variances and covariances of $\beta, K,$ and n are estimated from the local Fisher matrix (evaluated at $\widehat{\beta}, \widehat{K}, \widehat{n})$ as follows:

$$\begin{bmatrix} Var(\widehat{\beta}) & Cov(\widehat{\beta},\widehat{K}) & Cov(\widehat{\beta},\widehat{n}) \\ Cov(\widehat{K},\widehat{\beta}) & Var(\widehat{K}) & Cov(\widehat{K},\widehat{n}) \\ Cov(\widehat{n},\widehat{\beta}) & Cov(\widehat{n},\widehat{K}) & Var(\widehat{n}) \end{bmatrix} = \begin{bmatrix} -\frac{\partial^2\Lambda}{\partial\beta^2} & -\frac{\partial^2\Lambda}{\partial\beta\partial K} & -\frac{\partial^2\Lambda}{\partial\beta\partial n} \\ -\frac{\partial^2\Lambda}{\partial K\partial\beta} & -\frac{\partial^2\Lambda}{\partial K^2} & -\frac{\partial^2\Lambda}{\partial K\partial n} \\ -\frac{\partial^2\Lambda}{\partial n\partial\beta} & -\frac{\partial^2\Lambda}{\partial n\partial A} & -\frac{\partial^2\Lambda}{\partial n^2} \end{bmatrix}^{-1}$$

Confidence Bounds on Reliability

The reliability function (ML estimate) for the IPL-Weibull model is given by:

$$\widehat{R}(T,V) = e^{-\left(\widehat{K}V^{\widehat{n}}T\right)^{\widehat{\beta}}}$$

or:

$$\widehat{R}(T,V) = e^{-e^{\ln\left[\left(\widehat{K}V^{\widehat{n}}T\right)^{\widehat{\beta}}\right]}}$$

Setting:

$$\widehat{u} = \ln\left[\left(\widehat{K}V^{\widehat{n}}T\right)^{\widehat{\beta}}\right]$$

or:

$$\widehat{u} = \widehat{\beta}\left[\ln(T) + \ln(\widehat{K}) + \widehat{n}\ln(V)\right]$$

The reliability function now becomes:

$$\widehat{R}(T,V) = e^{-e^{\widehat{u}}}$$

The next step is to find the upper and lower bounds on $\widehat{u}$:

$$u_U = \widehat{u} + K_\alpha\sqrt{Var(\widehat{u})}$$

$$u_L = \widehat{u} - K_\alpha \sqrt{Var(\widehat{u})}$$

where:

$$Var(\widehat{u}) = \left(\frac{\partial \widehat{u}}{\partial \beta}\right)^2 Var(\widehat{\beta}) + \left(\frac{\partial \widehat{u}}{\partial K}\right)^2 Var(\widehat{K}) + \left(\frac{\partial \widehat{u}}{\partial n}\right)^2 Var(\widehat{n}) + 2\left(\frac{\partial \widehat{u}}{\partial \beta}\right)\left(\frac{\partial \widehat{u}}{\partial K}\right) Cov(\widehat{\beta}, \widehat{K})$$
$$+ 2\left(\frac{\partial \widehat{u}}{\partial \beta}\right)\left(\frac{\partial \widehat{u}}{\partial n}\right) Cov(\widehat{\beta}, \widehat{n}) + 2\left(\frac{\partial \widehat{u}}{\partial K}\right)\left(\frac{\partial \widehat{u}}{\partial n}\right) Cov(\widehat{K}, \widehat{n})$$

or:

$$Var(\widehat{u}) = \left(\frac{\widehat{u}}{\widehat{\beta}}\right)^2 Var(\widehat{\beta}) + \left(\frac{\widehat{\beta}}{\widehat{K}}\right)^2 Var(\widehat{K}) + \widehat{\beta}^2 [\ln(V)]^2 Var(\widehat{n}) + \frac{2\widehat{u}}{\widehat{K}} Cov(\widehat{\beta}, \widehat{K}) + 2\widehat{u}\ln(V) Cov(\widehat{\beta}, \widehat{n}) + \frac{2\widehat{\beta}^2 \ln(V)}{\widehat{K}} Cov(\widehat{K}, \widehat{n})$$

The upper and lower bounds on reliability are:

$$R_U = e^{-e^{(u_L)}}$$
$$R_L = e^{-e^{(u_U)}}$$

Confidence Bounds on Time

The bounds on time for a given reliability (ML estimate of time) are estimated by first solving the reliability function with respect to time:

$$\ln(R) = -\left(\widehat{K} V^{\widehat{n}} \widehat{T}\right)^{\widehat{\beta}}$$
$$\ln(-\ln(R)) = \widehat{\beta}\left[\ln(\widehat{T}) + \ln(\widehat{K}) + \widehat{n}\ln(V)\right]$$

or:

$$\widehat{u} = \frac{1}{\widehat{\beta}}\ln(-\ln(R)) - \ln(\widehat{K}) - \widehat{n}\ln(V)$$

where $\widehat{u} = \ln \widehat{T}$. The upper and lower bounds on u are estimated from:

$$u_U = \widehat{u} + K_\alpha \sqrt{Var(\widehat{u})}$$
$$u_L = \widehat{u} - K_\alpha \sqrt{Var(\widehat{u})}$$

where:

$$Var(\widehat{u}) = \left(\frac{\partial \widehat{u}}{\partial \beta}\right)^2 Var(\widehat{\beta}) + \left(\frac{\partial \widehat{u}}{\partial K}\right)^2 Var(\widehat{K}) + \left(\frac{\partial \widehat{u}}{\partial n}\right)^2 Var(\widehat{n}) + 2\left(\frac{\partial \widehat{u}}{\partial \beta}\right)\left(\frac{\partial \widehat{u}}{\partial K}\right) Cov(\widehat{\beta},\widehat{K})$$
$$+2\left(\frac{\partial \widehat{u}}{\partial \beta}\right)\left(\frac{\partial \widehat{u}}{\partial n}\right) Cov(\widehat{\beta},\widehat{n}) + 2\left(\frac{\partial \widehat{u}}{\partial K}\right)\left(\frac{\partial \widehat{u}}{\partial n}\right) Cov(\widehat{K},\widehat{n})$$

or:

$$Var(\widehat{u}) = \frac{1}{\widehat{\beta}^4}[\ln(-\ln(R))]^2 Var(\widehat{\beta}) + \frac{1}{\widehat{K}^2} Var(\widehat{K}) + [\ln(V)]^2 Var(\widehat{n}) + \frac{2\ln(-\ln(R))}{\widehat{\beta}^2 \widehat{K}} Cov(\widehat{\beta},\widehat{K})$$
$$+\frac{2\ln(-\ln(R))}{\widehat{\beta}^2}\ln(V) Cov(\widehat{\beta},\widehat{n}) + \frac{2\ln(V)}{\widehat{K}} Cov(\widehat{K},\widehat{n})$$

The upper and lower bounds on time are then found by:

$$T_U = e^{u_U}$$
$$T_L = e^{u_L}$$

Approximate Confidence Bounds on IPL-Lognormal

Bounds on the Parameters

Since the standard deviation, $\widehat{\sigma}_T$, and $\widehat{K}$ are positive parameters, then $\ln(\widehat{\sigma}_{T'})$ and $\ln(\widehat{K})$ are treated as normally distributed, and the bounds are estimated from:

$$\sigma_U = \widehat{\sigma}_{T'} \cdot e^{\frac{K_\alpha \sqrt{Var(\widehat{\sigma}_{T'})}}{\widehat{\sigma}_{T'}}} \quad \text{(Upper bound)}$$
$$\sigma_L = \frac{\widehat{\sigma}_{T'}}{e^{\frac{K_\alpha \sqrt{Var(\widehat{\sigma}_{T'})}}{\widehat{\sigma}_{T'}}}} \quad \text{(Lower bound)}$$

and:

$$K_U = \widehat{K} \cdot e^{\frac{K_\alpha \sqrt{Var(\widehat{K})}}{\widehat{K}}} \quad \text{(Upper bound)}$$
$$K_L = \frac{\widehat{K}}{e^{\frac{K_\alpha \sqrt{Var(\widehat{K})}}{\widehat{K}}}} \quad \text{(Lower bound)}$$

The lower and upper bounds on n , are estimated from:

$$\begin{aligned} n_U = & \quad \widehat{n} + K_\alpha \sqrt{Var(\widehat{n})} \text{ (Upper bound)} \\ n_L = & \quad \widehat{n} - K_\alpha \sqrt{Var(\widehat{n})} \text{ (Lower bound)} \end{aligned}$$

The variances and covariances of $A, B,$ and $\sigma_{T'}$ are estimated from the local Fisher matrix (evaluated at $\widehat{A}, \widehat{B}, \widehat{\sigma}_{T'}),$ as follows:

$$\begin{bmatrix} Var(\widehat{\sigma}_{T'}) & Cov(\widehat{K}, \widehat{\sigma}_{T'}) & Cov(\widehat{n}, \widehat{\sigma}_{T'}) \\ Cov(\widehat{\sigma}_{T'}, \widehat{K}) & Var(\widehat{K}) & Cov(\widehat{K}, \widehat{n}) \\ Cov(\widehat{\sigma}_{T'}, \widehat{n}) & Cov(\widehat{n}, \widehat{K}) & Var\,(\widehat{n}) \end{bmatrix} = [F]^{-1}$$

where:

$$F = \begin{bmatrix} -\frac{\partial^2 \Lambda}{\partial \sigma_{T'}^2} & -\frac{\partial^2 \Lambda}{\partial \sigma_{T'} \partial K} & -\frac{\partial^2 \Lambda}{\partial \sigma_{T'} \partial n} \\ -\frac{\partial^2 \Lambda}{\partial K \partial \sigma_{T'}} & -\frac{\partial^2 \Lambda}{\partial K^2} & -\frac{\partial^2 \Lambda}{\partial K \partial n} \\ -\frac{\partial^2 \Lambda}{\partial n \partial \sigma_{T'}} & -\frac{\partial^2 \Lambda}{\partial n \partial K} & -\frac{\partial^2 \Lambda}{\partial n^2} \end{bmatrix}$$

Bounds on Reliability

The reliability of the lognormal distribution is:

$$R(T', V; K, n, \sigma_{T'}) = \int_{T'}^{\infty} \frac{1}{\widehat{\sigma}_{T'} \sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{t + \ln(\widehat{K}) + \widehat{n} \ln(V)}{\widehat{\sigma}_{T'}}\right)^2} dt$$

Let $\hat{z}(t, V; K, n, \sigma_T) = \frac{t + \ln(\widehat{K}) + \widehat{n} \ln(V)}{\widehat{\sigma}_{T'}},$ then $\frac{d\hat{z}}{dt} = \frac{1}{\widehat{\sigma}_{T'}}.$

For $t = T'$, $\hat{z} = \frac{T' + \ln(\widehat{K}) + \widehat{n} \ln(V)}{\widehat{\sigma}_{T'}}$, and for $t = \infty, \hat{z} = \infty.$ The above equation then becomes:

$$R(\hat{z}) = \int_{\hat{z}(T', V)}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z^2} dz$$

The bounds on z are estimated from:

$$z_U = \hat{z} + K_\alpha \sqrt{Var(\hat{z})}$$
$$z_L = \hat{z} - K_\alpha \sqrt{Var(\hat{z})}$$

where:

$$Var(\hat{z}) = \left(\frac{\partial \hat{z}}{\partial K}\right)^2_{\widehat{K}} Var(\widehat{K}) + \left(\frac{\partial \hat{z}}{\partial n}\right)^2_{\widehat{n}} Var(\widehat{n}) + \left(\frac{\partial \hat{z}}{\partial \sigma_{T'}}\right)^2_{\widehat{\sigma}_{T'}} Var(\widehat{\sigma}_T) + 2\left(\frac{\partial \hat{z}}{\partial K}\right)_{\widehat{K}} \left(\frac{\partial \hat{z}}{\partial n}\right)_{\widehat{n}} Cov\left(\widehat{K}, \widehat{n}\right)$$
$$+ 2\left(\frac{\partial \hat{z}}{\partial K}\right)_{\widehat{K}} \left(\frac{\partial \hat{z}}{\partial \sigma_{T'}}\right)_{\widehat{\sigma}_{T'}} Cov\left(\widehat{K}, \widehat{\sigma}_T\right) + 2\left(\frac{\partial \hat{z}}{\partial n}\right)_{\widehat{n}} \left(\frac{\partial \hat{z}}{\partial \sigma_{T'}}\right)_{\widehat{\sigma}_{T'}} Cov\left(\widehat{n}, \widehat{\sigma}_T\right)$$

.

or:

$$Var(\hat{z}) = \frac{1}{\widehat{\sigma}^2_{T'}}[\frac{1}{K^2} Var(\widehat{K}) + \ln(V)^2 Var(\widehat{n}) + \hat{z}^2 Var(\widehat{\sigma}_{T'}) + \frac{2\ln(V)}{K} Cov\left(\widehat{K}, \widehat{n}\right) - \frac{2\hat{z}}{K} Cov\left(\widehat{K}, \widehat{\sigma}_{T'}\right) - 2\hat{z}\ln(V) Cov\left(\widehat{n}, \widehat{\sigma}_{T'}\right)]$$

The upper and lower bounds on reliability are:

$$R_U = \int_{z_L}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \text{ (Upper bound)}$$
$$R_L = \int_{z_U}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \text{ (Lower bound)}$$

Confidence Bounds on Time

The bounds around time, for a given lognormal percentile (unreliability), are estimated by first solving the reliability equation with respect to time, as follows:

$$T'(V; \widehat{K}, \widehat{n}, \widehat{\sigma}_{T'}) = -\ln(\widehat{K}) - \widehat{n}\ln(V) + z \cdot \widehat{\sigma}_{T'}$$

where:

$$T'(V; \widehat{K}, \widehat{n}, \widehat{\sigma}_{T'}) = \ln(T)$$

$$z = \Phi^{-1}\left[F(T')\right]$$

and:

$$\Phi(z)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z(T')} e^{-\frac{1}{2}z^2}dz$$

The next step is to calculate the variance of $T'(V;\widehat{K},\widehat{n},\widehat{\sigma}_{T'})$:

$$\begin{aligned}Var(T')=&\left(\frac{\partial T'}{\partial K}\right)^2 Var(\widehat{K})+\left(\frac{\partial T'}{\partial n}\right)^2 Var(\widehat{n})+\left(\frac{\partial T'}{\partial \sigma_{T'}}\right)^2 Var(\widehat{\sigma}_{T'})+2\left(\frac{\partial T'}{\partial K}\right)\left(\frac{\partial T'}{\partial n}\right)Cov\left(\widehat{K},\widehat{n}\right)\\&+2\left(\frac{\partial T'}{\partial K}\right)\left(\frac{\partial T'}{\partial \sigma_{T'}}\right)Cov\left(\widehat{K},\widehat{\sigma}_{T'}\right)+2\left(\frac{\partial T'}{\partial n}\right)\left(\frac{\partial T'}{\partial \sigma_{T'}}\right)Cov\left(\widehat{n},\widehat{\sigma}_{T'}\right)\end{aligned}$$

or:

$$\begin{aligned}Var(T')=&\frac{1}{K^2}Var(\widehat{K})+\ln(V)^2 Var(\widehat{n})+\hat{z}^2 Var(\widehat{\sigma}_{T'})+\frac{2\ln(V)}{K}Cov\left(\widehat{K},\widehat{n}\right)\\&-\frac{2\hat{z}}{K}Cov\left(\widehat{K},\widehat{\sigma}_{T'}\right)-2\hat{z}\ln(V)Cov\left(\widehat{n},\widehat{\sigma}_{T'}\right)\end{aligned}$$

The upper and lower bounds are then found by:

$$\begin{aligned}T'_U=&\quad \ln T_U=T'+K_\alpha\sqrt{Var(T')}\\T'_L=&\quad \ln T_L=T'-K_\alpha\sqrt{Var(T')}\end{aligned}$$

Solving for T_U and T_L yields:

$$\begin{aligned}T_U=&\quad e^{T'_U}\text{ (Upper bound)}\\T_L=&\quad e^{T'_L}\text{ (Lower bound)}\end{aligned}$$

Temperature-Humidity Relationship

IN THIS CHAPTER

The Temperature-Humidity (T-H) relationship, a variation of the Eyring relationship, has been proposed for predicting the life at use conditions when temperature and humidity are the accelerated stresses in a test. This combination model is given by:

$$L(V,U) = Ae^{\frac{\phi}{V}+\frac{b}{U}}$$

where:

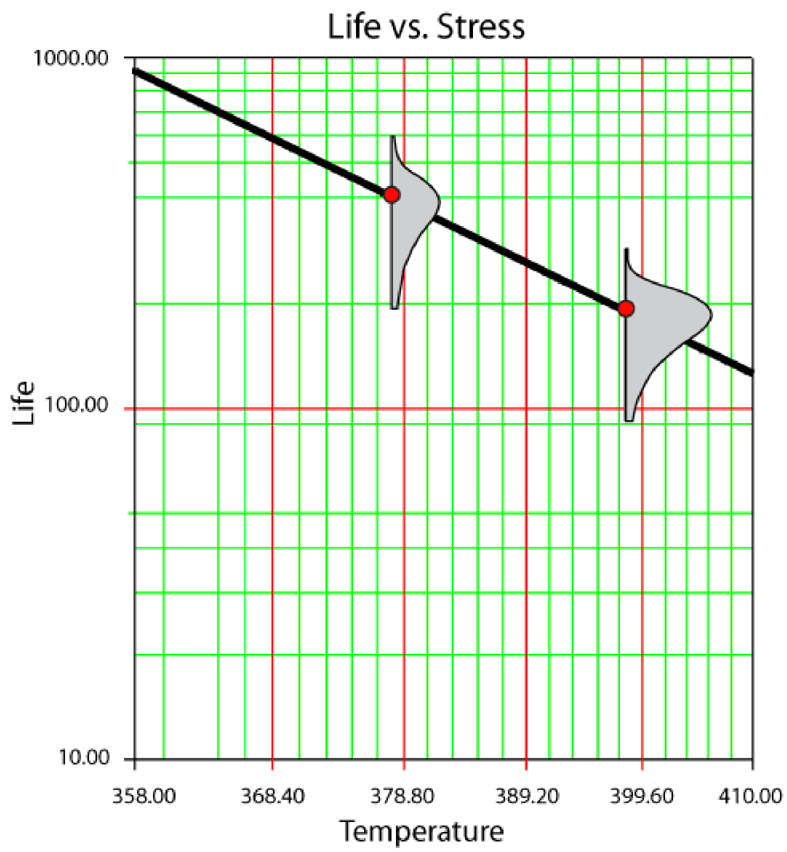
- ϕ is one of the three parameters to be determined.

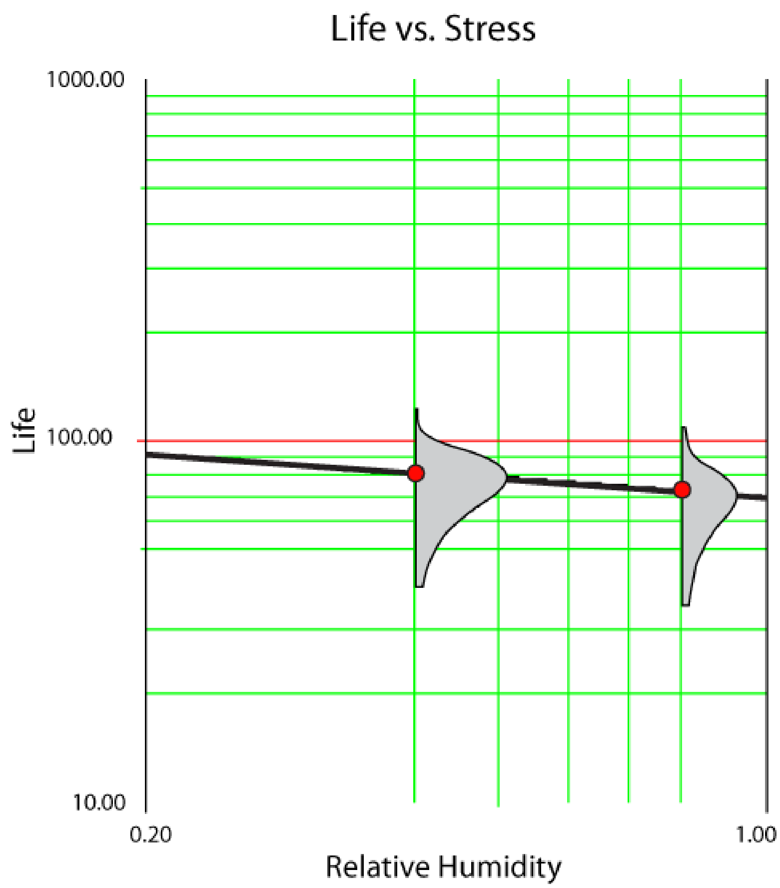
- b is the second of the three parameters to be determined (also known as the activation energy for humidity).
- A is a constant and the third of the three parameters to be determined.
- U is the relative humidity (decimal or percentage).
- V is temperature (**in absolute units**).

The T-H relationship can be linearized and plotted on a Life vs. Stress plot. The relationship is linearized by taking the natural logarithm of both sides in the T-H relationship, or:

$$ln(L(V,U)) = ln(A) + \frac{\phi}{V} + \frac{b}{U}$$

Since life is now a function of two stresses, a Life vs. Stress plot can only be obtained by keeping one of the two stresses constant and varying the other one. Doing so will yield a straight line where the term for the stress which is kept at a fixed value becomes another constant (in addition to the $\ln(A)$ constant). In the next two figures, data obtained from a temperature and humidity test were analyzed and plotted on Arrhenius paper. In the first figure, life is plotted versus temperature with relative humidity held at a fixed value. In the second figure, life is plotted versus relative humidity with temperature held at a fixed value.


Life vs. Stress
1000.00
100.00
10.00
Life
358.00
368.40
378.80
389.20
399.60
410.00
Temperature



Note that the Life vs. Stress plots are plotted on a log-reciprocal scale. Also note that the points shown in these plots represent the life characteristics at the test stress levels (the data set was fitted to a Weibull distribution, thus the points represent the scale parameter, η) . For example, the points shown in the first figure represent η at each of the test temperature levels (two temperature levels were considered in this test).

A look at the Parameters Phi and b

Depending on which stress type is kept constant, it can be seen from the linearized T-H relationship that either the parameter ϕ or the parameter b is the slope of the resulting line. If, for example, the humidity is kept constant then ϕ is the slope of the life line in a Life vs. Temperature plot. The steeper the slope, the greater the dependency of product life to the temperature. In other words, ϕ is a measure of the effect that temperature has on the life, and b is a measure of the effect that relative humidity has on the life. The larger the value of $\phi,$ the higher the dependency of the life on the temperature. Similarly, the larger the value of $b,$ the higher the dependency of the life on the humidity.

T-H Data

When using the T-H relationship, the effect of both temperature and humidity on life is sought. For this reason, the test must be performed in a combination manner between the different stress levels of the two stress types. For example, assume that an accelerated test is to be performed at two temperature and two humidity levels. The two temperature levels were chosen to be 300K and 343K. The two humidity levels were chosen to be 0.6 and 0.8. It would be wrong to perform the test at (300K, 0.6) and (343K, 0.8). Doing so would not provide information about the temperature-humidity effects on life. This is because both stresses are increased at the same time and therefore it is unknown which stress is causing the acceleration on life. A possible combination that would provide information about temperature-humidity effects on life would be (300K, 0.6), (300K, 0.8) and (343K, 0.8). It is clear that by testing at (300K, 0.6) and (300K, 0.8) the effect of humidity on life can be determined (since temperature remained constant). Similarly the effects of temperature on life can be determined by testing at (300K, 0.8) and (343K, 0.8) since humidity remained constant.

Acceleration Factor

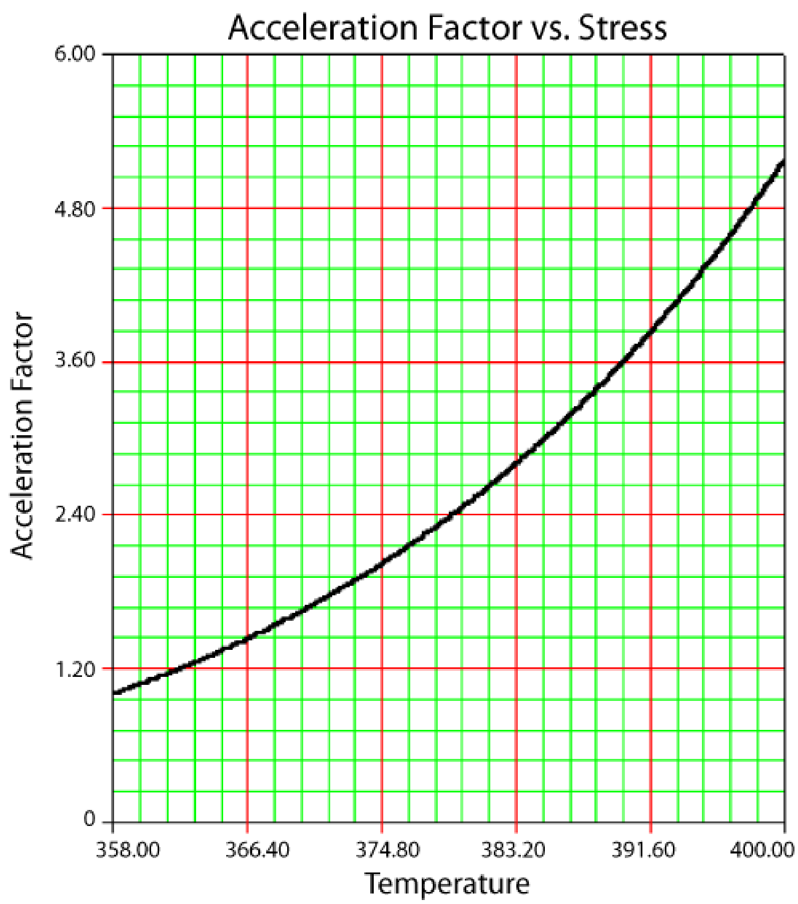
The acceleration factor for the T-H relationship is given by:

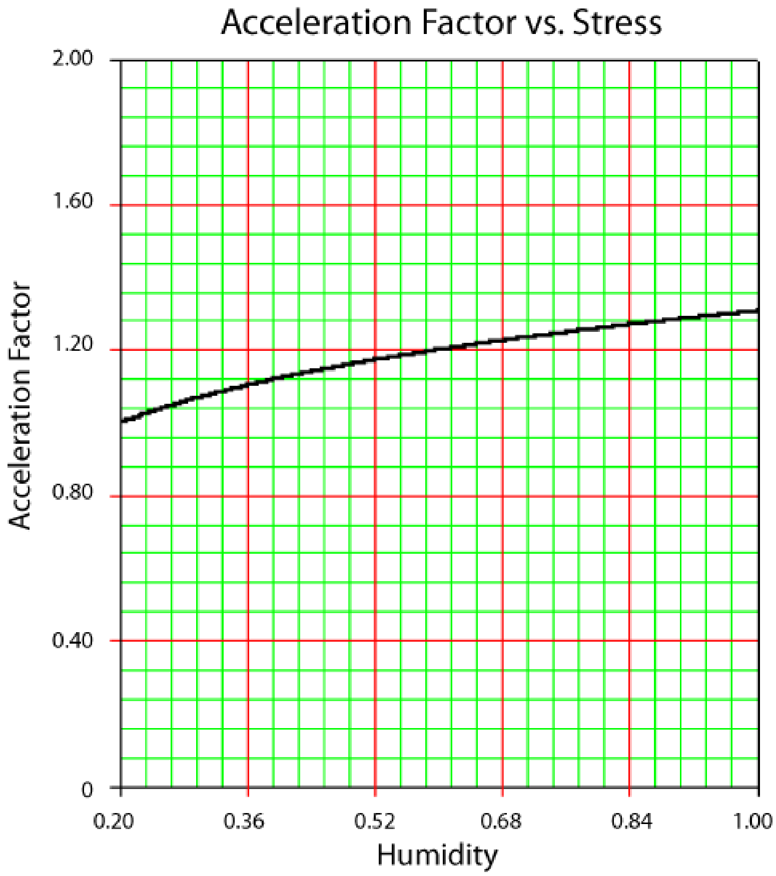
$$A_F = \frac{L_{USE}}{L_{Accelerated}} = \frac{Ae^{\frac{\phi}{V_u}+\frac{b}{U_u}}}{Ae^{\frac{\phi}{V_A}+\frac{b}{U_A}}} = e^{\phi\left(\frac{1}{V_u}-\frac{1}{V_A}\right)+b\left(\frac{1}{U_u}-\frac{1}{U_A}\right)}$$

where:

- L_{USE} is the life at use stress level.
- $L_{Accelerated}$ is the life at the accelerated stress level.
- V_u is the use temperature level.
- V_A is the accelerated temperature level.
- U_A is the accelerated humidity level.
- U_u is the use humidity level.

The acceleration Factor is plotted versus stress in the same manner used to create the Life vs. Stress plots. That is, one stress type is kept constant and the other is varied as shown in the next two figures.





T-H Exponential

By setting $m = L(U, V)$ in the exponential *pdf* we can obtain the T-H exponential *pdf*:

$$f(t, V, U) = \frac{1}{A} e^{-\left(\frac{\phi}{V} + \frac{b}{U}\right)} \cdot e^{-\frac{t}{A} \cdot e^{-\left(\frac{\phi}{V} + \frac{b}{U}\right)}}$$

T-H Exponential Statistical Properties Summary

Mean or MTTF

The mean, $\overline{T}$, or Mean Time To Failure (MTTF) for the T-H exponential model is given by:

$$\overline{T} = \int_0^\infty t \cdot f(t, V, U) dt$$

Substituting the T-H exponential *pdf* equation yields:

$$\overline{T} = \int_0^{\infty} t \cdot \frac{1}{A} e^{-\left(\frac{\phi}{V}+\frac{b}{U}\right)} e^{-\frac{t}{A} e^{-\left(\frac{\phi}{V}+\frac{b}{U}\right)}} dt = Ae^{\frac{\phi}{V}+\frac{b}{U}}$$

Median

The median, $\breve{T}$, for the T-H exponential model is given by:

$$\breve{T} = 0.693 \cdot Ae^{\frac{\phi}{V}+\frac{b}{U}}$$

Mode

The mode, $\tilde{T}$, for the T-H exponential model is given by:

$$\tilde{T} = 0$$

Standard Deviation

The standard deviation, σ_T , for the T-H exponential model is given by:

$$\sigma_T = Ae^{\frac{\phi}{V}+\frac{b}{U}}$$

T-H Exponential Reliability Function

The T-H exponential reliability function is given by:

$$R(T,V,U) = e^{-\frac{T}{A} e^{-\left(\frac{\phi}{V}+\frac{b}{U}\right)}}$$

This function is the complement of the T-H exponential cumulative distribution function or:

$$R(T,V,U) = 1 - Q(T,V,U) = 1 - \int_0^T f(T)dT$$

and:

$$R(T,V,U)=1-\int_0^T \frac{1}{A}e^{-\left(\frac{\phi}{V}+\frac{b}{U}\right)}e^{-\frac{T}{A}e^{-\left(\frac{\phi}{V}+\frac{b}{U}\right)}}dT=e^{-\frac{T}{A}e^{-\left(\frac{\phi}{V}+\frac{b}{U}\right)}}$$

Conditional Reliability

The conditional reliability function for the T-H exponential model is given by:

$$R((t|T),V,U)=\frac{R(T+t,V,U)}{R(T,V,U)}=\frac{e^{-\lambda(T+t)}}{e^{-\lambda T}}=e^{-\frac{t}{A}e^{-\left(\frac{\phi}{V}+\frac{b}{U}\right)}}$$

Reliable Life

For the T-H exponential model, the reliable life, or the mission duration for a desired reliability goal, t_R, is given by:

$$R(t_R,V,U)=e^{-\frac{t_R}{A}e^{-\left(\frac{\phi}{V}+\frac{b}{U}\right)}}$$

$$\ln[R(t_R,V,U)]=-\frac{t_R}{A}e^{-\left(\frac{\phi}{V}+\frac{b}{U}\right)}$$

or:

$$t_R=-Ae^{\frac{\phi}{V}+\frac{b}{U}}\ln[R(t_R,V,U)]$$

Parameter Estimation

Maximum Likelihood Estimation Method

Substituting the T-H model into the exponential log-likelihood equation yields:

$$\ln(L)=\quad \Lambda=\sum_{i=1}^{F_e}N_i\ln\left[\frac{1}{A}e^{-\left(\frac{\phi}{V_i}+\frac{b}{U_i}\right)}\cdot e^{-\frac{T_i}{A}\cdot e^{-\left(\frac{\phi}{V_i}+\frac{b}{U_i}\right)}}\right]-\sum_{i=1}^{S}N_i'\frac{T_i'}{A}e^{-\left(\frac{\phi}{V_i}+\frac{b}{U_i}\right)}+\sum_{i=1}^{FI}N_i''\ln[R_{Li}''-R_{Ri}'']$$

where:

$$R''_{Li} = e^{-\frac{T''_{Li}}{A} e^{-\left(\frac{\phi}{V_i} + \frac{b}{U''_i}\right)}}$$

$$R''_{Ri} = e^{-\frac{T''_{Ri}}{A} e^{-\left(\frac{\phi}{V_i} + \frac{b}{U''_i}\right)}}$$

and:

- F_e is the number of groups of exact times-to-failure data points.
- N_i is the number of times-to-failure data points in the i^{th} time-to-failure data group.
- A is the T-H parameter (unknown, the first of three parameters to be estimated).
- ϕ is the second T-H parameter (unknown, the second of three parameters to be estimated).
- b is the third T-H parameter (unknown, the third of three parameters to be estimated).
- V_i is the temperature level of the i^{th} group.
- U_i is the relative humidity level of the i^{th} group.
- T_i is the exact failure time of the i^{th} group.
- S is the number of groups of suspension data points.
- N'_i is the number of suspensions in the i^{th} group of suspension data points.
- T'_i is the running time of the i^{th} suspension data group.
- FI is the number of interval data groups.
- N''_i is the number of intervals in the i^{th} group of data intervals.
- T''_{Li} is the beginning of the i^{th} interval.
- T''_{Ri} is the ending of the i^{th} interval.

The solution (parameter estimates) will be found by solving for the parameters A, ϕ and b so that $\frac{\partial \Lambda}{\partial A} = 0$, $\frac{\partial \Lambda}{\partial \phi} = 0$ and $\frac{\partial \Lambda}{\partial b} = 0$.

T-H Weibull

By setting $\eta = L(U,V)$ in the Weibull *pdf*, the T--H Weibull model's *pdf* is given by:

$$f(t,V,U)=\frac{\beta}{A}e^{-\left(\frac{\varphi}{V}+\frac{b}{U}\right)}\left(\frac{t}{A}e^{-\left(\frac{\varphi}{V}+\frac{b}{U}\right)}\right)^{\beta-1}e^{-\left(\frac{t}{A}e^{-\left(\frac{\varphi}{V}+\frac{b}{U}\right)}\right)^{\beta}}$$

T-H Weibull Statistical Properties Summary

Mean or MTTF

The mean, $\overline{T}$ (also called $MTTF$), of the T-H Weibull model is given by:

$$\overline{T}=Ae^{\frac{\phi}{V}+\frac{b}{U}}\cdot\Gamma\left(\frac{1}{\beta}+1\right)$$

where $\Gamma\left(\frac{1}{\beta}+1\right)$ is the gamma function evaluated at the value of $\left(\frac{1}{\beta}+1\right)$.

Median

The median, $\breve{T},$ of the T-H Weibull model is given by:

$$\breve{T}=Ae^{\frac{\phi}{V}+\frac{b}{U}}\left(\ln 2\right)^{\frac{1}{\beta}}$$

Mode

The mode, $\tilde{T},$ of the T-H Weibull model is given by:

$$\tilde{T}=Ae^{\frac{\phi}{V}+\frac{b}{U}}\left(1-\frac{1}{\beta}\right)^{\frac{1}{\beta}}$$

Standard Deviation

The standard deviation, $\sigma_T,$ of the T-H Weibull model is given by:

$$\sigma_T = Ae^{\frac{\phi}{V}+\frac{b}{U}} \cdot \sqrt{\Gamma\left(\frac{2}{\beta}+1\right) - \left(\Gamma\left(\frac{1}{\beta}+1\right)\right)^2}$$

T-H Weibull Reliability Function

The T-H Weibull reliability function is given by:

$$R(T,V,U) = e^{-\left(\frac{T}{A}e^{-\left(\frac{\phi}{V}+\frac{b}{U}\right)}\right)^{\beta}}$$

Conditional Reliability Function

The T-H Weibull conditional reliability function at a specified stress level is given by:

$$R((t|T),V,U) = \frac{R(T+t,V,U)}{R(T,V,U)} = \frac{e^{-\left(\frac{T+t}{A}e^{-\left(\frac{\phi}{V}+\frac{b}{U}\right)}\right)^{\beta}}}{e^{-\left(\frac{T}{A}e^{-\left(\frac{\phi}{V}+\frac{b}{U}\right)}\right)^{\beta}}}$$

or:

$$R((t|T),V,U) = e^{-\left[\left(\frac{T+t}{A}e^{-\left(\frac{\phi}{V}+\frac{b}{U}\right)}\right)^{\beta}-\left(\frac{T}{A}e^{-\left(\frac{\phi}{V}+\frac{b}{U}\right)}\right)^{\beta}\right]}$$

Reliable Life

For the T-H Weibull model, the reliable life, t_R , of a unit for a specified reliability and starting the mission at age zero is given by:

$$t_R = Ae^{\frac{\phi}{V}+\frac{b}{U}}\{-\ln[R(T_R,V,U)]\}^{\frac{1}{\beta}}$$

T-H Weibull Failure Rate Function

The T-H Weibull failure rate function, $\lambda(T,V,U)$, is given by:

$$\lambda(T,V,U)=\frac{f(T,V,U)}{R(T,V,U)}=\frac{\beta}{A}e^{-\left(\frac{\phi}{V}+\frac{b}{U}\right)}\left(\frac{T}{A}e^{-\left(\frac{\phi}{V}+\frac{b}{U}\right)}\right)^{\beta-1}$$

Parameter Estimation

Maximum Likelihood Estimation Method

Substituting the T-H model into the Weibull log-likelihood function yields:

$$\ln(L)=\quad \Lambda=\sum_{i=1}^{F_e}N_i\ln\left[\frac{\beta}{A}e^{-\left(\frac{\phi}{V_i}+\frac{b}{U_i}\right)}\left(\frac{T_i}{A}e^{-\left(\frac{\phi}{V_i}+\frac{b}{U_i}\right)}\right)^{\beta-1}e^{-\left(\frac{T_i}{A}e^{-\left(\frac{\phi}{V_i}+\frac{b}{U_i}\right)}\right)^{\beta}}\right]-\sum_{i=1}^{S}N_i'\left(\frac{T_i'}{A}e^{-\left(\frac{\phi}{V_i}+\frac{b}{U_i}\right)}\right)^{\beta}+\sum_{i=1}^{FI}N_i''\ln[R_{Li}''-R_{Ri}'']$$

where:

$$R_{Li}''=e^{-\left(\frac{T_{Li}''}{A}e^{-\left(\frac{\phi}{V_i}+\frac{b}{U_i''}\right)}\right)^{\beta}}$$

$$R_{Ri}''=e^{-\left(\frac{T_{Ri}''}{A}e^{-\left(\frac{\phi}{V_i}+\frac{b}{U_i''}\right)}\right)^{\beta}}$$

and:

- F_e is the number of groups of exact times-to-failure data points.
- N_i is the number of times-to-failure data points in the i^{th} time-to-failure data group.
- β is the Weibull shape parameter (unknown, the first of four parameters to be estimated).
- A is the T-H parameter (unknown, the second of four parameters to be estimated).

- ϕ is the second T-H parameter (unknown, the third of four parameters to be estimated).
- b is the third T-H parameter (unknown, the fourth of four parameters to be estimated).
- V_i is the temperature level of the i^{th} group.
- U_i is the relative humidity level of the i^{th} group.
- T_i is the exact failure time of the i^{th} group.
- S is the number of groups of suspension data points.
- N_i' is the number of suspensions in the i^{th} group of suspension data points.
- T_i' is the running time of the i^{th} suspension data group.
- FI is the number of interval data groups.
- N_i'' is the number of intervals in the i^{th} group of data intervals.
- T_{Li}'' is the beginning of the interval.
- T_{Ri}'' is the ending of the i^{th} interval.

The solution (parameter estimates) will be found by solving for the parameters A, ϕ, b and β so that $\frac{\partial \Lambda}{\partial \beta} = 0, \frac{\partial \Lambda}{\partial A} = 0, \frac{\partial \Lambda}{\partial \phi} = 0$ and $\frac{\partial \Lambda}{\partial b} = 0$.

T-H Weibull Example

The following data were collected after testing twelve electronic devices at different temperature and humidity conditions:

Time, hr	Temperature, K	Humidity
310	378	0.4
316	378	0.4
329	378	0.4
411	378	0.4
190	378	0.8
208	378	0.8
230	378	0.8
298	378	0.8
108	398	0.4
123	398	0.4
166	398	0.4
200	398	0.4

Using Weibull++, the following results were obtained:

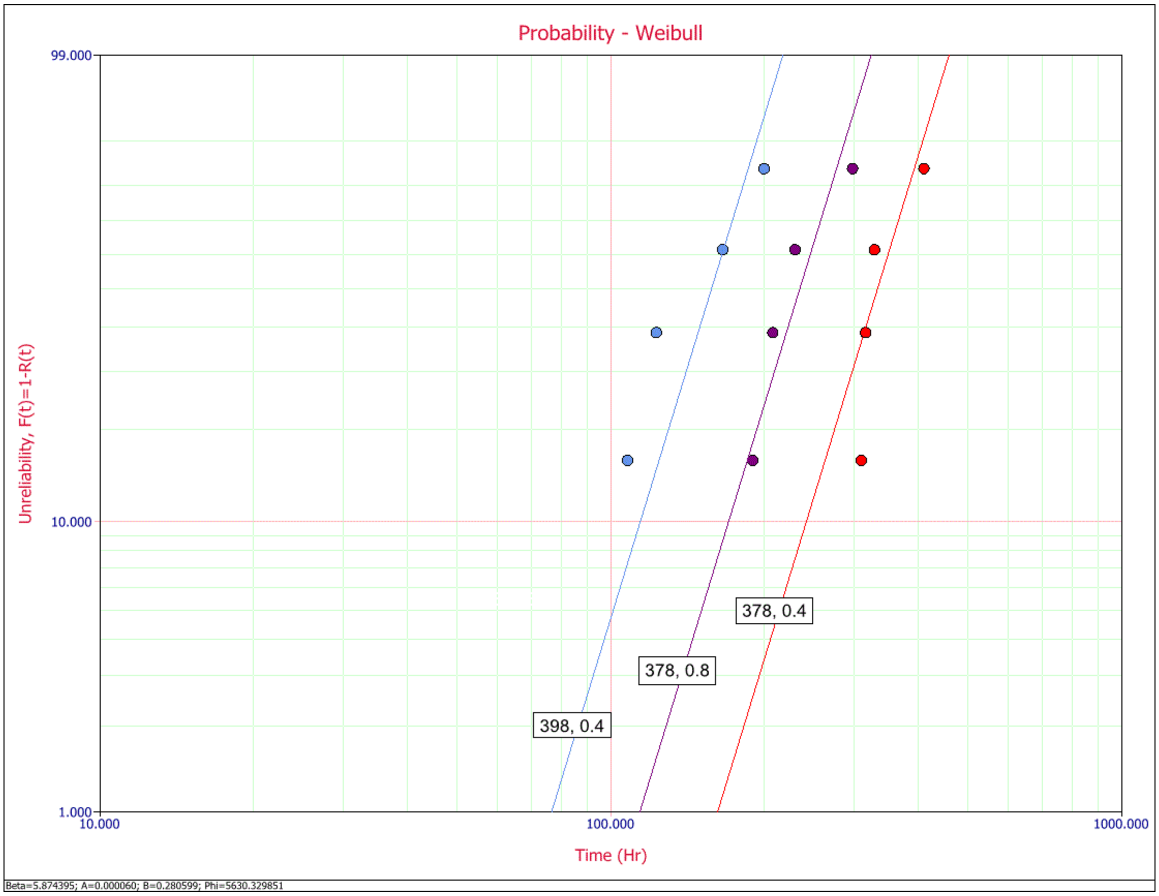
$$\widehat{\beta} = 5.874395$$

$$\widehat{A} = 0.000060$$

$$\hat{b} = 0.280599$$

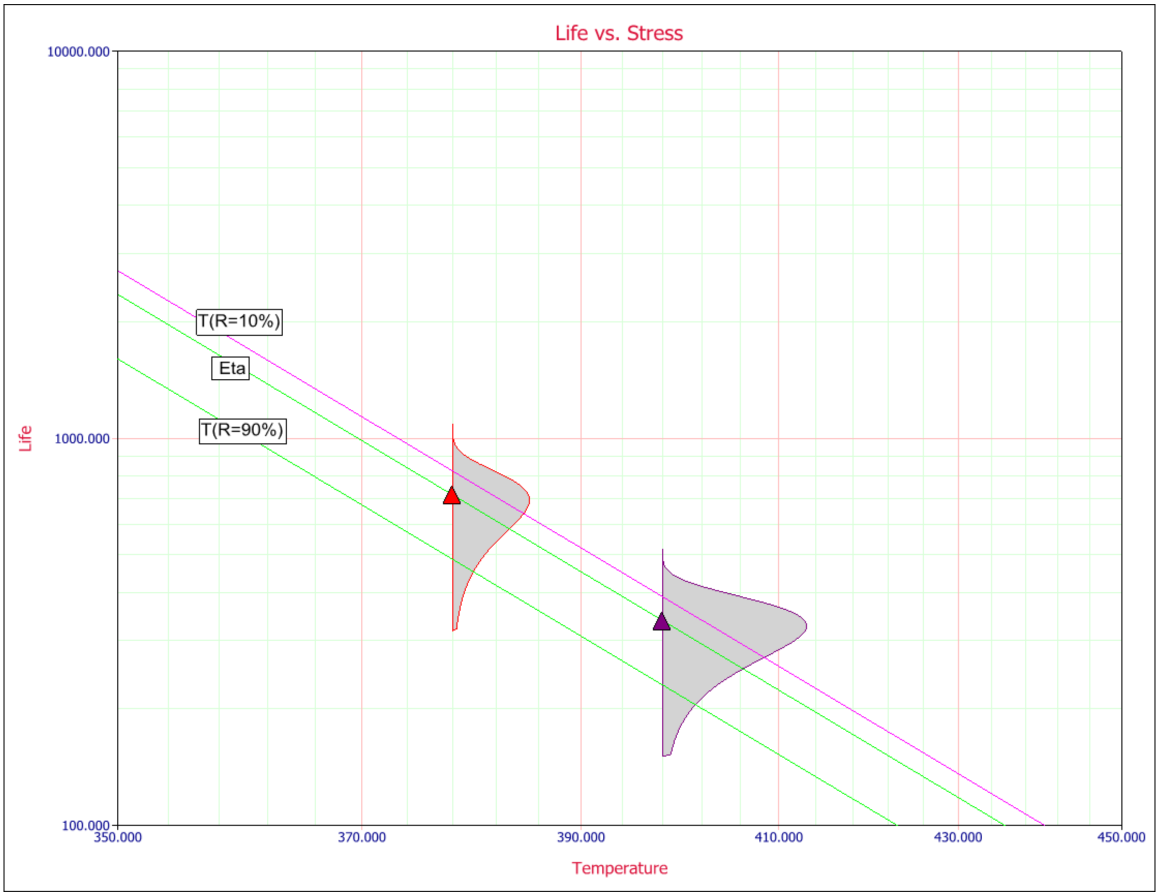
$$\widehat{\phi} = 5630.329851$$

A probability plot for the entered data is shown next.



Note that three lines are plotted because there are three combinations of stresses, namely, (398K, 0.4), (378K, 0.8) and (378K, 0.4).

Given the use stress levels, time estimates can be obtained for a specified probability. A Life vs. Stress plot can be obtained if one of the stresses is kept constant. For example, the following picture shows a Life vs. Temperature plot at a constant humidity of 0.4.



T-H Lognormal

The *pdf* of the lognormal distribution is given by:

$$f(T)=\frac{1}{T\,\sigma_{T'}\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{T'-\overline{T'}}{\sigma_{T'}}\right)^2}$$

where:

$$T'=\ln(T)$$

$$T=\text{times-to-failure}$$

and:

- $\overline{T'}=$ mean of the natural logarithms of the times-to-failure.

- $\sigma_{T'} =$ standard deviation of the natural logarithms of the times-to-failure.

The median of the lognormal distribution is given by:

$$\breve{T} = e^{\overline{T}'}$$

The T-H lognormal model *pdf* can be obtained first by setting $\breve{T} = L(V,U)$.
Therefore:

$$\breve{T} = L(V,U) = Ae^{\frac{\phi}{V}+\frac{b}{U}}$$

or:

$$e^{\overline{T}'} = Ae^{\frac{\phi}{V}+\frac{b}{U}}$$

Thus:

$$\overline{T}' = \ln(A) + \frac{\phi}{V} + \frac{b}{U}.$$

Substituting the above equation into the lognormal *pdf* yields the T-H lognormal model *pdf* or:

$$f(T,V,U) = \frac{1}{T\,\sigma_{T'}\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{T'-\ln(A)-\frac{\phi}{V}-\frac{b}{U}}{\sigma_{T'}}\right)^2}$$

T-H Lognormal Statistical Properties Summary

The Mean

- The mean life of the T-H lognormal model (mean of the times-to-failure), $\bar{T}$, is given by:

$$\bar{T} = e^{\bar{T}'+\frac{1}{2}\sigma_{T'}^2} = e^{\ln(A)+\frac{\phi}{V}+\frac{b}{U}+\frac{1}{2}\sigma_{T'}^2}$$

- The mean of the natural logarithms of the times-to-failure, $\bar{T}'$, in terms of $\bar{T}$ and σ_T is given by:

$$\bar{T}' = \ln(\bar{T}) - \frac{1}{2}\ln\left(\frac{\sigma_T^2}{\bar{T}^2} + 1\right)$$

The Standard Deviation

- The standard deviation of the T-H lognormal model (standard deviation of the times-to-failure), σ_T , is given by:

$$\sigma_T = \sqrt{\left(e^{2\bar{T}'+\sigma_{T'}^2}\right)\left(e^{\sigma_{T'}^2} - 1\right)} = \sqrt{\left(e^{2\left(\ln(A)+\frac{\phi}{V}+\frac{b}{U}\right)+\sigma_{T'}^2}\right)\left(e^{\sigma_{T'}^2} - 1\right)}$$

- The standard deviation of the natural logarithms of the times-to-failure, $\sigma_{T'}$, in terms of $\bar{T}$ and σ_T is given by:

$$\sigma_{T'} = \sqrt{\ln\left(\frac{\sigma_T^2}{\bar{T}^2} + 1\right)}$$

The Mode

- The mode of the T-H lognormal model is given by:

$$\tilde{T} = e^{\bar{T}'-\sigma_{T'}^2} = e^{\ln(A)+\frac{\phi}{V}+\frac{b}{U}-\sigma_{T'}^2}$$

T-H Lognormal Reliability

The reliability for a mission of time T , starting at age 0, for the T-H lognormal model is determined by:

$$R(T,V,U) = \int_T^{\infty} f(t,V,U)dt$$

or:

$$R(T,V,U) = \int_{T'}^{\infty} \frac{1}{\sigma_{T'}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{t-\ln(A)-\frac{\phi}{V}-\frac{b}{U}}{\sigma_{T'}}\right)^2} dt$$

There is no closed form solution for the lognormal reliability function. Solutions can be obtained via the use of standard normal tables. Since the application automatically solves for the reliability, we will not discuss manual solution methods.

Reliable Life

For the T-H lognormal model, the reliable life, or the mission duration for a desired reliability goal, $t_R,$ is estimated by first solving the reliability equation with respect to time, as follows:

$$T'_R = \ln(A) + \frac{\phi}{V} + \frac{b}{U} + z \cdot \sigma_{T'}$$

where:

$$z = \Phi^{-1}\left[F\left(T'_R, V, U\right)\right]$$

and:

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z(T',V,U)} e^{-\frac{t^2}{2}} dt$$

Since $T' = \ln(T),$ the reliable life, $t_R,$ is given by:

$$t_R = e^{T'_R}$$

T-H Lognormal Failure Rate

The lognormal failure rate is given by:

$$\lambda(T,V,U) = \frac{f(T,V,U)}{R(T,V,U)} = \frac{\frac{1}{T\ \sigma_{T'}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{T' - \ln(A) - \frac{\phi}{V} - \frac{b}{U}}{\sigma_{T'}}\right)^2}}{\int_{T'}^{\infty} \frac{1}{\sigma_{T'}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{T' - \ln(A) - \frac{\phi}{V} - \frac{b}{U}}{\sigma_{T'}}\right)^2} dt}$$

Parameter Estimation

Maximum Likelihood Estimation Method

The complete T-H lognormal log-likelihood function is:

$$\ln(L) = \quad \Lambda = \sum_{i=1}^{F_e} N_i \ln\left[\frac{1}{\sigma_{T'} T_i}\phi_{pdf}\left(\frac{\ln(T_i) - \ln(A) - \frac{\phi}{V_i} - \frac{b}{U_i}}{\sigma_{T'}}\right)\right] + \sum_{i=1}^{S} N_i' \ln\left[1 - \Phi\left(\frac{\ln(T_i') - \ln(A) - \frac{\phi}{V_i} - \frac{b}{U_i}}{\sigma_{T'}}\right)\right] + \sum_{i=1}^{FI} N_i'' \ln[\Phi(z_{Ri}'') - \Phi(z_{Li}'')]$$

where:

$$z_{Li}'' = \frac{\ln T_{Li}'' - \ln A - \frac{\phi}{V_i} - \frac{b}{U_i''}}{\sigma_T'}$$

$$z_{Ri}'' = \frac{\ln T_{Ri}'' - \ln A - \frac{\phi}{V_i} - \frac{b}{U_i''}}{\sigma_T'}$$

$$\phi_{pdf}(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}(x)^2}$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt$$

and:

- F_e is the number of groups of exact times-to-failure data points.
- N_i is the number of times-to-failure data points in the i^{th} time-to-failure data group.
- $\sigma_{T'}$ is the standard deviation of the natural logarithm of the times-to-failure (unknown, the first of four parameters to be estimated).
- A is the first T-H parameter (unknown, the second of four parameters to be estimated).
- ϕ is the second T-H parameter (unknown, the third of four parameters to be estimated).
- b is the third T-H parameter (unknown, the fourth of four parameters to be estimated).
- V_i is the stress level for the first stress type (i.e., temperature) of the i^{th} group.

- U_i is the stress level for the second stress type (i.e., relative humidity) of the i^{th} group.
- T_i is the exact failure time of the i^{th} group.
- S is the number of groups of suspension data points.
- N_i' is the number of suspensions in the i^{th} group of suspension data points.
- T_i' is the running time of the i^{th} suspension data group.
- FI is the number of interval data groups.
- N_i'' is the number of intervals in the i^{th} group of data intervals.
- T_{Li}'' is the beginning of the i^{th} interval.
- T_{Ri}'' is the ending of the i^{th} interval.

The solution (parameter estimates) will be found by solving for $\widehat{\sigma}_{T'}$, $\widehat{A}, \widehat{\phi}, \hat{b}$ so that $\frac{\partial \Lambda}{\partial \sigma_{T'}} = 0, \frac{\partial \Lambda}{\partial A} = 0, \frac{\partial \Lambda}{\partial \phi} = 0$ and $\frac{\partial \Lambda}{\partial b} = 0$.

T-H Confidence Bounds

Approximate Confidence Bounds for the T-H Exponential

Confidence Bounds on the Mean Life

The mean life for the T-H exponential distribution is given by Eqn. (Temp-Hum) by setting $m = L(V)$. The upper (m_U) and lower (m_L) bounds on the mean life (ML estimate of the mean life) are estimated by:

$$m_U = \widehat{m} \cdot e^{\frac{K_\alpha \sqrt{Var(\widehat{m})}}{\widehat{m}}}$$

$$m_L = \widehat{m} \cdot e^{-\frac{K_\alpha \sqrt{Var(\widehat{m})}}{\widehat{m}}}$$

where K_α is defined by:

$$\alpha = \frac{1}{\sqrt{2\pi}} \int_{K_\alpha}^{\infty} e^{-\frac{t^2}{2}} dt = 1 - \Phi(K_\alpha)$$

If δ is the confidence level, then $\alpha = \frac{1-\delta}{2}$ for the two-sided bounds, and $\alpha = 1 - \delta$ for the one-sided bounds. The variance of $\widehat{m}$ is given by:

$$Var(\widehat{m}) = \left(\frac{\partial m}{\partial A}\right)^2 Var(\widehat{A}) + \left(\frac{\partial m}{\partial \phi}\right)^2 Var(\widehat{\phi}) + \left(\frac{\partial m}{\partial b}\right)^2 Var(\hat{b}) + 2\left(\frac{\partial m}{\partial A}\right)\left(\frac{\partial m}{\partial \phi}\right) Cov(\widehat{A},\widehat{\phi})$$
$$+ 2\left(\frac{\partial m}{\partial A}\right)\left(\frac{\partial m}{\partial b}\right) Cov(\widehat{A},\hat{b}) + 2\left(\frac{\partial m}{\partial A}\right)\left(\frac{\partial m}{\partial \phi}\right) Cov(\widehat{\phi},\hat{b})$$

or:

$$Var(\widehat{m}) = e^{2\left(\frac{\widehat{\phi}}{V}+\frac{\hat{b}}{U}\right)}[Var(\widehat{A}) + \frac{\widehat{A}^2}{V^2} Var(\widehat{\phi}) + \frac{\widehat{A}^2}{U^2} Var(\hat{b})$$
$$+ \frac{2\widehat{A}}{V} Cov(\widehat{A},\widehat{\phi}) + \frac{2\widehat{A}}{U} Cov(\widehat{A},\hat{b}) + \frac{2\widehat{A}^2}{V \cdot U} Cov(\widehat{\phi},\hat{b})]$$

The variances and covariance of A, b and ϕ are estimated from the local Fisher matrix (evaluated at $\widehat{A}, \hat{b}, \widehat{\phi})$ as follows:

$$\begin{bmatrix} Var(\widehat{A}) & Cov(\widehat{A},\widehat{\phi}) & Cov(\widehat{A},\hat{b}) \\ Cov(\widehat{\phi},\widehat{A}) & Var(\widehat{\phi}) & Cov(\widehat{\phi},\hat{b}) \\ Cov(\hat{b},\widehat{A}) & Cov(\hat{b},\widehat{\phi}) & Var(\hat{b}) \end{bmatrix} = \begin{bmatrix} -\frac{\partial^2 \Lambda}{\partial A^2} & -\frac{\partial^2 \Lambda}{\partial A \partial \phi} & -\frac{\partial^2 \Lambda}{\partial A \partial b} \\ -\frac{\partial^2 \Lambda}{\partial \phi \partial A} & -\frac{\partial^2 \Lambda}{\partial \phi^2} & -\frac{\partial^2 \Lambda}{\partial \phi \partial b} \\ -\frac{\partial^2 \Lambda}{\partial b \partial A} & -\frac{\partial^2 \Lambda}{\partial b \partial \phi} & -\frac{\partial^2 \Lambda}{\partial b^2} \end{bmatrix}^{-1}$$

Confidence Bounds on Reliability

The bounds on reliability at a given time, T, are estimated by:

$$R_U = e^{-\frac{T}{m_U}}$$
$$R_L = e^{-\frac{T}{m_L}}$$

Confidence Bounds on Time

The bounds on time (ML estimate of time) for a given reliability are estimated by first solving the reliability function with respect to time or:

$$\widehat{T} = -\widehat{m} \cdot \ln(R)$$

The corresponding confidence bounds are estimated from:

$$T_U = -m_U \cdot \ln(R)$$
$$T_L = -m_L \cdot \ln(R)$$

Approximate Confidence Bounds for the T-H Weibull

Bounds on the Parameters

Using the same approach as previously discussed ($\widehat{\beta}$ and $\widehat{A}$ positive parameters):

$$\beta_U = \widehat{\beta} \cdot e^{\frac{K_\alpha \sqrt{Var(\widehat{\beta})}}{\widehat{\beta}}}$$
$$\beta_L = \widehat{\beta} \cdot e^{-\frac{K_\alpha \sqrt{Var(\widehat{\beta})}}{\widehat{\beta}}}$$

$$A_U = \widehat{A} \cdot e^{\frac{K_\alpha \sqrt{Var(\widehat{A})}}{\widehat{A}}}$$
$$A_L = \widehat{A} \cdot e^{-\frac{K_\alpha \sqrt{Var(\widehat{A})}}{\widehat{A}}}$$

$$\begin{aligned} b_U = & \quad \hat{b} + K_\alpha \sqrt{Var(\hat{b})} \\ b_L = & \quad \hat{b} - K_\alpha \sqrt{Var(\hat{b})} \end{aligned}$$

and:

$$\begin{aligned} \phi_U = & \quad \widehat{\phi} + K_\alpha \sqrt{Var(\widehat{\phi})} \\ \phi_L = & \quad \widehat{\phi} - K_\alpha \sqrt{Var(\widehat{\phi})} \end{aligned}$$

The variances and covariances of $\beta, A, b,$ and ϕ are estimated from the local Fisher matrix (evaluated at $\widehat{\beta}, \widehat{A}, \hat{b}, \widehat{\phi})$ as follows:

$$\begin{bmatrix} Var(\widehat{\beta}) & Cov(\widehat{\beta}, \widehat{A}) & Cov(\widehat{\beta}, \hat{b}) & Cov(\widehat{\beta}, \widehat{\phi}) \\ Cov(\widehat{A}, \widehat{\beta}) & Var(\widehat{A}) & Cov(\widehat{A}, \hat{b}) & Cov(\widehat{A}, \widehat{\phi}) \\ Cov(\hat{b}, \widehat{\beta}) & Cov(\hat{b}, \widehat{A}) & Var(\hat{b}) & Cov(\hat{b}, \widehat{\phi}) \\ Cov(\widehat{\phi}, \widehat{\beta}) & Cov(\widehat{\phi}, \widehat{A}) & Cov(\widehat{\phi}, \hat{b}) & Var(\widehat{\phi}) \end{bmatrix} = [F]^{-1}$$

where:

$$F = \begin{bmatrix} -\frac{\partial^2 \Lambda}{\partial \beta^2} & -\frac{\partial^2 \Lambda}{\partial \beta \partial A} & -\frac{\partial^2 \Lambda}{\partial \beta \partial b} & -\frac{\partial^2 \Lambda}{\partial \beta \partial \phi} \\ -\frac{\partial^2 \Lambda}{\partial A \partial \beta} & -\frac{\partial^2 \Lambda}{\partial A^2} & -\frac{\partial^2 \Lambda}{\partial A \partial b} & -\frac{\partial^2 \Lambda}{\partial A \partial \phi} \\ -\frac{\partial^2 \Lambda}{\partial b \partial \beta} & -\frac{\partial^2 \Lambda}{\partial b \partial A} & -\frac{\partial^2 \Lambda}{\partial b^2} & -\frac{\partial^2 \Lambda}{\partial b \partial \phi} \\ -\frac{\partial^2 \Lambda}{\partial \phi \partial \beta} & -\frac{\partial^2 \Lambda}{\partial \phi \partial A} & -\frac{\partial^2 \Lambda}{\partial \phi \partial b} & -\frac{\partial^2 \Lambda}{\partial \phi^2} \end{bmatrix}$$

Confidence Bounds on Reliability

The reliability function (ML estimate) for the T-H Weibull model is given by:

$$\widehat{R}(T,V,U)=e^{-\left(\frac{T}{\widehat{A}}e^{-\left(\frac{\widehat{\phi}}{V}+\frac{\hat{b}}{U}\right)}\right)^{\widehat{\beta}}}$$

or:

$$\widehat{R}(T,V,U)=e^{-e^{\ln\left[\left(\frac{T}{\widehat{A}}e^{-\left(\frac{\widehat{\phi}}{V}+\frac{\hat{b}}{U}\right)}\right)^{\widehat{\beta}}\right]}}$$

Setting:

$$\widehat{u}=\ln\left[\left(\frac{T}{\widehat{A}}e^{-\left(\frac{\widehat{\phi}}{V}+\frac{\hat{b}}{U}\right)}\right)^{\widehat{\beta}}\right]$$

or:

$$\widehat{u}=\widehat{\beta}\left[\ln(T)-\ln(\widehat{A})-\frac{\widehat{\phi}}{V}-\frac{\hat{b}}{U}\right]$$

The reliability function now becomes:

$$\widehat{R}(T,V,U)=e^{-e^{\widehat{u}}}$$

The next step is to find the upper and lower bounds on u :

$$\widehat{u}_U=\widehat{u}+K_\alpha\sqrt{Var(\widehat{u})}$$

$$\widehat{u}_L=\widehat{u}-K_\alpha\sqrt{Var(\widehat{u})}$$

where:

$$Var(\widehat{u}) = \left(\frac{\partial \widehat{u}}{\partial \beta}\right)^2 Var(\widehat{\beta}) + \left(\frac{\partial \widehat{u}}{\partial A}\right)^2 Var(\widehat{A}) + \left(\frac{\partial \widehat{u}}{\partial b}\right)^2 Var(\hat{b}) + \left(\frac{\partial \widehat{u}}{\partial \phi}\right)^2 Var(\widehat{\phi}) + 2\left(\frac{\partial \widehat{u}}{\partial \beta}\right)\left(\frac{\partial \widehat{u}}{\partial A}\right) Cov(\widehat{\beta}, \widehat{A}) + 2\left(\frac{\partial \widehat{u}}{\partial \beta}\right)\left(\frac{\partial \widehat{u}}{\partial b}\right) Cov(\widehat{\beta}, \hat{b})$$
$$+ 2\left(\frac{\partial \widehat{u}}{\partial \beta}\right)\left(\frac{\partial \widehat{u}}{\partial \phi}\right) Cov(\widehat{\beta}, \widehat{\phi}) + 2\left(\frac{\partial \widehat{u}}{\partial A}\right)\left(\frac{\partial \widehat{u}}{\partial b}\right) Cov(\widehat{A}, \hat{b}) + 2\left(\frac{\partial \widehat{u}}{\partial A}\right)\left(\frac{\partial \widehat{u}}{\partial \phi}\right) Cov(\widehat{A}, \widehat{\phi}) + 2\left(\frac{\partial \widehat{u}}{\partial b}\right)\left(\frac{\partial \widehat{u}}{\partial \phi}\right) Cov(\hat{b}, \widehat{\phi})$$

or:

$$Var(\widehat{u}) = \left(\frac{\widehat{u}}{\widehat{\beta}}\right)^2 Var(\widehat{\beta}) + \left(\frac{\widehat{\beta}}{\widehat{A}}\right)^2 Var(\widehat{A}) + \left(\frac{\widehat{\beta}}{U}\right)^2 Var(\hat{b}) + \left(\frac{\widehat{\beta}}{V}\right)^2 Var(\widehat{\phi}) - \frac{2\widehat{u}}{\widehat{A}} Cov(\widehat{\beta}, \widehat{A}) - \frac{2\widehat{u}}{U} Cov(\widehat{\beta}, \hat{b}) - \frac{2\widehat{u}}{V} Cov(\widehat{\beta}, \widehat{\phi})$$
$$+ \frac{2\widehat{\beta}^2}{\widehat{A}U} Cov(\widehat{A}, \hat{b}) + \frac{2\widehat{\beta}^2}{\widehat{A}V} Cov(\widehat{A}, \widehat{\phi}) + \frac{2\widehat{\beta}^2}{UV} Cov(\widehat{\phi}, \hat{b})$$

The upper and lower bounds on reliability are:

$$\begin{aligned} R_U &= e^{-e^{(u_L)}} \\ R_L &= e^{-e^{(u_U)}} \end{aligned}$$

Confidence Bounds on Time

The bounds on time (ML estimate of time) for a given reliability are estimated by first solving the reliability function with respect to time as follows:

$$\begin{aligned} \ln(R) &= -\left(\frac{\widehat{T}}{\widehat{A}} e^{-\left(\frac{\widehat{\phi}}{V} + \frac{\hat{b}}{U}\right)}\right)^{\widehat{\beta}} \\ \ln(-\ln(R)) &= \widehat{\beta}\left(\ln \widehat{T} - \ln \widehat{A} - \frac{\widehat{\phi}}{V} - \frac{\hat{b}}{U}\right) \end{aligned}$$

or:

$$\widehat{u} = \frac{1}{\widehat{\beta}} \ln(-\ln(R)) + \ln \widehat{A} + \frac{\widehat{\phi}}{V} + \frac{\hat{b}}{U}$$

where $\widehat{u} = \ln \widehat{T}$.

The upper and lower bounds on u are estimated from:

$$u_U = \widehat{u} + K_\alpha \sqrt{Var(\widehat{u})}$$

$$u_L = \widehat{u} - K_\alpha\sqrt{Var(\widehat{u})}$$

where:

$$Var(\widehat{u}) = \left(\frac{\partial\widehat{u}}{\partial\beta}\right)^2 Var(\widehat{\beta}) + \left(\frac{\partial\widehat{u}}{\partial A}\right)^2 Var(\widehat{A}) + \left(\frac{\partial\widehat{u}}{\partial b}\right)^2 Var(\widehat{b}) + \left(\frac{\partial\widehat{u}}{\partial\phi}\right)^2 Var(\widehat{\phi}) + 2\left(\frac{\partial\widehat{u}}{\partial\beta}\right)\left(\frac{\partial\widehat{u}}{\partial A}\right) Cov(\widehat{\beta},\widehat{A}) + 2\left(\frac{\partial\widehat{u}}{\partial\beta}\right)\left(\frac{\partial\widehat{u}}{\partial b}\right) Cov(\widehat{\beta},\widehat{b})$$
$$+ 2\left(\frac{\partial\widehat{u}}{\partial\beta}\right)\left(\frac{\partial\widehat{u}}{\partial\phi}\right) Cov(\widehat{\beta},\widehat{\phi}) + 2\left(\frac{\partial\widehat{u}}{\partial A}\right)\left(\frac{\partial\widehat{u}}{\partial b}\right) Cov(\widehat{A},\widehat{b}) + 2\left(\frac{\partial\widehat{u}}{\partial A}\right)\left(\frac{\partial\widehat{u}}{\partial\phi}\right) Cov(\widehat{A},\widehat{\phi}) + 2\left(\frac{\partial\widehat{u}}{\partial b}\right)\left(\frac{\partial\widehat{u}}{\partial\phi}\right) Cov(\widehat{b},\widehat{\phi})$$

or:

$$Var(\widehat{u}) = \frac{1}{\widehat{\beta}^4}[\ln(-\ln(R))]^2 Var(\widehat{\beta}) + \frac{1}{\widehat{A}^2} Var(\widehat{A}) + \frac{1}{U^2} Var(\widehat{b}) + \frac{1}{V^2} Var(\widehat{\phi}) + \frac{2\ln(-\ln(R))}{\widehat{\beta}^2\widehat{A}} Cov(\widehat{\beta},\widehat{A}) - \frac{2\ln(-\ln(R))}{\widehat{\beta}^2 U} Cov(\widehat{\beta},\widehat{b})$$
$$- \frac{2\ln(-\ln(R))}{\widehat{\beta}^2 V} Cov(\widehat{\beta},\widehat{\phi}) + \frac{2}{\widehat{A}U} Cov(\widehat{A},\widehat{b}) + \frac{2}{\widehat{A}V} Cov(\widehat{A},\widehat{\phi}) + \frac{2}{VU} Cov(\widehat{b},\widehat{\phi})$$

The upper and lower bounds on time are then found by:

$$T_U = e^{u_U}$$
$$T_L = e^{u_L}$$

Approximate Confidence Bounds for the T-H Lognormal

Bounds on the Parameters

Since the standard deviation, $\widehat{\sigma}_{T'}$, and $\widehat{A}$ are positive parameters, $\ln(\widehat{\sigma}_{T'})$ and $\ln(\widehat{A})$ are treated as normally distributed and the bounds are estimated from:

$$\sigma_U = \widehat{\sigma}_{T'} \cdot e^{\frac{K_\alpha\sqrt{Var(\widehat{\sigma}_{T'})}}{\widehat{\sigma}_{T'}}} \quad \text{(Upper bound)}$$
$$\sigma_L = \frac{\widehat{\sigma}_{T'}}{e^{\frac{K_\alpha\sqrt{Var(\widehat{\sigma}_{T'})}}{\widehat{\sigma}_{T'}}}} \quad \text{(Lower bound)}$$

and:

$$A_U = \widehat{A}\cdot e^{\frac{K_\alpha\sqrt{Var(\widehat{A})}}{\widehat{A}}} \quad \text{(Upper bound)}$$
$$A_L = \frac{\widehat{A}}{e^{\frac{K_\alpha\sqrt{Var(\widehat{A})}}{\widehat{A}}}} \quad \text{(Lower bound)}$$

The lower and upper bounds on ϕ and b are estimated from:

$$\phi_U = \widehat{\phi}+K_\alpha\sqrt{Var(\widehat{\phi})} \text{ (Upper bound)}$$
$$\phi_L = \widehat{\phi}-K_\alpha\sqrt{Var(\widehat{\phi})} \text{ (Lower bound)}$$

and:

$$b_U = \hat{b}+K_\alpha\sqrt{Var(\hat{b})} \text{ (Upper bound)}$$
$$b_L = \hat{b}-K_\alpha\sqrt{Var(\hat{b})} \text{ (Lower bound)}$$

The variances and covariances of A, $\phi, b,$ and $\sigma_{T'}$ are estimated from the local Fisher matrix (evaluated at $\widehat{A}$, $\widehat{\phi}, \hat{b}$, $\widehat{\sigma}_{T'}$), as follows:

$$\begin{pmatrix} Var\left(\widehat{\sigma}_{T'}\right) & Cov\left(\widehat{A},\widehat{\sigma}_{T'}\right) & Cov\left(\widehat{\phi},\widehat{\sigma}_{T'}\right) & Cov\left(\hat{b},\widehat{\sigma}_{T'}\right) \\ Cov\left(\widehat{\sigma}_{T'},\widehat{A}\right) & Var\left(\widehat{A}\right) & Cov\left(\widehat{A},\widehat{\phi}\right) & Cov\left(\widehat{A},\hat{b}\right) \\ Cov\left(\widehat{\sigma}_{T'},\widehat{\phi}\right) & Cov\left(\widehat{\phi},\widehat{A}\right) & Var\left(\widehat{\phi}\right) & Cov\left(\widehat{\phi},\hat{b}\right) \\ Cov\left(\hat{b},\widehat{\sigma}_{T'}\right) & Cov\left(\hat{b},\widehat{A}\right) & Cov\left(\hat{b},\widehat{\phi}\right) & Var\left(\hat{b}\right) \end{pmatrix} = F^{-1}$$

where:

$$F^{-1}=\begin{pmatrix} -\frac{\partial^2\Lambda}{\partial\sigma_{T'}^2} & -\frac{\partial^2\Lambda}{\partial\sigma_{T'}\partial A} & -\frac{\partial^2\Lambda}{\partial\sigma_{T'}\partial\phi} & -\frac{\partial^2\Lambda}{\partial\sigma_{T'}\partial b} \\ -\frac{\partial^2\Lambda}{\partial A\partial\sigma_{T'}} & -\frac{\partial^2\Lambda}{\partial A^2} & -\frac{\partial^2\Lambda}{\partial A\partial\phi} & -\frac{\partial^2\Lambda}{\partial A\partial b} \\ -\frac{\partial^2\Lambda}{\partial\phi\partial\sigma_{T'}} & -\frac{\partial^2\Lambda}{\partial\phi\partial A} & -\frac{\partial^2\Lambda}{\partial\phi^2} & -\frac{\partial^2\Lambda}{\partial\phi\partial b} \\ -\frac{\partial^2\Lambda}{\partial b\partial\sigma_{T'}} & -\frac{\partial^2\Lambda}{\partial b\partial A} & -\frac{\partial^2\Lambda}{\partial b\partial\phi} & -\frac{\partial^2\Lambda}{\partial b^2} \end{pmatrix}^{-1}$$

Bounds on Reliability

The reliability of the lognormal distribution is given by:

$$R(T',V,U;A,\phi,b,\sigma_{T'})=\int_{T'}^{\infty}\frac{1}{\widehat{\sigma}_{T'}\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{t-\ln(\widehat{A})-\frac{\widehat{\phi}}{V}-\frac{\hat{b}}{U}}{\widehat{\sigma}_{T'}}\right)^2}dt$$

Let $\hat{z}(t,V,U;A,\phi,b,\sigma_T)=\frac{t-\ln(\widehat{A})-\frac{\widehat{\phi}}{V}-\frac{\hat{b}}{U}}{\widehat{\sigma}_{T'}}$, then $\frac{d\hat{z}}{dt}=\frac{1}{\widehat{\sigma}_{T'}}$. For $t=T'$, $\hat{z}=\frac{T'-\ln(\widehat{A})-\frac{\widehat{\phi}}{V}-\frac{\hat{b}}{U}}{\widehat{\sigma}_{T'}}$, and for $t=\infty,\hat{z}=\infty.$ The above equation then becomes:

$$R(\hat{z})=\int_{\hat{z}(T',V,U)}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz$$

The bounds on z are estimated from:

$$z_U = \hat{z}+K_\alpha\sqrt{Var(\hat{z})}$$
$$z_L = \hat{z}-K_\alpha\sqrt{Var(\hat{z})}$$

where:

$$\begin{aligned}Var(\hat{z})=&\left(\frac{\partial\hat{z}}{\partial A}\right)_{\widehat{A}}^2Var(\widehat{A})+\left(\frac{\partial\hat{z}}{\partial\phi}\right)_{\widehat{\phi}}^2Var(\widehat{\phi})+\left(\frac{\partial\hat{z}}{\partial b}\right)_{\hat{b}}^2Var(\hat{b})+\left(\frac{\partial\hat{z}}{\partial\sigma_{T'}}\right)_{\widehat{\sigma}_{T'}}^2Var(\widehat{\sigma}_{T'})+2\left(\frac{\partial\hat{z}}{\partial A}\right)_{\widehat{A}}\left(\frac{\partial\hat{z}}{\partial\phi}\right)_{\widehat{\phi}}Cov\left(\widehat{A},\widehat{\phi}\right)+2\left(\frac{\partial\hat{z}}{\partial A}\right)_{\widehat{A}}\left(\frac{\partial\hat{z}}{\partial b}\right)_{\hat{b}}Cov\left(\widehat{A},\hat{b}\right)\\&+2\left(\frac{\partial\hat{z}}{\partial\phi}\right)_{\widehat{\phi}}\left(\frac{\partial\hat{z}}{\partial b}\right)_{\hat{b}}Cov\left(\widehat{\phi},\hat{b}\right)+2\left(\frac{\partial\hat{z}}{\partial A}\right)_{\widehat{A}}\left(\frac{\partial\hat{z}}{\partial\sigma_{T'}}\right)_{\widehat{\sigma}_{T'}}Cov\left(\widehat{A},\widehat{\sigma}_{T'}\right)+2\left(\frac{\partial\hat{z}}{\partial\phi}\right)_{\widehat{\phi}}\left(\frac{\partial\hat{z}}{\partial\sigma_{T'}}\right)_{\widehat{\sigma}_{T'}}Cov\left(\widehat{\phi},\widehat{\sigma}_{T'}\right)+2\left(\frac{\partial\hat{z}}{\partial b}\right)_{\hat{b}}\left(\frac{\partial\hat{z}}{\partial\sigma_{T'}}\right)_{\widehat{\sigma}_{T'}}Cov\left(\hat{b},\widehat{\sigma}_{T'}\right)\end{aligned}$$

or:

$$\begin{aligned} Var(\hat{z}) = & \frac{1}{\widehat{\sigma}_{T'}^2}[\frac{1}{A^2}Var(\widehat{A}) + \frac{1}{V^2}Var(\widehat{\phi}) + \frac{1}{U^2}Var(\hat{b}) + \hat{z}^2Var(\widehat{\sigma}_{T'}) + \frac{2}{A\cdot V}Cov\left(\widehat{A},\widehat{\phi}\right) + \frac{2}{A\cdot U}Cov\left(\widehat{A},\hat{b}\right) \\ & + \frac{2}{V\cdot U}Cov\left(\widehat{\phi},\hat{b}\right) + \frac{2\hat{z}}{A}Cov\left(\widehat{A},\widehat{\sigma}_{T'}\right) + \frac{2\hat{z}}{V}Cov\left(\widehat{\phi},\widehat{\sigma}_{T'}\right) + \frac{2\hat{z}}{U}Cov\left(\hat{b},\widehat{\sigma}_{T'}\right)] \end{aligned}$$

The upper and lower bounds on reliability are:

$$\begin{aligned} R_U = & \int_{z_L}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \text{ (Upper bound)} \\ R_L = & \int_{z_U}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \text{ (Lower bound)} \end{aligned}$$

Confidence Bounds on Time

The bounds around time, for a given lognormal percentile (unreliability), are estimated by first solving the reliability equation with respect to time, as follows:

$$T'(V,U;\widehat{A},\widehat{\phi},\hat{b},\widehat{\sigma}_{T'}) = \ln(\widehat{A}) + \frac{\widehat{\phi}}{V} + \frac{\hat{b}}{U} + z\cdot\widehat{\sigma}_{T'}$$

where:

$$\begin{aligned} T'(V,U;\widehat{A},\widehat{\phi},\hat{b},\widehat{\sigma}_{T'}) = & \ln(T) \\ z = & \Phi^{-1}\left[F(T')\right] \end{aligned}$$

and:

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z(T')} e^{-\frac{1}{2}z^2} dz$$

The next step is to calculate the variance of $T'(V,U;\widehat{A},\widehat{\phi},\hat{b},\widehat{\sigma}_{T'})$ as follows:

$$\begin{aligned} Var(T') = & \left(\frac{\partial T'}{\partial A}\right)^2 Var(\widehat{A}) + \left(\frac{\partial T'}{\partial \phi}\right)^2 Var(\widehat{\phi}) + \left(\frac{\partial T'}{\partial b}\right)^2 Var(\hat{b}) + \left(\frac{\partial T'}{\partial \sigma_{T'}}\right)^2 Var(\widehat{\sigma}_{T'}) + 2\left(\frac{\partial T'}{\partial A}\right)\left(\frac{\partial T'}{\partial \phi}\right) Cov\left(\widehat{A},\widehat{\phi}\right) \\ & + 2\left(\frac{\partial T'}{\partial A}\right)\left(\frac{\partial T'}{\partial b}\right) Cov\left(\widehat{A},\hat{b}\right) + 2\left(\frac{\partial T'}{\partial \phi}\right)\left(\frac{\partial T'}{\partial b}\right) Cov\left(\widehat{\phi},\hat{b}\right) + 2\left(\frac{\partial T'}{\partial A}\right)\left(\frac{\partial T'}{\partial \sigma_{T'}}\right) Cov\left(\widehat{A},\widehat{\sigma}_{T'}\right) \\ & + 2\left(\frac{\partial T'}{\partial \phi}\right)\left(\frac{\partial T'}{\partial \sigma_{T'}}\right) Cov\left(\widehat{\phi},\widehat{\sigma}_{T'}\right) + 2\left(\frac{\partial T'}{\partial b}\right)\left(\frac{\partial T'}{\partial \sigma_{T'}}\right) Cov\left(\hat{b},\widehat{\sigma}_{T'}\right) \end{aligned}$$

or:

$$\begin{aligned} Var(T') = & \frac{1}{A^2}Var(\widehat{A}) + \frac{1}{V^2}Var(\widehat{\phi}) + \frac{1}{U^2}Var(\hat{b}) + \hat{z}^2 Var(\widehat{\sigma}_{T'}) + \frac{2}{A \cdot V}Cov\left(\widehat{A}, \widehat{\phi}\right) + \frac{2}{A \cdot U}Cov\left(\widehat{A}, \hat{b}\right) \\ & + \frac{2}{V \cdot U}Cov\left(\widehat{\phi}, \hat{b}\right) + \frac{2\hat{z}}{A}Cov\left(\widehat{A}, \widehat{\sigma}_{T'}\right) + \frac{2\hat{z}}{V}Cov\left(\widehat{\phi}, \widehat{\sigma}_{T'}\right) + \frac{2\hat{z}}{U}Cov\left(\hat{b}, \widehat{\sigma}_{T'}\right) \end{aligned}$$

The upper and lower bounds are then found by:

$$\begin{aligned} T'_U = & \quad \ln T_U = T' + K_\alpha \sqrt{Var(T')} \\ T'_L = & \quad \ln T_L = T' - K_\alpha \sqrt{Var(T')} \end{aligned}$$

Solving for T_U and T_L yields:

$$\begin{aligned} T_U = & \quad e^{T'_U} \text{ (Upper bound)} \\ T_L = & \quad e^{T'_L} \text{ (Lower bound)} \end{aligned}$$

Temperature-NonThermal Relationship

IN THIS CHAPTER

When temperature and a second non-thermal stress (e.g., voltage) are the accelerated stresses of a test, then the Arrhenius and the inverse power law relationships can be combined to yield the Temperature-NonThermal (T-NT) relationship. This relationship is given by:

$$L(U,V)=\frac{C}{U^{n}e^{-\frac{B}{V}}}$$

where:

- U is the non-thermal stress (i.e., voltage, vibration, etc.)
- V is the temperature (**in absolute units**).
- B, C, n are the parameters to be determined.

The T-NT relationship can be linearized and plotted on a Life vs. Stress plot. The relationship is linearized by taking the natural logarithm of both sides in the T-NT relationship or:

$$\ln(L(V,U)) = \ln(C) - n\ln(U) + \frac{B}{V}$$

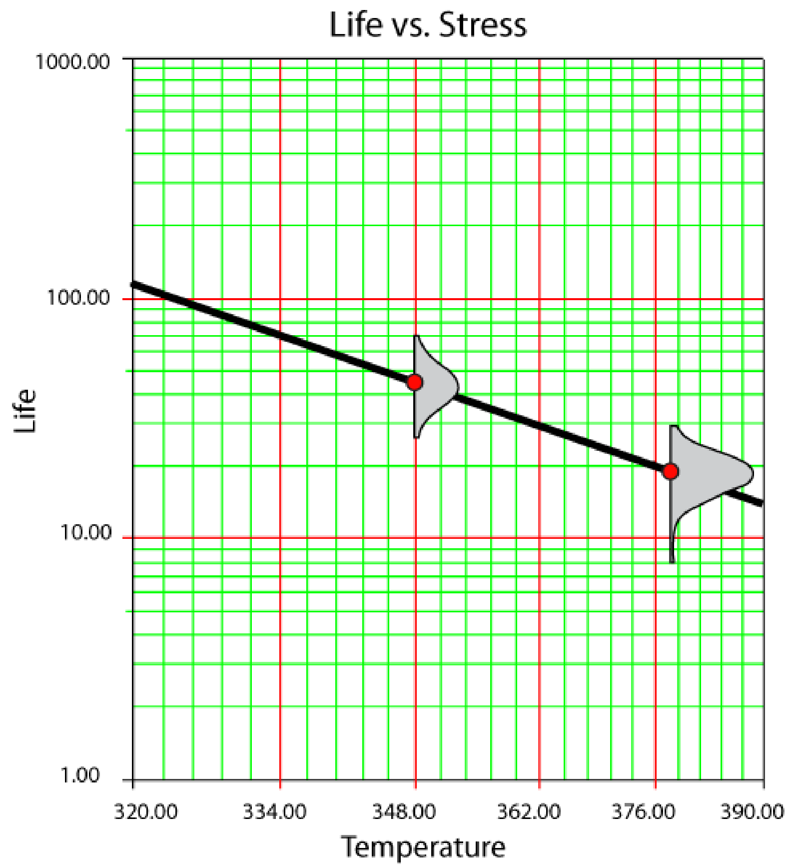
Since life is now a function of two stresses, a Life vs. Stress plot can only be obtained by keeping one of the two stresses constant and varying the other one. Doing so will yield the straight line described by the above equation, where the term for the stress which is kept at a fixed value becomes another constant (in addition to the $\ln(C)$ constant). When the non-thermal stress is kept constant, then the linearized T-NT relationship becomes:

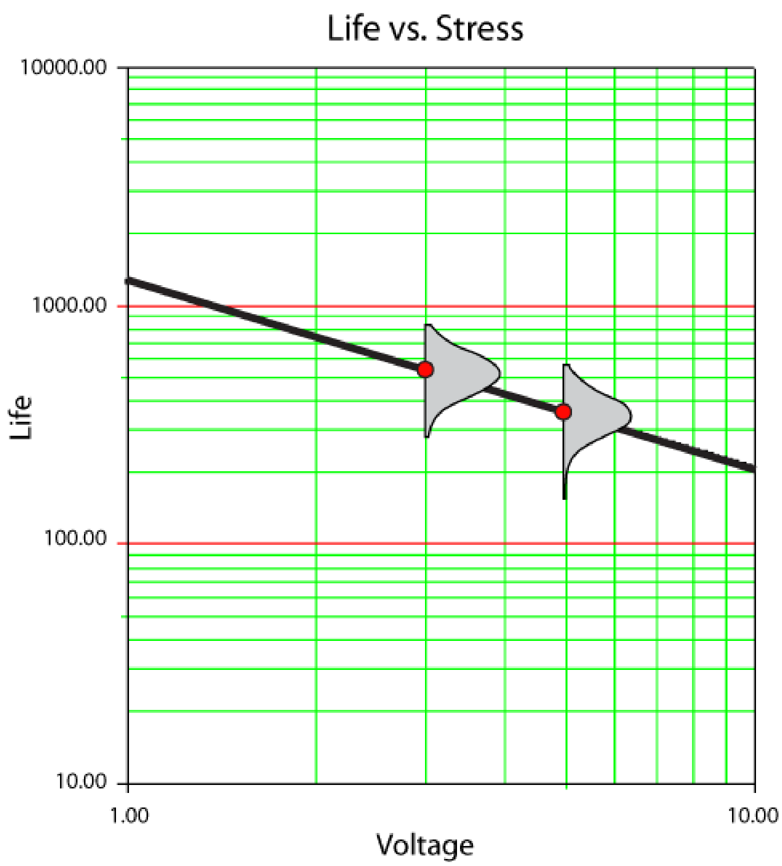
$$\ln(L(V)) = const. + \frac{B}{V}$$

This is the Arrhenius equation and it is plotted on a log-reciprocal scale. When the thermal stress is kept constant, then the linearized T-NT relationship becomes:

$$\ln(L(U)) = const. - n\ln(U)$$

This is the inverse power law equation and it is plotted on a log-log scale. In the next two figures, data obtained from a temperature and voltage test were analyzed and plotted on a log-reciprocal scale. In the first figure, life is plotted versus temperature, with voltage held at a fixed value. In the second figure, life is plotted versus voltage, with temperature held at a fixed value.


Life vs. Stress
1000.00
100.00
10.00
1.00
Life
320.00
334.00
348.00
362.00
376.00
390.00
Temperature



A look at the Parameters *B* and *n*

Depending on which stress type is kept constant, it can be seen from the linearized T-NT relationship that either the parameter B or the parameter n is the slope of the resulting line. If, for example, the non-thermal stress is kept constant then B is the slope of the life line in a Life vs. Temperature plot. The steeper the slope, the greater the dependency of the product's life to the temperature. In other words, B is a measure of the effect that temperature has on the life and n is a measure of the effect that the non-thermal stress has on the life. The larger the value of $B,$ the higher the dependency of the life on the temperature. Similarly, the larger the value of $n,$ the higher the dependency of the life on the non-thermal stress.

Acceleration Factor

The acceleration factor for the T-NT relationship is given by:

$$A_F = \frac{L_{USE}}{L_{Accelerated}} = \frac{\frac{C}{U_u^n} e^{\frac{B}{V_u}}}{\frac{C}{U_A^n} e^{\frac{B}{V_A}}} = \left(\frac{U_A}{U_u}\right)^n e^{B\left(\frac{1}{V_u} - \frac{1}{V_A}\right)}$$

where:

- L_{USE} is the life at use stress level.
- $L_{Accelerated}$ is the life at the accelerated stress level.
- V_u is the use temperature level.
- V_A is the accelerated temperature level.
- U_A is the accelerated non-thermal level.
- U_u is the use non-thermal level.

The acceleration factor is plotted versus stress in the same manner used to create the Life vs. Stress plots. That is, one stress type is kept constant and the other is varied.

$$\overline{T} = \int_0^\infty t \cdot f(t, U, V) dt = \int_0^\infty t \cdot \frac{U^n e^{-\frac{B}{V}}}{C} e^{-\frac{t \cdot U^n e^{-\frac{B}{V}}}{C}} dt = \frac{C}{U^n e^{-\frac{B}{V}}}$$

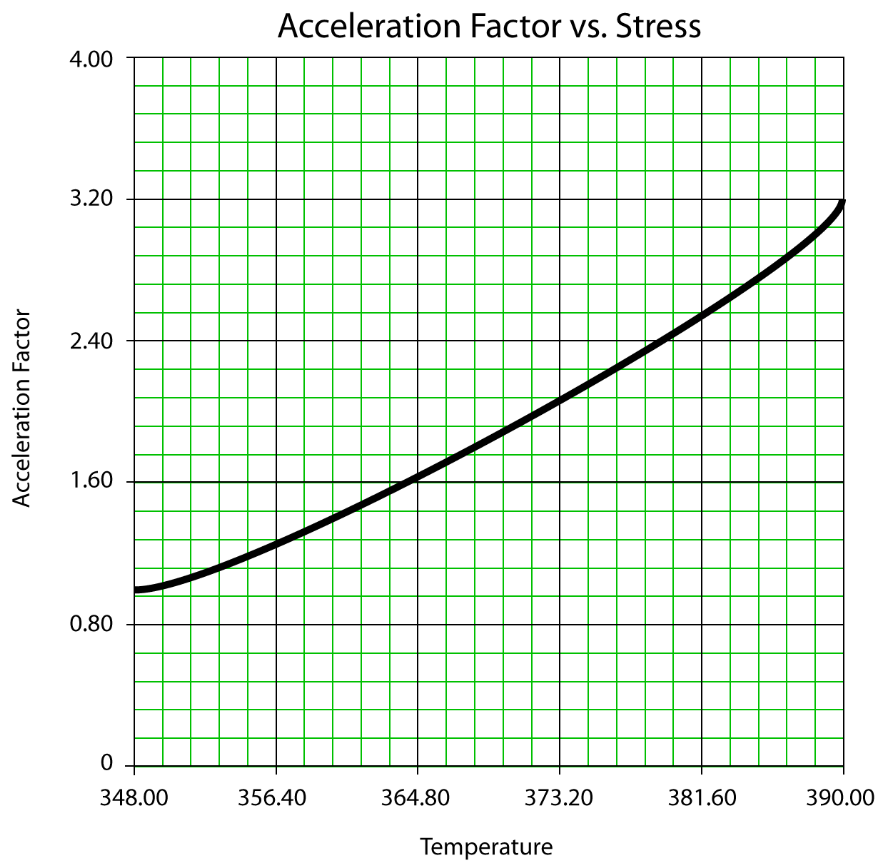
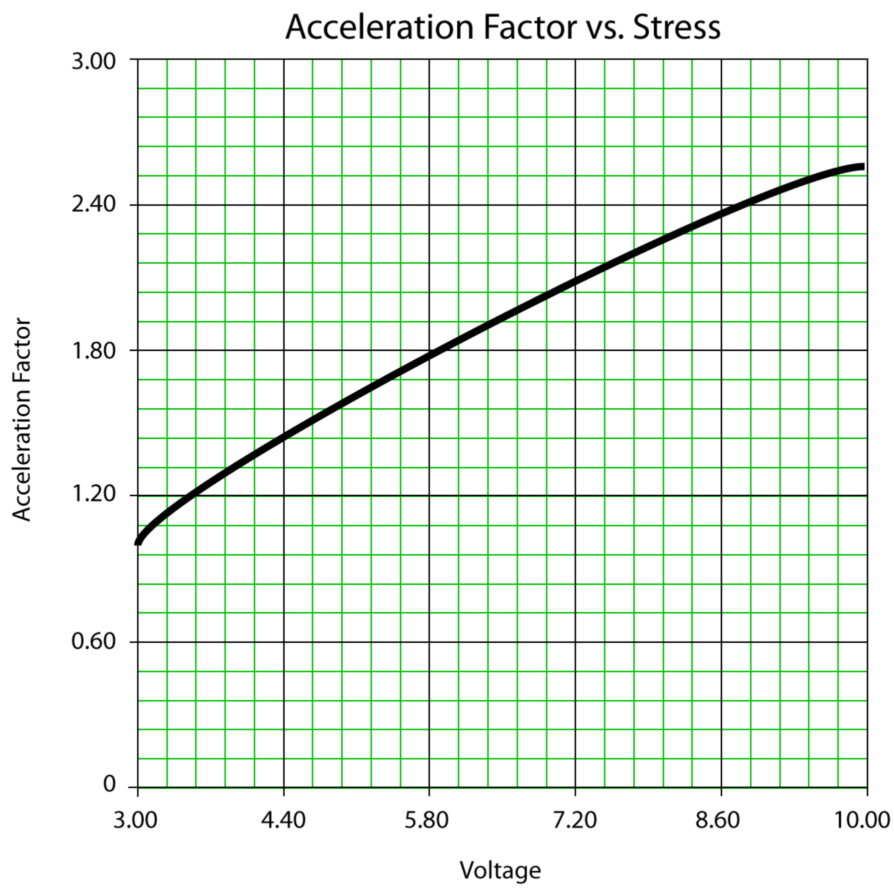


Fig. 3: Acceleration Factor vs. Temperature at a fixed voltage



T-NT Exponential

By setting $m = L(U, V)$, the exponential *pdf* becomes:

$$f(t, U, V) = \frac{U^n}{C} e^{-\frac{B}{V}} \cdot e^{-\frac{U^n}{C}\left(e^{-\frac{B}{V}}\right)t}$$

T-NT Exponential Statistical Properties Summary

Mean or MTTF

The mean, $\overline{T}$, or Mean Time To Failure (MTTF) for the T-NT exponential model is given by:

$$\overline{T} = \int_0^{\infty} t \cdot f(t,U,V)dt = \int_0^{\infty} t \cdot \frac{U^n e^{-\frac{B}{V}}}{C} e^{-\frac{t \cdot U^n e^{-\frac{B}{V}}}{C}} dt = \frac{C}{U^n e^{-\frac{B}{V}}}$$

Median

The median, $\breve{T}$, for the T-NT exponential model is given by:

$$\breve{T} = \frac{1}{\lambda} 0.693 = 0.693 \frac{C}{U^n e^{-\frac{B}{V}}}$$

Mode

The mode, $\tilde{T}$, for the T-NT exponential model is given by:

$$\tilde{T} = 0$$

Standard Deviation

The standard deviation, σ_T , for the T-NT exponential model is given by:

$$\sigma_T = \frac{1}{\lambda} = m = \frac{C}{U^n e^{-\frac{B}{V}}}$$

T-NT Exponential Reliability Function

The T-NT exponential reliability function is given by:

$$R(T,U,V) = e^{-\frac{T \cdot U^n e^{-\frac{B}{V}}}{C}}$$

This function is the complement of the T-NT exponential cumulative distribution function or:

$$R(T,U,V) = 1 - Q(T,U,V) = 1 - \int_0^T f(T)dT$$

and,

$$R(T,U,V)=1-\int_0^T \frac{U^n e^{-\frac{B}{V}}}{C} e^{-\frac{T\cdot U^n e^{-\frac{B}{V}}}{C}} dT = e^{-\frac{T\cdot U^n e^{-\frac{B}{V}}}{C}}$$

Conditional Reliability

The conditional reliability function for the T-NT exponential model is given by,

$$R((t|T),U,V)=\frac{R(T+t,U,V)}{R(T,U,V)}=\frac{e^{-\lambda(T+t)}}{e^{-\lambda T}}=e^{-\frac{t\cdot U^n e^{-\frac{B}{V}}}{C}}$$

Reliable Life

For the T-NT exponential model, the reliable life, or the mission duration for a desired reliability goal, t_R , is given by:

$$R(t_R,U,V)=e^{-\frac{t_R\cdot U^n e^{-\frac{B}{V}}}{C}}$$

$$\ln[R(t_R,U,V)]=^{-\frac{t_R\cdot U^n e^{-\frac{B}{V}}}{C}}$$

or:

$$t_R=-\frac{C}{U^n e^{-\frac{B}{V}}}\ln[R(t_R,U,V)]$$

Parameter Estimation

Maximum Likelihood Estimation Method

Substituting the T-NT relationship into the exponential log-likelihood equation yields:

$$\ln(L)= \qquad \Lambda=\sum_{i=1}^{F_e} N_i \ln\left[\frac{U_i^n}{C}e^{-\frac{B}{V_i}}\cdot e^{-\frac{U_i^n}{C}\left(e^{-\frac{B}{V_i}}\right)T_i}\right]-\sum_{i=1}^{S} N_i' \frac{U_i^n}{C}\left(e^{-\frac{B}{V_i}}\right)T_i' + \sum_{i=1}^{FI} N_i'' \ln[R_{Li}''-R_{Ri}'']$$

where:

$$R''_{Li} = e^{-\frac{T''_{Li}}{C} U_i^{\prime\prime n} e^{-\frac{B}{V_i}}}$$

$$R''_{Ri} = e^{-\frac{T''_{Ri}}{C} U_i^{\prime\prime m} e^{-\frac{B}{V_i}}}$$

and:

- F_e is the number of groups of exact times-to-failure data points.
- N_i is the number of times-to-failure data points in the i^{th} time-to-failure data group.
- B is the T-NT parameter (unknown, the first of three parameters to be estimated).
- C is the second T-NT parameter (unknown, the second of three parameters to be estimated).
- n is the third T-NT parameter (unknown, the third of three parameters to be estimated).
- V_i is the temperature level of the i^{th} group.
- U_i is the non-thermal stress level of the i^{th} group.
- T_i is the exact failure time of the i^{th} group.
- S is the number of groups of suspension data points.
- N'_i is the number of suspensions in the i^{th} group of suspension data points.
- T'_i is the running time of the i^{th} suspension data group.
- FI is the number of interval data groups.
- N''_i is the number of intervals in the i^{th} group of data intervals.
- T''_{Li} is the beginning of the i^{th} interval.
- T''_{Ri} is the ending of the i^{th} interval.

The solution (parameter estimates) will be found by solving for the parameters B, C and n so that $\frac{\partial \Lambda}{\partial B} = 0, \frac{\partial \Lambda}{\partial C} = 0$ and $\frac{\partial \Lambda}{\partial n} = 0$.

T-NT Weibull

By setting $\eta = L(U, V)$, the T-NT Weibull model is given by:

$$f(t, U, V) = \frac{\beta U^n e^{-\frac{B}{V}}}{C} \left(\frac{t \cdot U^n e^{-\frac{B}{V}}}{C} \right)^{\beta - 1} e^{-\left(\frac{t \cdot U^n e^{-\frac{B}{V}}}{C} \right)^{\beta}}$$

T-NT Weibull Statistical Properties Summary

Mean or MTTF

The mean, $\overline{T}$, for the T-NT Weibull model is given by:

$$\overline{T} = \frac{C}{U^n e^{-\frac{B}{V}}} \cdot \Gamma \left(\frac{1}{\beta} + 1 \right)$$

where $\Gamma \left(\frac{1}{\beta} + 1 \right)$ is the gamma function evaluated at the value of $\left(\frac{1}{\beta} + 1 \right)$.

Median

The median, $\breve{T}$, for the T-NT Weibull model is given by:

$$\breve{T} = \frac{C}{U^n e^{-\frac{B}{V}}} (\ln 2)^{\frac{1}{\beta}}$$

Mode

The mode, $\tilde{T}$, for the T-NT Weibull model is given by:

$$\tilde{T}=\frac{C}{U^{n}e^{-\frac{B}{V}}}\left(1-\frac{1}{\beta}\right)^{\frac{1}{\beta}}$$

Standard Deviation

The standard deviation, σ_T, for the T-NT Weibull model is given by:

$$\sigma_T=\frac{C}{U^{n}e^{-\frac{B}{V}}}\cdot\sqrt{\Gamma\left(\frac{2}{\beta}+1\right)-\left(\Gamma\left(\frac{1}{\beta}+1\right)\right)^{2}}$$

T-NT Weibull Reliability Function

The T-NT Weibull reliability function is given by:

$$R(T,U,V)=e^{-\left(\frac{TU^{n}e^{-\frac{B}{V}}}{C}\right)^{\beta}}$$

Conditional Reliability Function

The T-NT Weibull conditional reliability function at a specified stress level is given by:

$$R((t|T),U,V)=\frac{R(T+t,U,V)}{R(T,U,V)}=\frac{e^{-\left(\frac{(T+t)U^{n}e^{-\frac{B}{V}}}{C}\right)^{\beta}}}{e^{-\left(\frac{TU^{n}e^{-\frac{B}{V}}}{C}\right)^{\beta}}}$$

or:

$$R((t|T),U,V)=e^{-\left[\left(\frac{(T+t)U^{n}e^{-\frac{B}{V}}}{C}\right)^{\beta}-\left(\frac{TU^{n}e^{-\frac{B}{V}}}{C}\right)^{\beta}\right]}$$

Reliable Life

For the T-NT Weibull model, the reliable life, T_R , of a unit for a specified reliability and starting the mission at age zero is given by:

$$T_R = \frac{C}{U^n e^{-\frac{B}{V}}}\{-\ln[R(T_R, U, V)]\}^{\frac{1}{\beta}}$$

T-NT Weibull Failure Rate Function

The T-NT Weibull failure rate function, $\lambda(T)$, is given by:

$$\lambda(T, U, V) = \frac{f(T, U, V)}{R(T, U, V)} = \frac{\beta U^n e^{-\frac{B}{V}}}{C}\left(\frac{TU^n e^{-\frac{B}{V}}}{C}\right)^{\beta-1}$$

Parameter Estimation

Maximum Likelihood Estimation Method

Substituting the T-NT relationship into the Weibull log-likelihood function yields:

$$\ln(L) = \quad \Lambda = \sum_{i=1}^{F_e} N_i \ln\left[\frac{\beta U_i^n e^{-\frac{B}{V_i}}}{C}\left(\frac{U_i^n e^{-\frac{B}{V_i}}}{C} T_i\right)^{\beta-1} e^{-\left(\frac{U_i^n e^{-\frac{B}{V_i}}}{C} T_i\right)^{\beta}}\right] - \sum_{i=1}^{S} N_i' \left(\frac{U_i^n e^{-\frac{B}{V_i}}}{C} T_i'\right)^{\beta} + \sum_{i=1}^{FI} N_i'' \ln[R_{Li}'' - R_{Ri}'']$$

where:

$$R_{Li}'' = e^{-\left(\frac{T_{Li}''}{C} U_i''^n e^{-\frac{B}{V_i}}\right)^{\beta}}$$

$$R_{Ri}'' = e^{-\left(\frac{T_{Ri}''}{C} U_i''^n e^{-\frac{B}{V_i}}\right)^{\beta}}$$

and:

- F_e is the number of groups of exact times-to-failure data points.
- N_i is the number of times-to-failure data points in the i^{th} time-to-failure data group.
- β is the Weibull shape parameter (unknown, the first of four parameters to be estimated).
- B is the first T-NT parameter (unknown, the second of four parameters to be estimated).
- C is the second T-NT parameter (unknown, the third of four parameters to be estimated).
- n is the third T-NT parameter (unknown, the fourth of four parameters to be estimated).
- V_i is the temperature level of the i^{th} group.
- U_i is the non-thermal stress level of the i^{th} group.
- T_i is the exact failure time of the i^{th} group.
- S is the number of groups of suspension data points.
- N_i' is the number of suspensions in the i^{th} group of suspension data points.
- T_i' is the running time of the i^{th} suspension data group.
- FI is the number of interval data groups.
- N_i'' is the number of intervals in the i^{th} group of data intervals.
- T_{Li}'' is the beginning of the i^{th} interval.
- T_{Ri}'' is the ending of the i^{th} interval.

The solution (parameter estimates) will be found by solving for the parameters B, C, n and β so that $\frac{\partial \Lambda}{\partial B} = 0, \frac{\partial \Lambda}{\partial C} = 0, \frac{\partial \Lambda}{\partial n} = 0$ and $\frac{\partial \Lambda}{\partial \beta} = 0$.

T-NT Lognormal

The *pdf* of the lognormal distribution is given by:

$$f(T)=\frac{1}{T\,\sigma_{T'}\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{T'-\overline{T'}}{\sigma_{T'}}\right)^2}$$

where:

$$T'=\ln(T)$$

and:

- $T=$ times-to-failure.
- $\overline{T'}=$ mean of the natural logarithms of the times-to-failure.
- $\sigma_{T'}=$ standard deviation of the natural logarithms of the times-to-failure.

The median of the lognormal distribution is given by:

$$\breve{T}=e^{\overline{T}'}$$

The T-NT lognormal model *pdf* can be obtained by setting $\breve{T}=L(V)$. Therefore:

$$\breve{T}=L(V)=\frac{C}{U^n}e^{\frac{B}{V}}$$

or:

$$e^{\overline{T}'}=\frac{C}{U^n}e^{\frac{B}{V}}$$

Thus:

$$\overline{T}'=\ln(C)-n\ln(U)+\frac{B}{V}$$

Substituting the above equation into the lognormal *pdf* yields the T-NT lognormal model *pdf* or:

$$f(T,U,V)=\frac{1}{T\,\sigma_{T'}\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{T'-\ln(C)+n\ln(U)-\frac{B}{V}}{\sigma_{T'}}\right)^{2}}$$

T-N-T Lognormal Statistical Properties Summary

The Mean

The mean life of the T-NT lognormal model (mean of the times-to-failure), $\bar{T}$, is given by:

$$\bar{T}=\quad e^{\bar{T}'+\frac{1}{2}\sigma_{T'}^{2}}=e^{\ln(C)-n\ln(U)+\frac{B}{V}+\frac{1}{2}\sigma_{T'}^{2}}$$

The mean of the natural logarithms of the times-to-failure, $\bar{T}'$, in terms of $\bar{T}$ and σ_T is given by:

$$\bar{T}'=\ln(\bar{T})-\frac{1}{2}\ln\left(\frac{\sigma_T^2}{\bar{T}^2}+1\right)$$

The Standard Deviation

The standard deviation of the T-NT lognormal model (standard deviation of the times-to-failure), σ_T , is given by:

$$\sigma_T=\quad\sqrt{\left(e^{2\bar{T}'+\sigma_{T'}^{2}}\right)\left(e^{\sigma_{T'}^{2}}-1\right)}=\sqrt{\left(e^{2\left(\ln(C)-n\ln(U)+\frac{B}{V}\right)+\sigma_{T'}^{2}}\right)\left(e^{\sigma_{T'}^{2}}-1\right)}$$

The standard deviation of the natural logarithms of the times-to-failure, $\sigma_{T'}$, in terms of $\bar{T}$ and σ_T is given by:

$$\sigma_{T'}=\sqrt{\ln\left(\frac{\sigma_T^2}{\bar{T}^2}+1\right)}$$

The Mode

The mode of the T-NT lognormal model is given by:

$$\tilde{T} = \quad e^{\overline{T}'-\sigma_{T'}^2} = e^{\ln(C)-n\ln(U)+\frac{B}{V}-\sigma_{T'}^2}$$

T-NT Lognormal Reliability

For the T-NT lognormal model, the reliability for a mission of time T, starting at age 0, for the T-NT lognormal model is determined by:

$$R(T,U,V)=\int_{T}^{\infty} f(t,U,V)dt$$

or:

$$R(T,U,V)=\int_{T'}^{\infty} \frac{1}{\sigma_{T'}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{t-\ln(C)+n\ln(U)-\frac{B}{V}}{\sigma_{T'}}\right)^2} dt$$

Reliable Life

For the T-NT lognormal model, the reliable life, or the mission duration for a desired reliability goal, t_R, is estimated by first solving the reliability equation with respect to time, as follows:

$$T'_R=\ln(C)-n\ln(U)+\frac{B}{V}+z\cdot\sigma_{T'}$$

where:

$$z=\Phi^{-1}\left[F\left(T'_R,U,V\right)\right]$$

and:

$$\Phi(z)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z(T',U,V)} e^{-\frac{t^2}{2}}dt$$

Since $T'=\ln(T)$ the reliable life, t_R, is given by:

$$t_R=e^{T'_R}$$

Lognormal Failure Rate

The T-NT lognormal failure rate is given by:

$$\lambda(T,U,V)=\frac{f(T,U,V)}{R(T,U,V)}=\frac{\frac{1}{T\ \sigma_{T'}\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{T'-\ln(C)+n\ln(U)-\frac{B}{V}}{\sigma_{T'}}\right)^2}}{\int_{T'}^{\infty}\frac{1}{\sigma_{T'}\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{T'-\ln(C)+n\ln(U)-\frac{B}{V}}{\sigma_{T'}}\right)^2}dt}$$

Parameter Estimation

Maximum Likelihood Estimation Method

The complete T-NT lognormal log-likelihood function is:

$$\ln(L)=\quad\Lambda=\sum_{i=1}^{F_e}N_i\ln\left[\frac{1}{\sigma_{T'}T_i}\phi_{pdf}\left(\frac{\ln(T_i)-\ln(C)+n\ln(U_i)-\frac{B}{V_i}}{\sigma_{T'}}\right)\right]+\sum_{i=1}^{S}N_i'\ln\left[1-\Phi\left(\frac{\ln(T_i')-\ln(C)+n\ln(U_i)-\frac{B}{V_i}}{\sigma_{T'}}\right)\right]+\sum_{i=1}^{FI}N_i''\ln[\Phi(z_{Ri}'')-\Phi(z_{Li}'')]$$

where:

$$z_{Ri}''=\frac{\ln T_{Ri}''-\ln C+n\ln U_i''-\frac{B}{V_i}}{\sigma_T'}$$

$$z_{Li}''=\frac{\ln T_{Li}''-\ln C+n\ln U_i''-\frac{B}{V_i}}{\sigma_T'}$$

$$\phi(x)=\frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{1}{2}(x)^2}$$

$$\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{t^2}{2}}dt$$

and:

- F_e is the number of groups of exact times-to-failure data points.
- N_i is the number of times-to-failure data points in the i^{th} time-to-failure data group.
- $\sigma_{T'}$ is the standard deviation of the natural logarithm of the times-to-failure (unknown, the first of four parameters to be estimated).
- B is the first T-NT parameter (unknown, the second of four parameters to be estimated).
- C is the second T-NT parameter (unknown, the third of four parameters to be estimated).
- n is the third T-NT parameter (unknown, the fourth of four parameters to be estimated).
- V_i is the stress level for the first stress type (i.e., temperature) of the i^{th} group.
- U_i is the stress level for the second stress type (i.e., non-thermal) of the i^{th} group.
- T_i is the exact failure time of the i^{th} group.
- S is the number of groups of suspension data points.
- N'_i is the number of suspensions in the i^{th} group of suspension data points.
- T'_i is the running time of the i^{th} suspension data group.
- FI is the number of interval data groups.
- N''_i is the number of intervals in the i^{th} group of data intervals.
- T''_{Li} is the beginning of the i^{th} interval.
- T''_{Ri} is the ending of the i^{th} interval.

The solution (parameter estimates) will be found by solving for $\widehat{\sigma}_{T'}, \widehat{B}, \widehat{C}, \widehat{n}$ so that $\frac{\partial \Lambda}{\partial \sigma_{T'}} = 0$, $\frac{\partial \Lambda}{\partial B} = 0$, $\frac{\partial \Lambda}{\partial C} = 0$ and $\frac{\partial \Lambda}{\partial n} = 0$.

T-NT Lognormal Example

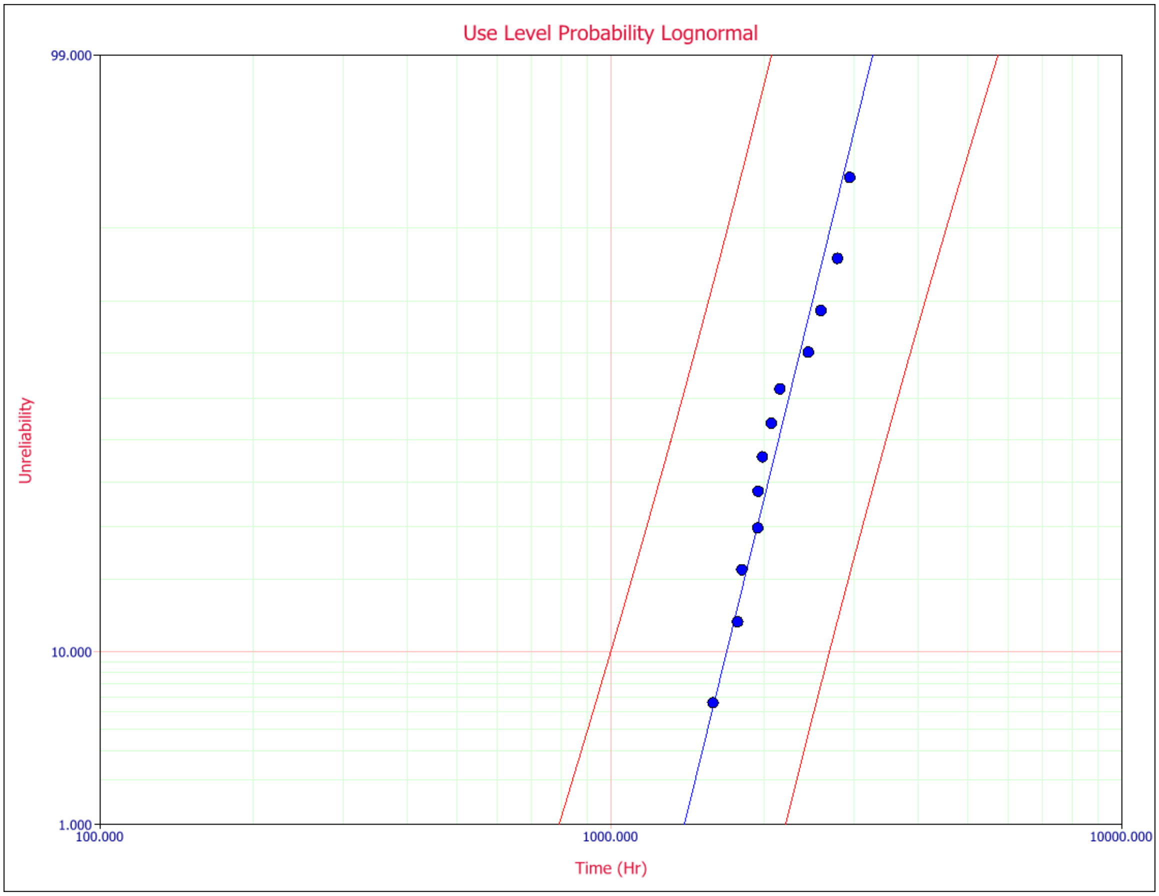
12 electronic devices were put into a continuous accelerated life test and the following data were collected.

Time, hr	Temperature, K	Voltage, V
620	348	3
632	348	3
685	348	3
822	348	3
380	348	5
416	348	5
460	348	5
596	348	5
216	378	3
146	378	3
332	378	3
400	378	3

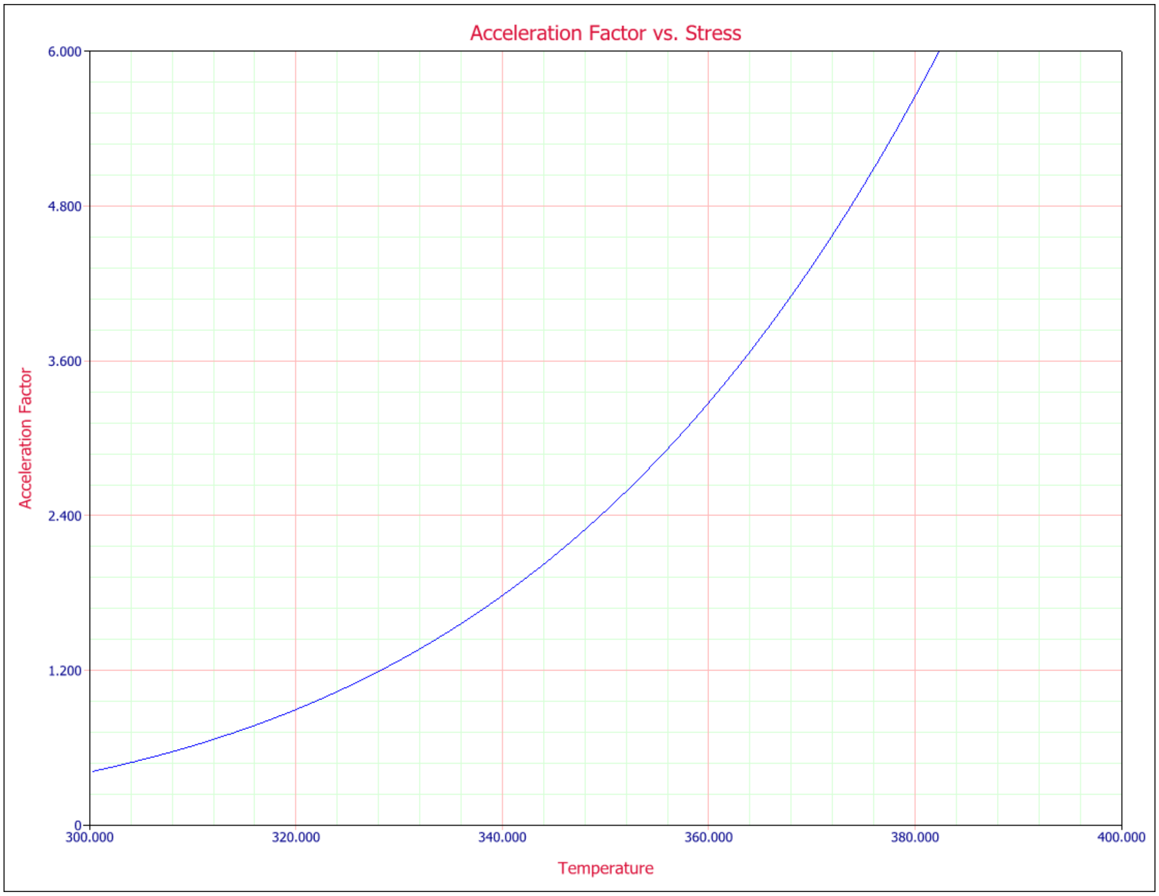
Using Weibull++ and the T-NT lognormal model, the following parameters were obtained:

$$\begin{aligned} \widehat{Std} &= 0.182558 \\ \widehat{B} &= 3729.650303 \\ \widehat{C} &= 0.035292 \\ \widehat{n} &= 0.776797 \end{aligned}$$

A probability plot, with the 2-sided 90% confidence bounds for the use stress levels of 323K and 2V, is shown next.



An acceleration factor plot, in which one of the stresses must be kept constant, can also be obtained. For example, in the following plot, the acceleration factor is plotted versus temperature given a constant voltage of 2V.



T-NT Confidence Bounds

Approximate Confidence Bounds for the T-NT Exponential

Confidence Bounds on the Mean Life

The mean life for the T-NT model is given by setting $m = L(V)$. The upper (m_U) and lower (m_L) bounds on the mean life (ML estimate of the mean life) are estimated by:

$$m_U = \widehat{m} \cdot e^{\frac{K_\alpha \sqrt{Var(\widehat{m})}}{\widehat{m}}}$$

$$m_L = \widehat{m} \cdot e^{-\frac{K_\alpha \sqrt{Var(\widehat{m})}}{\widehat{m}}}$$

where K_α is defined by:

$$\alpha = \frac{1}{\sqrt{2\pi}} \int_{K_\alpha}^{\infty} e^{-\frac{t^2}{2}} dt = 1 - \Phi(K_\alpha)$$

If δ is the confidence level, then $\alpha = \frac{1-\delta}{2}$ for the two-sided bounds, and $\alpha = 1 - \delta$ for the one-sided bounds. The variance of $\widehat{m}$ is given by:

$$\begin{aligned} Var(\widehat{m}) = & \left(\frac{\partial m}{\partial B}\right)^2 Var(\widehat{B}) + \left(\frac{\partial m}{\partial C}\right)^2 Var(\widehat{C}) + \left(\frac{\partial m}{\partial n}\right)^2 Var(\hat{b}) + 2\left(\frac{\partial m}{\partial B}\right)\left(\frac{\partial m}{\partial C}\right) Cov(\widehat{B}, \widehat{C}) \\ & + 2\left(\frac{\partial m}{\partial B}\right)\left(\frac{\partial m}{\partial n}\right) Cov(\widehat{B}, \widehat{n}) + 2\left(\frac{\partial m}{\partial C}\right)\left(\frac{\partial m}{\partial n}\right) Cov(\widehat{C}, \widehat{n}) \end{aligned}$$

or:

$$\begin{aligned} Var(\widehat{m}) = & \frac{1}{U^{2\widehat{n}}} e^{2\frac{\widehat{B}}{V}} [\frac{\widehat{C}^2}{V^2} Var(\widehat{B}) + Var(\widehat{C}) + \widehat{C}^2 (\ln(U))^2 Var(\widehat{n}) + \frac{2\widehat{C}}{V} Cov(\widehat{B}, \widehat{C}) \\ & - \frac{2\widehat{C}^2 \ln(U)}{V} Cov(\widehat{B}, \widehat{n}) - 2\widehat{C} \ln(U) Cov(\widehat{C}, \widehat{n})] \end{aligned}$$

The variances and covariance of B, C and n are estimated from the local Fisher matrix (evaluated at $\widehat{B}, \widehat{C}, \widehat{n})$ as follows:

$$\begin{bmatrix} Var(\widehat{B}) & Cov(\widehat{B}, \widehat{C}) & Cov(\widehat{B}, \widehat{n}) \\ Cov(\widehat{C}, \widehat{B}) & Var(\widehat{C}) & Cov(\widehat{C}, \widehat{n}) \\ Cov(\widehat{n}, \widehat{B}) & Cov(\widehat{n}, \widehat{C}) & Var(\widehat{n}) \end{bmatrix} = [F]^{-1}$$

where:

$$F = \begin{bmatrix} -\frac{\partial^2 \Lambda}{\partial B^2} & -\frac{\partial^2 \Lambda}{\partial B \partial C} & -\frac{\partial^2 \Lambda}{\partial B \partial n} \\ -\frac{\partial^2 \Lambda}{\partial C \partial B} & -\frac{\partial^2 \Lambda}{\partial C^2} & -\frac{\partial^2 \Lambda}{\partial C \partial n} \\ -\frac{\partial^2 \Lambda}{\partial n \partial B} & -\frac{\partial^2 \Lambda}{\partial n \partial C} & -\frac{\partial^2 \Lambda}{\partial n^2} \end{bmatrix}.$$

Confidence Bounds on Reliability

The bounds on reliability at a given time, T, are estimated by:

$$R_U = e^{-\frac{T}{m_U}}$$
$$R_L = e^{-\frac{T}{m_L}}$$

Confidence Bounds on Time

The bounds on time for a given reliability (ML estimate of time) are estimated by first solving the reliability function with respect to time:

$$\widehat{T} = -\widehat{m} \cdot \ln(R)$$

The corresponding confidence bounds are estimated from:

$$T_U = -m_U \cdot \ln(R)$$
$$T_L = -m_L \cdot \ln(R)$$

Approximate Confidence Bounds for the T-NT Weibull

Bounds on the Parameters

Using the same approach as previously discussed ($\widehat{\beta}$ and $\widehat{C}$ positive parameters):

$$\beta_U = \widehat{\beta} \cdot e^{\frac{K_\alpha \sqrt{Var(\widehat{\beta})}}{\widehat{\beta}}}$$
$$\beta_L = \widehat{\beta} \cdot e^{-\frac{K_\alpha \sqrt{Var(\widehat{\beta})}}{\widehat{\beta}}}$$

$$B_U = \widehat{B} + K_\alpha \sqrt{Var(\widehat{B})}$$
$$B_L = \widehat{B} - K_\alpha \sqrt{Var(\widehat{A})}$$

$$C_U = \widehat{C} \cdot e^{\frac{K_\alpha \sqrt{Var(\widehat{C})}}{\widehat{C}}}$$
$$C_L = \widehat{C} \cdot e^{-\frac{K_\alpha \sqrt{Var(\widehat{C})}}{\widehat{C}}}$$

and:

$$n_U = \widehat{n} + K_\alpha \sqrt{Var(\widehat{n})}$$
$$n_L = \widehat{n} - K_\alpha \sqrt{Var(\widehat{n})}$$

The variances and covariances of $\beta, B, C,$ and n are estimated from the Fisher matrix (evaluated at $\widehat{\beta}, \widehat{B}, \widehat{C}, \widehat{n})$ as follows:

$$\begin{bmatrix} Var(\widehat{\beta}) & Cov(\widehat{\beta}, \widehat{B}) & Cov(\widehat{\beta}, \widehat{C}) & Cov(\widehat{\beta}, \widehat{n}) \\ Cov(\widehat{B}, \widehat{\beta}) & Var(\widehat{B}) & Cov(\widehat{B}, \widehat{C}) & Cov(\widehat{B}, \widehat{n}) \\ Cov(\widehat{C}, \widehat{\beta}) & Cov(\widehat{C}, \widehat{B}) & Var(\widehat{C}) & Cov(\widehat{C}, \widehat{n}) \\ Cov(\widehat{n}, \widehat{\beta}) & Cov(\widehat{n}, \widehat{B}) & Cov(\widehat{n}, \widehat{C}) & Var(\widehat{n}) \end{bmatrix} = [F]^{-1}$$

where:

$$F = \begin{bmatrix} -\frac{\partial^2 \Lambda}{\partial \beta^2} & -\frac{\partial^2 \Lambda}{\partial \beta \partial B} & -\frac{\partial^2 \Lambda}{\partial \beta \partial C} & -\frac{\partial^2 \Lambda}{\partial \beta \partial n} \\ -\frac{\partial^2 \Lambda}{\partial B \partial \beta} & -\frac{\partial^2 \Lambda}{\partial B^2} & -\frac{\partial^2 \Lambda}{\partial B \partial C} & -\frac{\partial^2 \Lambda}{\partial B \partial n} \\ -\frac{\partial^2 \Lambda}{\partial C \partial \beta} & -\frac{\partial^2 \Lambda}{\partial C \partial B} & -\frac{\partial^2 \Lambda}{\partial C^2} & -\frac{\partial^2 \Lambda}{\partial C \partial n} \\ -\frac{\partial^2 \Lambda}{\partial n \partial \beta} & -\frac{\partial^2 \Lambda}{\partial n \partial B} & -\frac{\partial^2 \Lambda}{\partial n \partial C} & -\frac{\partial^2 \Lambda}{\partial n^2} \end{bmatrix}$$

Confidence Bounds on Reliability

The reliability function (ML estimate) for the T-NT Weibull model is given by:

$$\widehat{R}(T,U,V)=e^{-\left(\frac{U^{\widehat{n}}e^{-\frac{\widehat{B}}{V}}}{\widehat{C}}T\right)^{\widehat{\beta}}}$$

or:

$$\widehat{R}(T,U,V)=e^{-e^{\ln\left[\left(\frac{U^{\widehat{n}}e^{-\frac{\widehat{B}}{V}}}{\widehat{C}}T\right)^{\widehat{\beta}}\right]}}$$

Setting:

$$\widehat{u}=\ln\left[\left(\frac{U^{\widehat{n}}e^{-\frac{\widehat{B}}{V}}}{\widehat{C}}T\right)^{\widehat{\beta}}\right]$$

or:

$$\widehat{u}=\widehat{\beta}\left[\ln(T)-\frac{\widehat{B}}{V}-\ln(\widehat{C})+\widehat{n}\ln(U)\right]$$

The reliability function now becomes:

$$\widehat{R}(T,U,V)=e^{-e^{\widehat{u}}}$$

The next step is to find the upper and lower bounds on u :

$$u_U=\widehat{u}+K_\alpha\sqrt{Var(\widehat{u})}$$

$$u_L=\widehat{u}-K_\alpha\sqrt{Var(\widehat{u})}$$

where:

$$Var(\widehat{u})=\left(\frac{\partial\widehat{u}}{\partial\beta}\right)^2Var(\widehat{\beta})+\left(\frac{\partial\widehat{u}}{\partial B}\right)^2Var(\widehat{B})+\left(\frac{\partial\widehat{u}}{\partial C}\right)^2Var(\widehat{C})+\left(\frac{\partial\widehat{u}}{\partial n}\right)^2Var(\widehat{n})+2\left(\frac{\partial\widehat{u}}{\partial\beta}\right)\left(\frac{\partial\widehat{u}}{\partial C}\right)Cov(\widehat{\beta},\widehat{B})+2\left(\frac{\partial\widehat{u}}{\partial\beta}\right)\left(\frac{\partial\widehat{u}}{\partial C}\right)Cov(\widehat{\beta},\widehat{C})$$
$$+2\left(\frac{\partial\widehat{u}}{\partial\beta}\right)\left(\frac{\partial\widehat{u}}{\partial n}\right)Cov(\widehat{\beta},\widehat{n})+2\left(\frac{\partial\widehat{u}}{\partial B}\right)\left(\frac{\partial\widehat{u}}{\partial C}\right)Cov(\widehat{B},\widehat{C})+2\left(\frac{\partial\widehat{u}}{\partial B}\right)\left(\frac{\partial\widehat{u}}{\partial n}\right)Cov(\widehat{B},\widehat{n})+2\left(\frac{\partial\widehat{u}}{\partial C}\right)\left(\frac{\partial\widehat{u}}{\partial n}\right)Cov(\widehat{C},\widehat{n})$$

or:

$$Var(\widehat{u})=\left(\frac{\widehat{u}}{\widehat{\beta}}\right)^2Var(\widehat{\beta})+\left(\frac{\widehat{\beta}}{V}\right)^2Var(\widehat{B})+\left(\frac{\widehat{\beta}}{\widehat{C}}\right)^2Var(\widehat{C})+\left(\widehat{\beta}\ln(U)\right)^2Var(\widehat{n})-\frac{2\widehat{u}}{V}Cov(\widehat{\beta},\widehat{B})-\frac{2\widehat{u}}{\widehat{C}}Cov(\widehat{\beta},\widehat{C})$$
$$+2\widehat{u}\ln(U)Cov(\widehat{\beta},\widehat{n})+\frac{2\widehat{\beta}^2}{\widehat{C}V}Cov(\widehat{B},\widehat{C})-\frac{2\widehat{\beta}^2\ln(U)}{V}Cov(\widehat{B},\widehat{n})-\frac{2\widehat{\beta}^2\ln(U)}{\widehat{C}}Cov(\widehat{C},\widehat{n})$$

The upper and lower bounds on reliability are:

$$R_U = e^{-e^{(u_L)}}$$
$$R_L = e^{-e^{(u_U)}}$$

Confidence Bounds on Time

The bounds on time (ML estimate of time) for a given reliability are estimated by first solving the reliability function with respect to time as follows:

$$\ln(R) = -\left(\frac{U^{\widehat{n}}e^{-\frac{\widehat{B}}{V}}}{\widehat{C}}\widehat{T}\right)^{\widehat{\beta}}$$
$$\ln(-\ln(R)) = \widehat{\beta}\left(\ln(\widehat{T})-\frac{\widehat{B}}{V}-\ln(\widehat{C})+\widehat{n}\ln(U)\right)$$

or:

$$\widehat{u}=\frac{1}{\widehat{\beta}}\ln(-\ln(R))+\frac{\widehat{B}}{V}+\ln(\widehat{C})-\widehat{n}\ln(U)$$

where $\widehat{u}=\ln\widehat{T}$.

The upper and lower bounds on u are estimated from:

$$u_U=\widehat{u}+K_\alpha\sqrt{Var(\widehat{u})}$$

$$u_L = \widehat{u} - K_\alpha\sqrt{Var(\widehat{u})}$$

where:

$$Var(\widehat{u}) = \left(\frac{\partial\widehat{u}}{\partial\beta}\right)^2 Var(\widehat{\beta}) + \left(\frac{\partial\widehat{u}}{\partial B}\right)^2 Var(\widehat{B}) + \left(\frac{\partial\widehat{u}}{\partial C}\right)^2 Var(\widehat{C}) + \left(\frac{\partial\widehat{u}}{\partial n}\right)^2 Var(\widehat{n}) + 2\left(\frac{\partial\widehat{u}}{\partial\beta}\right)\left(\frac{\partial\widehat{u}}{\partial C}\right)Cov(\widehat{\beta},\widehat{B}) + 2\left(\frac{\partial\widehat{u}}{\partial\beta}\right)\left(\frac{\partial\widehat{u}}{\partial C}\right)Cov(\widehat{\beta},\widehat{C})$$
$$+2\left(\frac{\partial\widehat{u}}{\partial\beta}\right)\left(\frac{\partial\widehat{u}}{\partial n}\right)Cov(\widehat{\beta},\widehat{n}) + 2\left(\frac{\partial\widehat{u}}{\partial B}\right)\left(\frac{\partial\widehat{u}}{\partial C}\right)Cov(\widehat{B},\widehat{C}) + 2\left(\frac{\partial\widehat{u}}{\partial B}\right)\left(\frac{\partial\widehat{u}}{\partial n}\right)Cov(\widehat{B},\widehat{n}) + 2\left(\frac{\partial\widehat{u}}{\partial C}\right)\left(\frac{\partial\widehat{u}}{\partial n}\right)Cov(\widehat{C},\widehat{n})$$

or:

$$Var(\widehat{u}) = \frac{1}{\widehat{\beta}^4}[\ln(-\ln(R))]^2 Var(\widehat{\beta}) + \frac{1}{V^2}Var(\widehat{B}) + \frac{1}{\widehat{C}^2}Var(\widehat{C}) + [\ln(U)]^2 Var(\widehat{n}) - \frac{2\ln(-\ln(R))}{\widehat{\beta}^2 V}Cov(\widehat{\beta},\widehat{B}) - \frac{2\ln(-\ln(R))}{\widehat{\beta}^2\widehat{C}}Cov(\widehat{\beta},\widehat{C})$$
$$+\frac{2\ln(-\ln(R))\ln(U)}{\widehat{\beta}^2}Cov(\widehat{\beta},\widehat{n}) + \frac{2}{\widehat{C}V}Cov(\widehat{B},\widehat{C}) - \frac{2\ln(U)}{V}Cov(\widehat{B},\widehat{n}) - \frac{2\ln(U)}{\widehat{C}}Cov(\widehat{C},\widehat{n})$$

The upper and lower bounds on time are then found by:

$$T_U = e^{u_U}$$
$$T_L = e^{u_L}$$

Approximate Confidence Bounds for the T-NT Lognormal

Bounds on the Parameters

Since the standard deviation, $\widehat{\sigma}_{T'}$, and $\widehat{C}$ are positive parameters, $\ln(\widehat{\sigma}_{T'})$ and $\ln(\widehat{C})$ are treated as normally distributed and the bounds are estimated from:

$$\sigma_U = \widehat{\sigma}_{T'} \cdot e^{\frac{K_\alpha\sqrt{Var(\widehat{\sigma}_{T'})}}{\widehat{\sigma}_{T'}}} \quad \text{(Upper bound)}$$

$$\sigma_L = \frac{\widehat{\sigma}_{T'}}{e^{\frac{K_\alpha\sqrt{Var(\widehat{\sigma}_{T'})}}{\widehat{\sigma}_{T'}}}} \quad \text{(Lower bound)}$$

and:

$$C_U = \widehat{C} \cdot e^{\frac{K_\alpha \sqrt{Var(\widehat{C})}}{\widehat{C}}} \quad \text{(Upper bound)}$$

$$C_L = \frac{\widehat{A}}{e^{\frac{K_\alpha \sqrt{Var(\widehat{C})}}{\widehat{C}}}} \quad \text{(Lower bound)}$$

The lower and upper bounds on B and n are estimated from:

$$B_U = \widehat{B} + K_\alpha \sqrt{Var(\widehat{B})} \text{ (Upper bound)}$$

$$B_L = \widehat{B} - K_\alpha \sqrt{Var(\widehat{B})} \text{ (Lower bound)}$$

and:

$$n_U = \widehat{n} + K_\alpha \sqrt{Var(\widehat{n})} \text{ (Upper bound)}$$

$$n_L = \widehat{n} - K_\alpha \sqrt{Var(\widehat{n})} \text{ (Lower bound)}$$

The variances and covariances of B, $C, n,$ and $\sigma_{T'}$ are estimated from the local Fisher matrix (evaluated at $\widehat{B}, \widehat{C}, \widehat{n}, \widehat{\sigma}_{T'}$) as follows:

$$\begin{pmatrix} Var\left(\widehat{\sigma}_{T'}\right) & Cov\left(\widehat{B}, \widehat{\sigma}_{T'}\right) & Cov\left(\widehat{C}, \widehat{\sigma}_{T'}\right) & Cov\left(\widehat{n}, \widehat{\sigma}_{T'}\right) \\ Cov\left(\widehat{\sigma}_{T'}, \widehat{B}\right) & Var\left(\widehat{B}\right) & Cov\left(\widehat{B}, \widehat{C}\right) & Cov\left(\widehat{B}, \widehat{n}\right) \\ Cov\left(\widehat{\sigma}_{T'}, \widehat{C}\right) & Cov\left(\widehat{C}, \widehat{B}\right) & Var\left(\widehat{C}\right) & Cov\left(\widehat{C}, \widehat{n}\right) \\ Cov\left(\widehat{n}, \widehat{\sigma}_{T'}\right) & Cov\left(\widehat{n}, \widehat{B}\right) & Cov\left(\widehat{n}, \widehat{C}\right) & Var\left(\widehat{n}\right) \end{pmatrix} = [F]^{-1}$$

where:

$$F=\begin{pmatrix} -\frac{\partial^2\Lambda}{\partial\sigma_{T'}^2} & -\frac{\partial^2\Lambda}{\partial\sigma_{T'}\partial B} & -\frac{\partial^2\Lambda}{\partial\sigma_{T'}\partial C} & -\frac{\partial^2\Lambda}{\partial\sigma_{T'}\partial n} \\ -\frac{\partial^2\Lambda}{\partial B\partial\sigma_{T'}} & -\frac{\partial^2\Lambda}{\partial B^2} & -\frac{\partial^2\Lambda}{\partial B\partial C} & -\frac{\partial^2\Lambda}{\partial B\partial n} \\ -\frac{\partial^2\Lambda}{\partial C\partial\sigma_{T'}} & -\frac{\partial^2\Lambda}{\partial C\partial B} & -\frac{\partial^2\Lambda}{\partial C^2} & -\frac{\partial^2\Lambda}{\partial C\partial n} \\ -\frac{\partial^2\Lambda}{\partial n\partial\sigma_{T'}} & -\frac{\partial^2\Lambda}{\partial n\partial B} & -\frac{\partial^2\Lambda}{\partial n\partial C} & -\frac{\partial^2\Lambda}{\partial n^2} \end{pmatrix}$$

Bounds on Reliability

The reliability of the lognormal distribution is given by:

$$R(T',U,V;B,C,n,\sigma_{T'})=\int_{T'}^{\infty}\frac{1}{\widehat{\sigma}_{T'}\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{t-\ln(\widehat{C})+\widehat{n}\ln(U_i)-\frac{\widehat{B}}{V_i}}{\widehat{\sigma}_{T'}}\right)^2}dt$$

Let $\hat{z}(t,U,V;B,C,n,\sigma_T)=\frac{t-\ln(\widehat{C})+\widehat{n}\ln(U)-\frac{\widehat{B}}{V}}{\widehat{\sigma}_{T'}}$, then $\frac{d\hat{z}}{dt}=\frac{1}{\widehat{\sigma}_{T'}}$. For $t=T'$,
$\hat{z}=\frac{T'-\ln(\widehat{C})+\widehat{n}\ln(U)-\frac{\widehat{B}}{V}}{\widehat{\sigma}_{T'}}$, and for $t=\infty,\hat{z}=\infty.$

The above equation then becomes:

$$R(\hat{z})=\int_{\hat{z}(T',U,V)}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz$$

The bounds on z are estimated from:

$$z_U=\hat{z}+K_\alpha\sqrt{Var(\hat{z})}$$
$$z_L=\hat{z}-K_\alpha\sqrt{Var(\hat{z})}$$

where:

$$\begin{aligned}Var(\hat{z}) = & \left(\frac{\partial \hat{z}}{\partial B}\right)^2_{\widehat{B}} Var(\widehat{B}) + \left(\frac{\partial \hat{z}}{\partial C}\right)^2_{\widehat{C}} Var(\widehat{C}) + \left(\frac{\partial \hat{z}}{\partial n}\right)^2_{\hat{b}} Var(\widehat{n}) + \left(\frac{\partial \hat{z}}{\partial \sigma_{T'}}\right)^2_{\widehat{\sigma}_{T'}} Var(\widehat{\sigma}_{T'}) + 2\left(\frac{\partial \hat{z}}{\partial B}\right)_{\widehat{B}} \left(\frac{\partial \hat{z}}{\partial C}\right)_{\widehat{C}} Cov\left(\widehat{B},\widehat{C}\right) \\ & + 2\left(\frac{\partial \hat{z}}{\partial B}\right)_{\widehat{B}} \left(\frac{\partial \hat{z}}{\partial b}\right)_{\widehat{n}} Cov\left(\widehat{B},\widehat{n}\right) + 2\left(\frac{\partial \hat{z}}{\partial C}\right)_{\widehat{C}} \left(\frac{\partial \hat{z}}{\partial n}\right)_{\widehat{n}} Cov\left(\widehat{C},\widehat{n}\right) + 2\left(\frac{\partial \hat{z}}{\partial B}\right)_{\widehat{B}} \left(\frac{\partial \hat{z}}{\partial \sigma_{T'}}\right)_{\widehat{\sigma}_{T'}} Cov\left(\widehat{B},\widehat{\sigma}_{T'}\right) \\ & + 2\left(\frac{\partial \hat{z}}{\partial C}\right)_{\widehat{C}} \left(\frac{\partial \hat{z}}{\partial \sigma_{T'}}\right)_{\widehat{\sigma}_{T'}} Cov\left(\widehat{C},\widehat{\sigma}_{T'}\right) + 2\left(\frac{\partial \hat{z}}{\partial n}\right)_{\widehat{n}} \left(\frac{\partial \hat{z}}{\partial \sigma_{T'}}\right)_{\widehat{\sigma}_{T'}} Cov\left(\widehat{n},\widehat{\sigma}_{T'}\right)\end{aligned}$$

or:

$$\begin{aligned}Var(\hat{z}) = & \frac{1}{\widehat{\sigma}^2_{T'}}[\frac{1}{V^2} Var(\widehat{B}) + \frac{1}{C^2} Var(\widehat{C}) + \ln(U)^2 Var(\widehat{n}) + \hat{z}^2 Var(\widehat{\sigma}_{T'}) + \frac{2}{C \cdot V} Cov\left(\widehat{B},\widehat{C}\right) - \frac{2\ln(U)}{V} Cov\left(\widehat{B},\widehat{n}\right) \\ & - \frac{2\ln(U)}{C} Cov\left(\widehat{C},\widehat{n}\right) + \frac{2\hat{z}}{V} Cov\left(\widehat{B},\widehat{\sigma}_{T'}\right) + \frac{2\hat{z}}{C} Cov\left(\widehat{C},\widehat{\sigma}_{T'}\right) - 2\hat{z}\ln(U) Cov\left(\widehat{n},\widehat{\sigma}_{T'}\right)]\end{aligned}$$

The upper and lower bounds on reliability are:

$$\begin{aligned}R_U = & \int_{z_L}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \text{ (Upper bound)} \\ R_L = & \int_{z_U}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \text{ (Lower bound)}\end{aligned}$$

Confidence Bounds on Time

The bounds around time for a given lognormal percentile (unreliability) are estimated by first solving the reliability equation with respect to time, as follows:

$$T'(U,V;\widehat{B},\widehat{C},\widehat{n},\widehat{\sigma}_{T'}) = \ln(\widehat{C}) + \widehat{n}\ln(U) - \frac{\widehat{B}}{V} + z \cdot \widehat{\sigma}_{T'}$$

where:

$$\begin{aligned}T'(U,V;\widehat{A},\widehat{\phi},\hat{b},\widehat{\sigma}_{T'}) = & \ln(T) \\ z = & \Phi^{-1}\left[F(T')\right]\end{aligned}$$

and:

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z(T',U,V)} e^{-\frac{1}{2}z^2} dz$$

The next step is to calculate the variance of $T'(U,V;\widehat{B},\widehat{C},\widehat{n},\widehat{\sigma}_{T'})$:

$$Var(T') = \left(\frac{\partial T'}{\partial B}\right)^2 Var(\widehat{B}) + \left(\frac{\partial T'}{\partial C}\right)^2 Var(\widehat{C}) + \left(\frac{\partial T'}{\partial n}\right)^2 Var(\widehat{n}) + \left(\frac{\partial T'}{\partial \sigma_{T'}}\right)^2 Var(\widehat{\sigma}_{T'}) + 2\left(\frac{\partial T'}{\partial B}\right)\left(\frac{\partial T'}{\partial C}\right) Cov\left(\widehat{B},\widehat{C}\right)$$
$$+ 2\left(\frac{\partial T'}{\partial B}\right)\left(\frac{\partial T'}{\partial n}\right) Cov\left(\widehat{B},\widehat{n}\right) + 2\left(\frac{\partial T'}{\partial C}\right)\left(\frac{\partial T'}{\partial n}\right) Cov\left(\widehat{C},\widehat{n}\right) + 2\left(\frac{\partial T'}{\partial B}\right)\left(\frac{\partial T'}{\partial \sigma_{T'}}\right) Cov\left(\widehat{B},\widehat{\sigma}_{T'}\right)$$
$$+ 2\left(\frac{\partial T'}{\partial C}\right)\left(\frac{\partial T'}{\partial \sigma_{T'}}\right) Cov\left(\widehat{C},\widehat{\sigma}_{T'}\right) + 2\left(\frac{\partial T'}{\partial n}\right)\left(\frac{\partial T'}{\partial \sigma_{T'}}\right) Cov\left(\widehat{n},\widehat{\sigma}_{T'}\right)$$

or:

The upper and lower bounds are then found by:

$$T'_U = \quad \ln T_U = T' + K_\alpha \sqrt{Var(T')}$$
$$T'_L = \quad \ln T_L = T' - K_\alpha \sqrt{Var(T')}$$

Solving for T_U and T_L yields:

$$T_U = \quad e^{T'_U} \text{ (Upper bound)}$$
$$T_L = \quad e^{T'_L} \text{ (Lower bound)}$$

Multivariable Relationships: General Log-Linear and Proportional Hazards

IN THIS CHAPTER

So far in this reference the life-stress relationships presented have been either single stress relationships or two stress relationships. In most practical applications, however, life is a function of more than one or two variables (stress types). In addition, there are many applications where the life of a product as a function of stress and of some engineering variable other than stress is sought. In this chapter, the general log-linear relationship and the proportional hazards model are presented for the analysis of such cases where more than two accelerated stresses (or variables) need to be considered.

General Log-Linear Relationship

When a test involves multiple accelerating stresses or requires the inclusion of an engineering variable, a general multivariable relationship is needed. Such a relationship is the general log-linear relationship, which describes a life characteristic as a function of a vector of n stresses, or $\underline{X} = (X_1, X_2 ... X_n)$. Weibull++ includes this relationship and allows up to eight stresses. Mathematically the relationship is given by:

$$L(\underline{X}) = e^{\alpha_0 + \sum^n{}_{j=1} \alpha_j X_j}$$

where:

- α_0 and α_j are model parameters.
- X is a vector of n stresses.

This relationship can be further modified through the use of transformations and can be reduced to the relationships discussed previously, if so desired. As an example, consider a single stress application of this relationship and an inverse transformation on $X,$ such that $V = 1/X$ or:

$$L(V) = e^{\alpha_0 + \frac{\alpha_1}{V}} = e^{\alpha_0} e^{\frac{\alpha_1}{V}}$$

It can be easily seen that the generalized log-linear relationship with a single stress and an inverse transformation has been reduced to the Arrhenius relationship, where:

$$C = e^{\alpha_0}$$
$$B = \alpha_1$$

or:

$$L(V) = Ce^{\frac{B}{V}}$$

Similarly, when one chooses to apply a logarithmic transformation on X such that $X = \ln(V)$, the relationship would reduce to the Inverse Power Law relationship. Furthermore, if more than one stress is present, one could choose to apply a different transformation to each stress to create combination relationships similar to the Temperature-Humidity and the Temperature-Non Thermal. Weibull++ has three built-in transformation options, namely:

None	$X = V$	Exponential LSR
Reciprocal	$X = 1/V$	Arrhenius LSR
Logarithmic	$X = \ln(V)$	Power LSR

The power of the relationship and this formulation becomes evident once one realizes that 6,561 unique life-stress relationships are possible (when allowing a maximum of eight stresses). When combined with the life distributions available in Weibull++, almost 20,000 models can be created.

Using the GLL Model

Like the previous relationships, the general log-linear relationship can be combined with any of the available life distributions by expressing a life characteristic from that distribution with the GLL relationship. A brief overview of the GLL-distribution models available in Weibull++ is presented next.

GLL Exponential

The GLL-exponential model can be derived by setting $m = L(\underline{X})$ in the exponential *pdf*, yielding the following GLL-exponential *pdf*:

$$f(t, \underline{X}) = e^{-\left(\alpha_0 + \sum^{n}{}_{j=1} \alpha_j X_j\right)} e^{-\left(\alpha_0 + \sum^{n}{}_{j=1} \alpha_j X_j\right) \cdot t}$$

The total number of unknowns to solve for in this model is $n + 1$ (i.e., $a_0, a_1, \ldots a_n$).

GLL Weibull

The GLL-Weibull model can be derived by setting $\eta = L(\underline{X})$ in Weibull *pdf*, yielding the following GLL-Weibull *pdf*:

$$f(t, \underline{X}) = \beta \cdot t^{\beta - 1} e^{-\beta\left(\alpha_0 + \sum^{n}{}_{j=1} \alpha_j X_j\right)} e^{-t^{\beta} e^{-\beta\left(\alpha_0 + \sum^{n}{}_{j=1} \alpha_j X_j\right)}}$$

The total number of unknowns to solve for in this model is $n + 2$ (i.e., $\beta, a_0, a_1, \ldots a_n$).

GLL Lognormal

The GLL-lognormal model can be derived by setting $\breve{T} = L(\underline{X})$ in the lognormal *pdf*, yielding the following GLL-lognormal *pdf*:

$$f(t, \underline{X}) = \frac{1}{t\, \sigma_{T'} \sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{T' - \alpha_0 - \sum^{n}{}_{j=1} \alpha_j X_j}{\sigma_{T'}}\right)^2}$$

The total number of unknowns to solve for in this model is $n + 2$ (i.e., $\sigma_{T'}, a_0, a_1, \ldots a_n$).

GLL Likelihood Function

The maximum likelihood estimation method can be used to determine the parameters for the GLL relationship and the selected life distribution. For each distribution, the likelihood function can be derived, and the parameters of model (the distribution parameters and the GLL parameters) can be obtained by maximizing the log-likelihood function. For example, the log-likelihood function for the Weibull distribution is given by:

$$\ln(L) = \Lambda = \sum_{i=1}^{F_e} N_i \ln\left[\beta \cdot T_i^{\beta-1} e^{-T_i^{\beta} \cdot e^{-\beta\left(\alpha_0+\sum_{j=1}^{n} a_j x_{i,j}\right)}} e^{-\beta\left(\alpha_0+\sum_{j=1}^{n} a_j x_{i,j}\right)}\right] - \sum_{i=1}^{S} N_i' \left(T_i'\right)^{\beta} e^{-\beta\left(\alpha_0+\sum_{j=1}^{n} a_j x_{i,j}\right)} + \sum_{i=1}^{FI} N_i'' \ln[R_{Li}'' - R_{Ri}'']$$

where:

$$R_{Li}'' = e^{-\left(T_{Li}'' e^{\alpha_0+\sum_{j=1}^{n} \alpha_j x_j}\right)^{\beta}}$$

$$R_{Ri}'' = e^{-\left(T_{Ri}'' e^{\alpha_0+\sum_{j=1}^{n} \alpha_j x_j}\right)^{\beta}}$$

and:

- F_e is the number of groups of exact times-to-failure data points.
- N_i is the number of times-to-failure in the i^{th} time-to-failure data group.
- λ is the failure rate parameter (unknown).
- T_i is the exact failure time of the i^{th} group.
- S is the number of groups of suspension data points.
- N_i' is the number of suspensions in the i^{th} group of suspension data points.
- T_i' is the running time of the i^{th} suspension data group.
- FI is the number of interval data groups.
- N_i'' is the number of intervals in the i^{th} group of data intervals.

- T''_{Li} is the beginning of the i^{th} interval.
- T''_{Ri} is the ending of the i^{th} interval.

GLL Example

Consider the data summarized in the following tables. These data illustrate a typical three-stress type accelerated test.

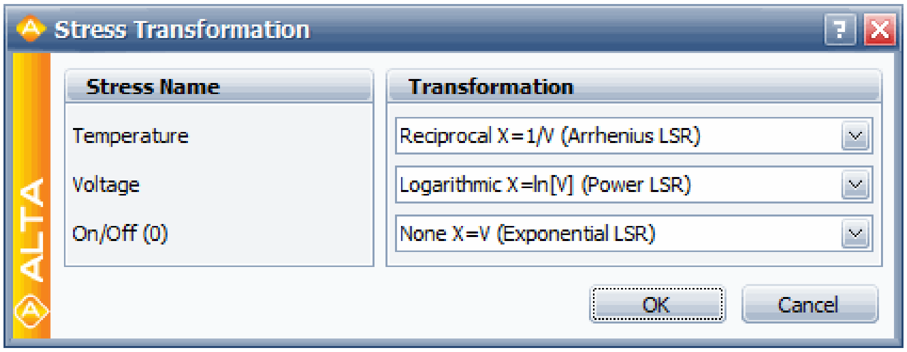
Stress Profile Summary

Profile	Temp (K)	Voltage (V)	Operation Type
A	358	12	On/Off
B	358	12	Continuous
C	378	12	On/Off
D	378	12	Continuous
E	378	16	On/Off
F	378	16	Continuous
G	398	12	On/Off
H	398	12	Continuous

Failure Data

Time (Profile)	Time (Profile)	Time (Profile)	Time (Profile)
498 (A)	211 (D)	249 (E)	134 (G)
750 (A)	266 (D)	145 (F)	163 (G)
445(B)	298 (D)	192 (F)	116 (H)
586 (B)	343 (D)	208 (F)	149 (H)
691 (B)	364 (D)	231 (F)	155 (H)
176 (C)	387 (D)	254 (F)	173 (H)
252 (C)	118 (E)	293 (F)	193 (H)
309 (C)	163 (E)	87 (G)	214 (H)
398 (C)	210 (E)	112 (G)	
20 units suspended at 750 (B) 14 units suspended at 445 (D)		10 units suspended at 300 (F) 7 units suspended at 228 (H)	

The data in the second table are analyzed assuming a Weibull distribution, an Arrhenius life-stress relationship for temperature and an inverse power life-stress relationship for voltage. No transformation is performed on the operation type. The operation type variable is treated as an indicator variable that takes a discrete value of 0 for an on/off operation and 1 for a continuous operation. The following figure shows the stress types and their transformations in Weibull++.



The GLL relationship then becomes:

$$\eta = e^{\alpha_0 + \alpha_1 \frac{1}{V_1} + \alpha_2 \ln(V_2) + \alpha_3 V_3}$$

The resulting relationship after performing these transformations is:

$$\eta = e^{\alpha_0} e^{\alpha_1 \frac{1}{V_1}} e^{\alpha_2 \ln(V_2)} e^{\alpha_3 V_3} = e^{\alpha_0} e^{\alpha_1 \frac{1}{V_1}} V_2^{\alpha_2} e^{\alpha_3 V_3}$$

Therefore, the parameter B of the Arrhenius relationship is equal to the log-linear coefficient α_1 , and the parameter n of the inverse power relationship is equal to ($-\alpha_2$). Therefore η can also be written as:

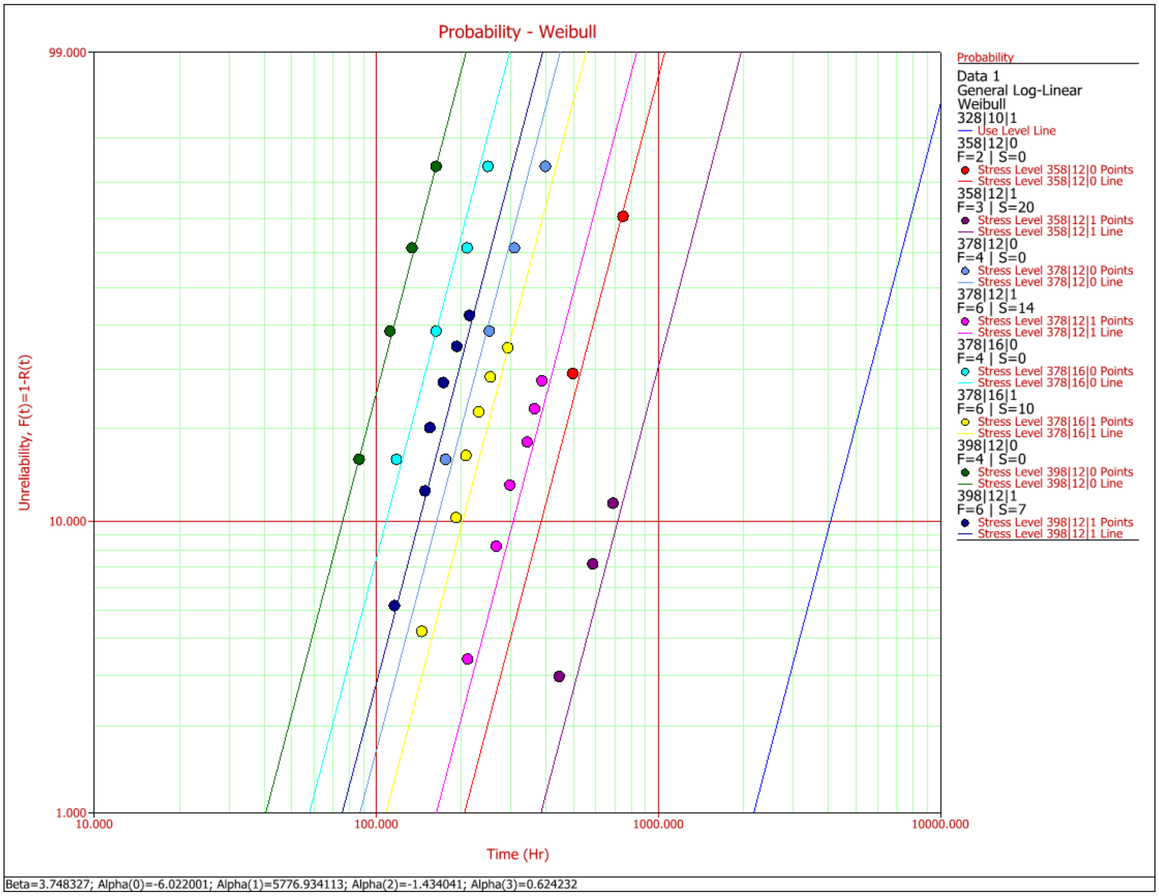
$$\eta = e^{\alpha_0} e^{\frac{B}{V_1}} V_2^n e^{\alpha_3 V_3}$$

The activation energy of the Arrhenius relationship can be calculated by multiplying B with Boltzmann's constant.

The best fit values for the parameters in this case are:

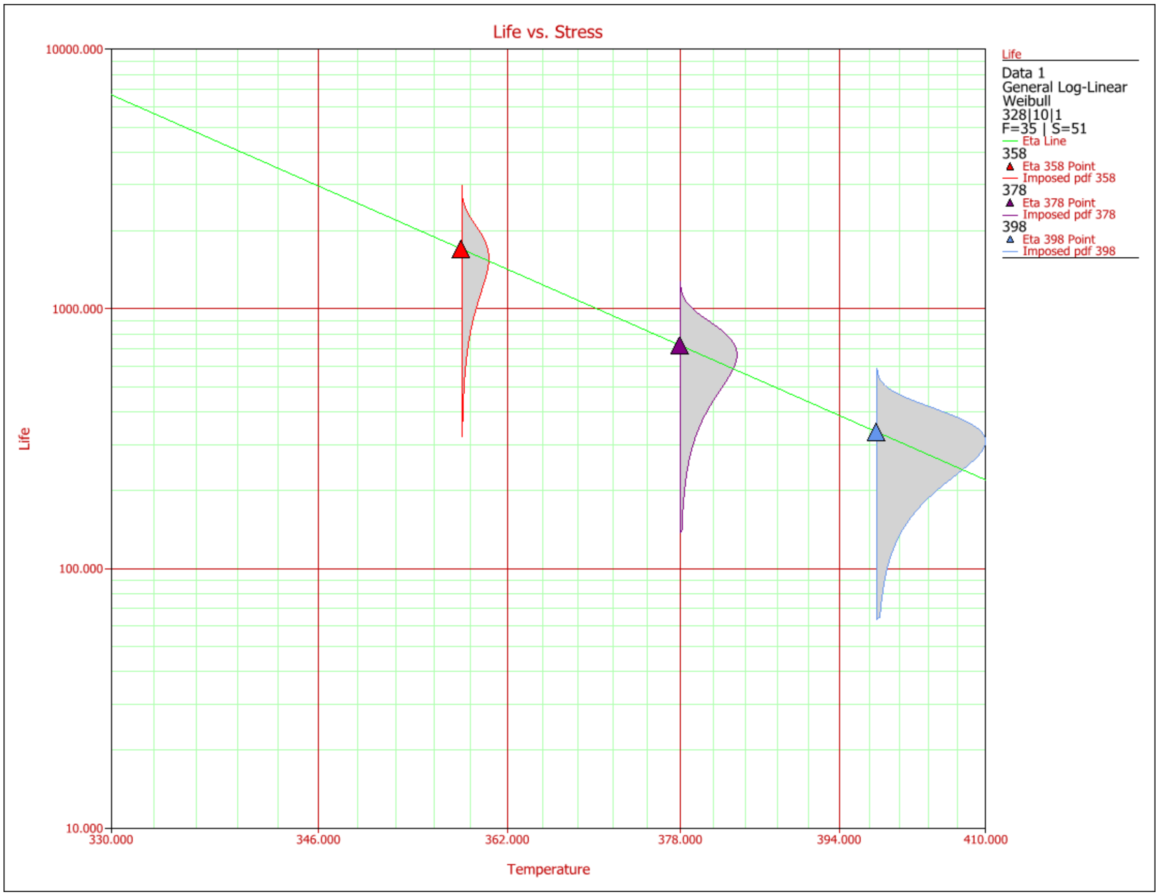
$$\beta = 3.7483;\ \alpha_0 = -6.0220;\ \alpha_1 = 5776.9341;$$
$$\alpha_2 = -1.4340;\ \alpha_3 = 0.6242.$$

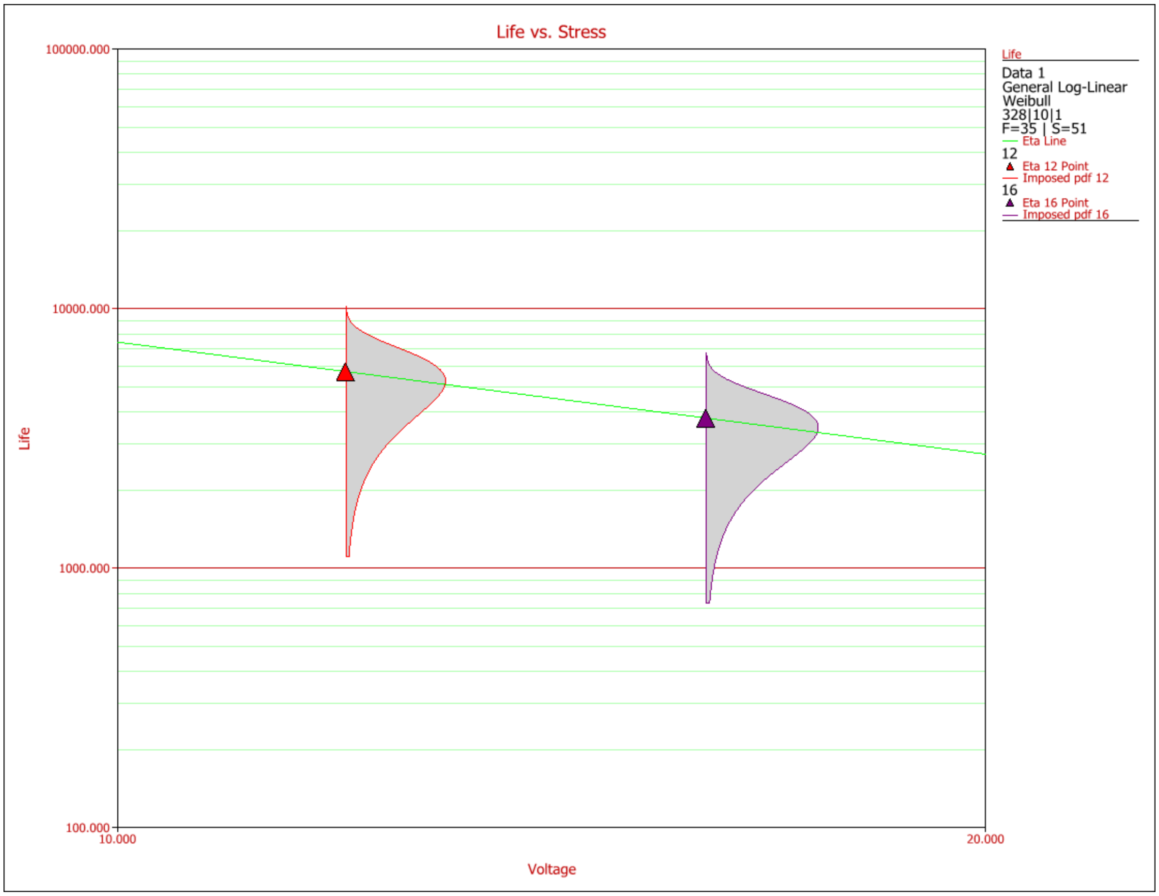
Once the parameters are estimated, further analysis on the data can be performed. First, using Weibull++, a Weibull probability plot of the data can be obtained, as shown next.



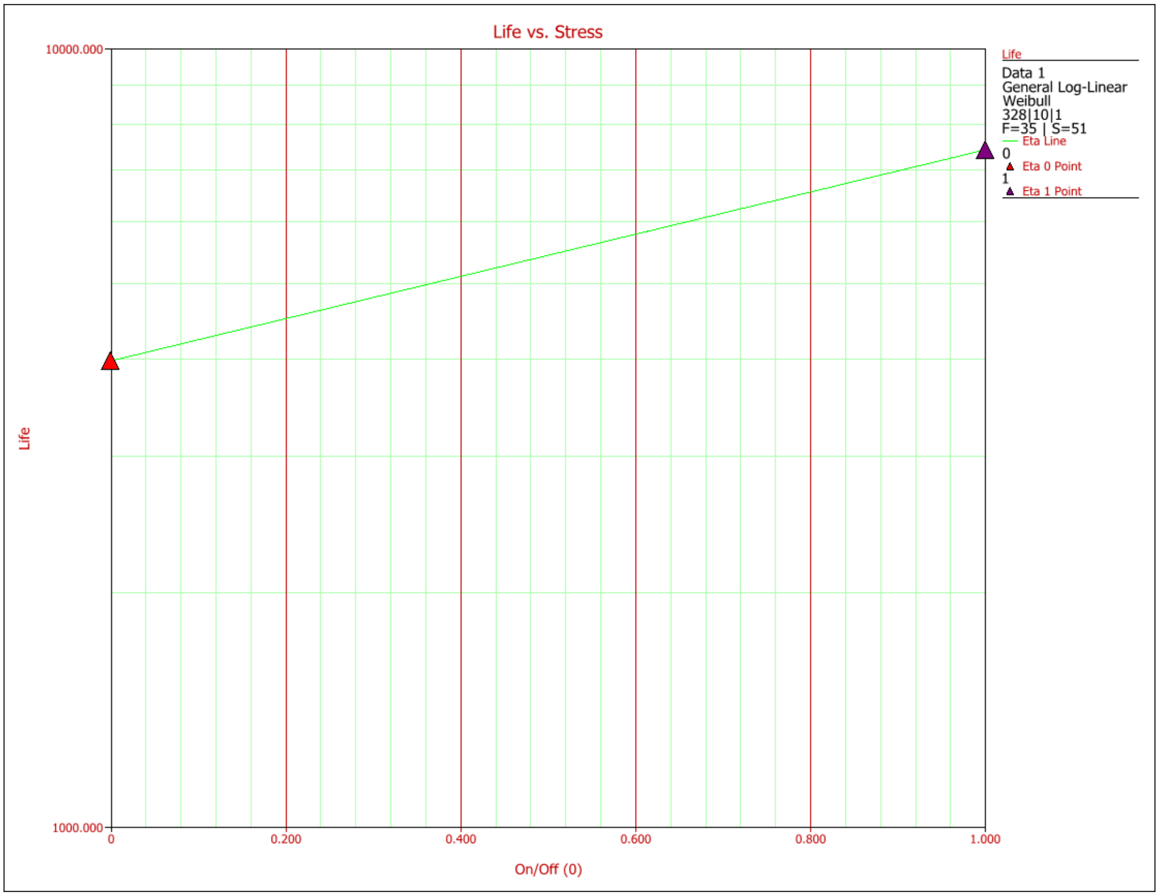
Several types of information about the model as well as the data can be obtained from a probability plot. For example, the choice of an underlying distribution and the assumption of a common slope (shape parameter) can be examined. In this example, the linearity of the data supports the use of the Weibull distribution. In addition, the data appear parallel on this plot, therefore reinforcing the assumption of a common beta. Further statistical analysis can and should be performed for these purposes as well.

The Life vs. Stress plot is a very common plot for the analysis of accelerated data. Life vs. Stress plots can be very useful in assessing the effect of each stress on a product's failure. In this case, since the life is a function of three stresses, three different plots can be created. Such plots are created by holding two of the stresses constant at the desired use level, and varying the remaining one. The use stress levels for this example are 328K for temperature and 10V for voltage. For the operation type, a decision has to be made by the engineers as to whether they implement an on/off or continuous operation. The next two figures display the effects of temperature and voltage on the life of the product.





The effects of the two different operation types on life can be observed in the next figure. It can be seen that the on/off cycling has a greater effect on the life of the product in terms of accelerating failure than the continuous operation. In other words, a higher reliability can be achieved by running the product continuously.



Proportional Hazards Model

Introduced by D. R. Cox, the Proportional Hazards (PH) model was developed in order to estimate the effects of different covariates influencing the times-to-failure of a system. The model has been widely used in the biomedical field, as discussed in Leemis [22], and recently there has been an increasing interest in its application in reliability engineering. In its original form, the model is non-parametric, (i.e., no assumptions are made about the nature or shape of the underlying failure distribution). In this reference, the original non-parametric formulation as well as a parametric form of the model will be considered utilizing a Weibull life distribution. In Weibull++, the proportional hazards model is included in its parametric form and can be used to analyze data with up to eight variables. The GLL-Weibull and GLL-exponential models are actually special cases of the proportional hazards model. However, when using the proportional hazards in Weibull++, no transformation on the covariates (or stresses) can be performed.

Non-Parametric Model Formulation

According to the PH model, the failure rate of a system is affected not only by its operation time, but also by the covariates under which it operates. For example, a unit may have been tested under a combination of different accelerated stresses such as humidity, temperature, voltage, etc. It is clear then that such factors affect the failure rate of a unit.

The instantaneous failure rate (or hazard rate) of a unit is given by:

$$\lambda(t)=\frac{f(t)}{R(t)}$$

where:

- $f(t)$ is the probability density function.
- $R(t)$ is the reliability function.

Note that for the case of the failure rate of a unit being dependent not only on time but also on other covariates, the above equation must be modified in order to be a function of time and of the covariates. The proportional hazards model assumes that the failure rate (hazard rate) of a unit is the product of:

- an arbitrary and unspecified baseline failure rate, $\lambda_0(t),$ which is a function of time only.
- a positive function $g(x,\underline{A})$, independent of time, which incorporates the effects of a number of covariates such as humidity, temperature, pressure, voltage, etc.

The failure rate of a unit is then given by:

$$\lambda(t,\underline{X})=\lambda_0(t)\cdot g(\underline{X},\underline{A})$$

where:

- $\underline{X}$ is a row vector consisting of the covariates:

$$\underline{X}=(x_1,x_2,\ldots,x_m)$$

- $\underline{A}$ is a column vector consisting of the unknown parameters (also called regression parameters) of the model:

$$\underline{A} = (a_1, a_2, \ldots a_m)^T$$

where:

m = number of stress related variates (time-independent).

It can be assumed that the form of $g(\underline{X}, \underline{A})$ is known and $\lambda_0(t)$ is unspecified. Different forms of $g(\underline{X}, \underline{A})$ can be used.

However, the exponential form is mostly used due to its simplicity and is given by:

$$g(\underline{X}, \underline{A}) = e^{\underline{A}^T \underline{X}^T} = e^{\sum_{j=1}^{m} a_j x_j}$$

The failure rate can then be written as:

$$\lambda(t, \underline{X}) = \lambda_0(t) \cdot e^{\sum_{j=1}^{m} a_j x_j}$$

Parametric Model Formulation

A parametric form of the proportional hazards model can be obtained by assuming an underlying distribution. In Weibull++, the Weibull and exponential distributions are available. In this section we will consider the Weibull distribution to formulate the parametric proportional hazards model. In other words, it is assumed that the baseline failure rate is parametric and given by the Weibull distribution. In this case, the baseline failure rate is given by:

$$\lambda_0(t) = \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1}$$

The PH failure rate then becomes:

$$\lambda(t, \underline{X}) = \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1} \cdot e^{\sum_{j=1}^{m} a_j x_j}$$

It is often more convenient to define an additional covariate, $x_0 = 1$, in order to allow the Weibull scale parameter raised to the beta (shape parameter) to be included in the vector of regression coefficients. The PH failure rate can then be written as:

$$\lambda(t, \underline{X}) = \beta \cdot t^{\beta-1} \cdot e^{\sum_{j=0}^{m} a_j x_j}$$

The PH reliability function is given by:

$$R(t,\underline{X}) = e^{-\int_0^t \lambda(u)du} = e^{-\int_0^t \lambda(u,\underline{X})du} = e^{-t^{\beta}\cdot e^{\sum_{j=0}^{m} a_j x_j}}$$

The *pdf* can be obtained by taking the partial derivative of the reliability function with respect to time. The PH *pdf* is:

$$f(t,\underline{X}) = \lambda(t,\underline{X})\cdot R(t,\underline{X}) = \beta\cdot t^{\beta-1} e^{\left[\sum_{j=0}^{m} a_j x_j - t^{\beta}\cdot e^{\sum_{j=0}^{m} a_j x_j}\right]}$$

The total number of unknowns to solve for in this model is $m+2$ (i.e., $\beta, a_0, a_1, \ldots a_m$).

The maximum likelihood estimation method can be used to determine these parameters. The log-likelihood function for this case is given by:

$$\ln(L) = \Lambda = \sum_{i=1}^{F_e} N_i \ln\left(\beta\cdot T_i^{\beta-1} e^{-T_i^{\beta}\cdot e^{\sum_{j=0}^{m} a_j x_{i,j}}} e^{\sum_{j=0}^{m} a_j x_{i,j}}\right) - \sum_{i=1}^{S} N_i' \left(T_i'\right)^{\beta} e^{\sum_{j=0}^{m} a_j x_{i,j}} + \sum_{i=1}^{FI} N_i'' \ln[R_{Li}'' - R_{Ri}'']$$

where:

$$R_{Li}'' = e^{-T_{Li}''^{\beta} e^{\sum^{n}{}_{j=0} \alpha_j x_j}}$$
$$R_{Ri}'' = e^{-T_{Ri}''^{\beta} e^{\sum^{n}{}_{j=0} \alpha_j x_j}}$$

Solving for the parameters that maximize the log-likelihood function will yield the parameters for the PH-Weibull model. Note that for $\beta = 1$, the log-likelihood function becomes the log-likelihood function for the PH-exponential model, which is similar to the original form of the proportional hazards model proposed by Cox and Oakes [39].

Note that the likelihood function of the GLL model is very similar to the likelihood function for the proportional hazards-Weibull model. In particular, the shape parameter of the Weibull distribution can be included in the regression coefficients as follows:

$$a_{i,PH} = -\beta\cdot a_{i,GLL}$$

where:

- $a_{i,PH}$ are the parameters of the PH model.
- $a_{i,GLL}$ are the parameters of the general log-linear model.

In this case, the likelihood functions are identical. Therefore, if no transformation on the covariates is performed, the parameter values that maximize the likelihood function of the GLL model also maximize the likelihood function for the proportional hazards-Weibull (PHW) model. Note that for $\beta = 1$ (exponential life distribution), the two likelihood functions are identical, and $a_{i,PH} = -a_{i,GLL}$.

Indicator Variables

Another advantage of the multivariable relationships included in Weibull++ is that they allow for simultaneous analysis of continuous and categorical variables. Categorical variables are variables that take on discrete values such as the lot designation for products from different manufacturing lots. In this example, lot is a categorical variable, and it can be expressed in terms of indicator variables. Indicator variables only take a value of 1 or 0. For example, consider a sample of test units. A number of these units were obtained from Lot 1, others from Lot 2, and the rest from Lot 3. These three lots can be represented with the use of indicator variables, as follows:

- Define two indicator variables, X_1 and X_2.
- For the units from Lot 1, $X_1 = 1,$ and $X_2 = 0.$
- For the units from Lot 2, $X_1 = 0,$ and $X_2 = 1.$
- For the units from Lot 3, $X_1 = 0,$ and $X_2 = 0.$

Assume that an accelerated test was performed with these units, and temperature was the accelerated stress. In this case, the GLL relationship can be used to analyze the data. From the GLL relationship we get:

$$L(\underline{X}) = e^{\alpha_0 + \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3}$$

where:

- X_1 and X_2 are the indicator variables, as defined above.
- $X_3 = \frac{1}{T},$ where T is the temperature.

The data can now be entered in Weibull++ and, with the assumption of an underlying life distribution and using MLE, the parameters of this model can be obtained.

Time-Varying Stress Models

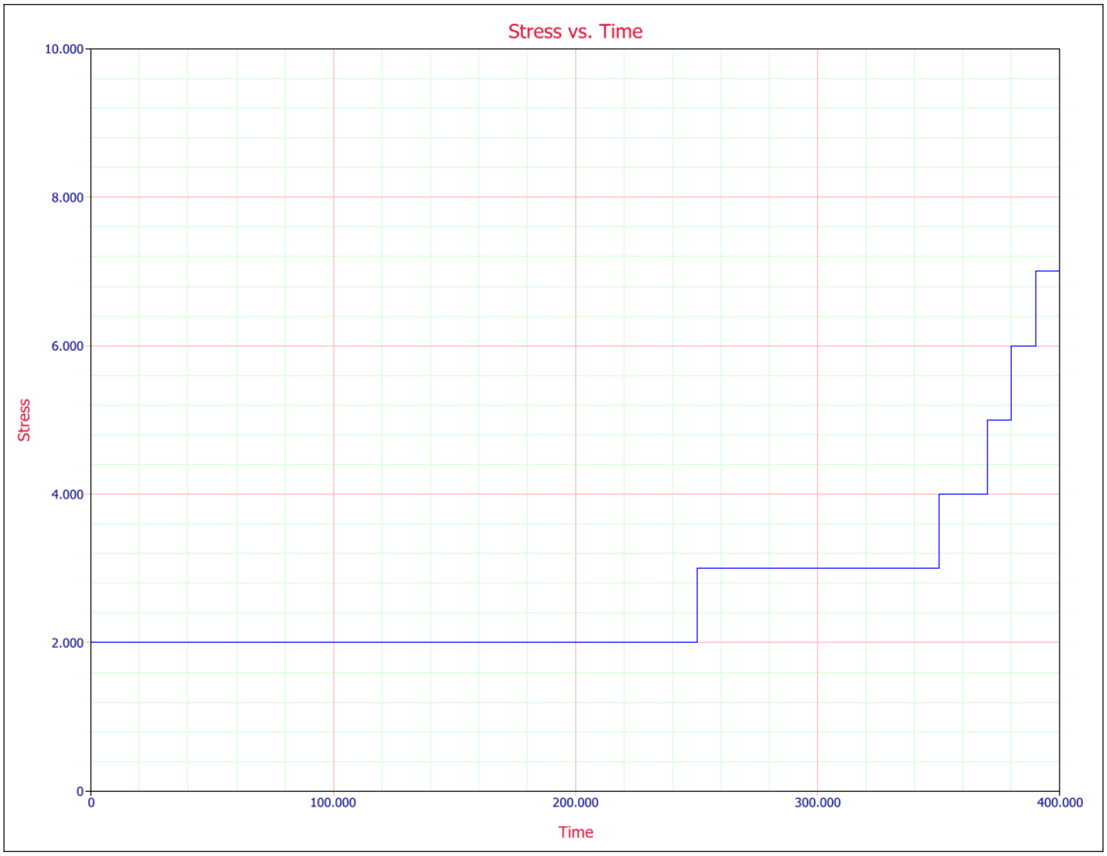
IN THIS CHAPTER

Traditionally, accelerated tests that use a time-varying stress application have been used to assure failures quickly. This is highly desirable given the pressure on industry today to shorten new product introduction time. The most basic type of time-varying stress test is a step-stress test. In step-stress accelerated testing, the test units are subjected to successively higher stress levels in predetermined stages, and thus follow a time-varying stress profile. The units usually start at a lower stress level and at a predetermined time, or failure number, the stress is increased and the test continues. The test is terminated when all units have failed, when a certain number of failures are observed or when a certain time has elapsed. Step-stress testing can substantially shorten the reliability test's duration. In addition to step-stress testing, there are many other types of time-varying stress profiles that can be used in accelerated life testing. However, it should be noted that there is more uncertainty in the results from such time-varying stress tests than from traditional constant stress tests of the same length and sample size.

When dealing with data from accelerated tests with time-varying stresses, the life-stress relationship must take into account the cumulative effect of the applied stresses. Such a model is commonly referred to as a *cumulative damage* or *cumulative exposure* model. Nelson [28] defines and presents the derivation and assumptions of such a model. Weibull++ includes the cumulative damage model for the analysis of time-varying stress data. This section presents an introduction to the model formulation and its application.

Model Formulation

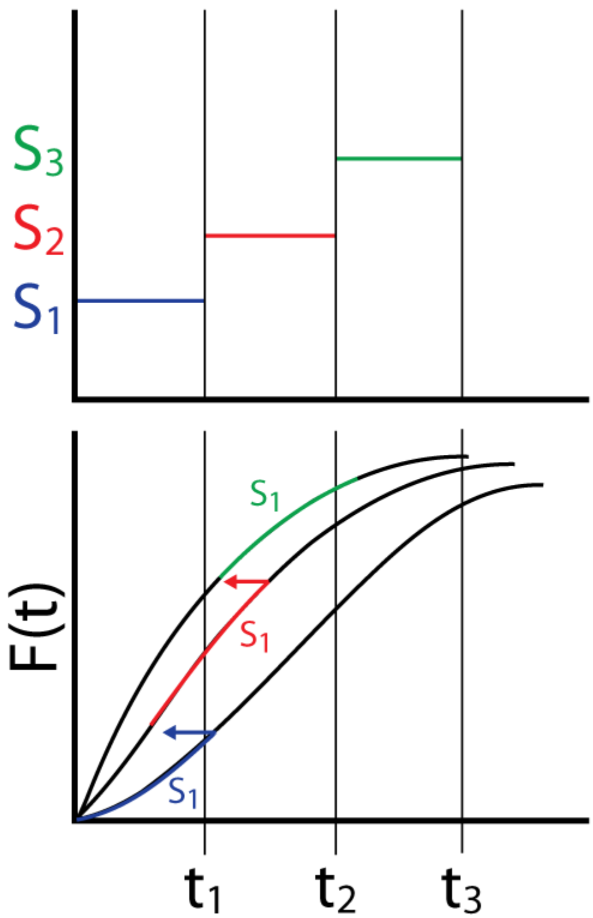
To formulate the cumulative exposure/damage model, consider a simple step-stress experiment where an electronic component was subjected to a voltage stress, starting at 2V (use stress level) and increased to 7V in stepwise increments, as shown in the next figure. The following steps, in hours, were used to apply stress to the products under test: 0 to 250, 2V; 250 to 350, 3V; 350 to 370, 4V; 370 to 380, 5V; 380 to 390, 6V; and 390 to 400, 7V.



In this example, 11 units were available for the test. All units were tested using this same stress profile. Units that failed were removed from the test and their total times on test were recorded. The following times-to-failure were observed in the test, in hours: 280, 310, 330, 352, 360, 366, 371, 374, 378, 381 and 385. The first failure in this test occurred at 280 hours when the stress was 3V. During the test, this unit experienced a period of time at 2V before failing at 3V. If the stress were 2V, one would expect the unit to fail at a time later than 280 hours, while if the unit were always at 3V, one would expect that failure time to be sooner than 280 hrs. The problem faced by the analyst in this case is to determine some equivalency between the stresses. In other words, what is the equivalent of 280 hours (with 250 hours spent at 2V and 30 hours spent at 3V) at a constant 2V stress or at a constant 3V stress?

Mathematical Formulation for a Step-Stress Model

To mathematically formulate the model, consider the step-stress test shown in the next figure, with stresses S1, S2 and S3. Furthermore, assume that the underlying life distribution is the Weibull distribution, and also assume an inverse power law relationship between the Weibull scale parameter and the applied stress.



From the inverse power law relationship, the scale parameter, η, of the Weibull distribution can be expressed as an inverse power function of the stress, V or:

$$\eta(V) = \frac{1}{KV^{\eta}}$$

where K and n are model parameters. The fraction of the units failing by time t_1 under a constant stress $V = S_1$, is given by:

$$F(t;V) = 1 - R(t;V)$$

where:

$$R(t;V) = e^{-\left[\frac{t}{\eta(V)}\right]^{\beta}}$$

The *cdf* for each constant stress level is:

$$F_1(t;S_1)= \quad 1-e^{-(KS_1^n t)^\beta}$$
$$F_2(t;S_2)= \quad 1-e^{-(KS_2^n t)^\beta}$$
$$F_3(t;S_3)= \quad 1-e^{-(KS_3^n t)^\beta}$$

The above equations would suffice if the units did not experience different stresses during the test, as they did in this case. To analyze the data from this step-stress test, a cumulative exposure model is needed. Such a model will relate the life distribution, in this case the Weibull distribution, of the units at one stress level to the distribution at the next stress level. In formulating this model, it is assumed that the remaining life of the test units depends only on the cumulative exposure the units have seen and that the units do not remember how such exposure was accumulated. Moreover, since the units are held at a constant stress at each step, the surviving units will fail according to the distribution at the current step, but with a starting age corresponding to the total accumulated time up to the beginning of the current step. This model can be formulated as follows:

- Units failing during the first step have not experienced any other stresses and will fail according to the S_1 *cdf*. Units that made it to the second step will fail according to the S_2 *cdf*, but will have accumulated some equivalent age, ε_1, at this stress level (given the fact that they have spent t_1 hours at S_1) or:

$$F_2(t;S_2)=1-e^{-[KS_2^n((t-t_1)+\varepsilon_1)]^\beta}$$

In other words, the probability that the units will fail at a time, t, while at S_2 and between t_1 and t_2 is equivalent to the probability that the units would fail after accumulating $(t-t_1)$ plus some equivalent time, ε_1, to account for the exposure the units have seen at S_1.

- The equivalent time, ε_1, will be the time by which the probability of failure at S_2 is equal to the probability of failure at S_1 after an exposure of t_1 or:

$$\begin{aligned} F_1(t_1;S_1) &= F_2(\varepsilon_1;S_2) \\ 1-e^{-(KS_1^n t_1)^\beta} &= 1-e^{-(KS_2^n \varepsilon_1)^\beta} \\ S_1^n t_1 &= S_2^n \varepsilon_1 \\ \varepsilon_1 &= t_1\left(\frac{S_1}{S_2}\right)^n \end{aligned}$$

- One would repeat this for step 3 taking into account the accumulated exposure during steps 1 and 2, or in more general terms and for the i^{th} step:

$$F_i(t;S_i)=1-e^{-[KS_i^n((t-t_{i-1})+\varepsilon_{i-1})]^\beta}$$

where:

$$\varepsilon_{i-1}=(t_{i-1}-t_{i-2}+\varepsilon_{i-2})\left(\frac{S_{i-1}}{S_i}\right)^n$$

- Once the *cdf* for each step has been obtained, the *pdf* can also then be determined utilizing:

$$f_i(t,S_i)=-\frac{d}{dt}\left[F_i(t,S_i)\right]$$

Once the model has been formulated, model parameters (i.e., K , n and β) can be computed utilizing maximum likelihood estimation methods.

The previous example can be expanded for any time-varying stress. Weibull++ allows you to define any stress profile. For example, the stress can be a ramp stress, a monotonically increasing stress, sinusoidal, etc. This section presents a generalized formulation of the cumulative damage model, where stress can be any function of time.

Confidence Intervals

Using the same methodology as in previous sections, approximate confidence intervals can be derived and applied to all results of interest using the Fisher Matrix approach discussed in Appendix A. Weibull++ utilizes such intervals on all results.

Notes on Trigonometric Functions

Trigonometric functions sometime are used in accelerated life tests. However Weibull++ does not include them. In fact, a trigonometric function can be defined by its frequency and magnitude. Frequency and magnitude then can be treated as two constant stresses. The GLL model discussed in General Log-Linear Relationship then can be applied for modeling.

Cumulative Damage Power Relationship

This section presents a generalized formulation of the cumulative damage model where stress can be any function of time and the life-stress relationship is based on the power relationship.

and assuming the power law relationship, the life-stress relationship is given by:

$$L(x(t))=\left(\frac{a}{x(t)}\right)^{n}$$

In Weibull++, the above relationship is actually presented in a format consistent with the general log-linear (GLL) relationship for the power law relationship:

$$L(x(t))=e^{\alpha_0+\alpha_1\ln(x(t))}$$

Therefore, instead of displaying a and n as the calculated parameters, the following reparameterization is used:

$$\begin{aligned}\alpha_0 &= \ln(a^n)\\ \alpha_1 &= -n\end{aligned}$$

Cumulative Damage Power - Exponential

Given a time-varying stress $x(t)$ and assuming the power law relationship, the mean life is given by:

$$\frac{1}{m(t,\, x)}=s(t,\, x)=\left(\frac{x(t)}{a}\right)^{n}$$

The reliability function of the unit under a single stress is given by:

$$R(t,\, x(t))=e^{-I(t,\, x)}$$

where:

$$I(t,\, x)=\int_0^t\left(\frac{x(u)}{a}\right)^{n}du$$

Therefore, the *pdf* is:

$$f(t,\, x)=s(t,\, x)e^{-I(t,\, x)}$$

Parameter estimation can be accomplished via maximum likelihood estimation methods, and confidence intervals can be approximated using the Fisher matrix approach. Once the parameters are determined, all other characteristics of interest (e.g., mean life, failure rate, etc.) can be obtained utilizing the statistical properties definitions presented in previous chapters. The log-likelihood equation is as follows:

$$\ln(L) = \quad \Lambda = \sum_{i=1}^{Fe} N_i \ln[s(T_i, x_i)] - \sum_{i=1}^{Fe} N_i \left(I(T_i, x_i)\right) - \sum_{i=1}^{S} N_i' \left(I(T_i', x_i')\right) + \sum_{i=1}^{FI} N_i'' \ln[R_{Li}'' - R_{Ri}'']$$

where:

$$R_{Li}''(T_{Li}'', x_i'') = \quad e^{-I(T_{Li}'', x_i'')}$$
$$R_{Ri}''(T_{Ri}'', x_i'') = \quad e^{-I(T_{Ri}'', x_i'')}$$

and:

- F_e is the number of groups of exact times-to-failure data points.
- N_i is the number of times-to-failure in the i^{th} time-to-failure data group.
- T_i is the exact failure time of the i^{th} group.
- S is the number of groups of suspension data points.
- N_i' is the number of suspensions in the i^{th} group of suspension data points.
- T_i' is the running time of the i^{th} suspension data group.
- FI is the number of interval data groups.
- N_i'' is the number of intervals in the i^{th} group of data intervals.
- T_{Li}'' is the beginning of the i^{th} interval.
- T_{Ri}'' is the ending of the i^{th} interval.

Cumulative Damage Power - Weibull

Given a time-varying stress $x(t)$ and assuming the power law relationship, the characteristic life is given by:

$$\frac{1}{\eta(t,x)} = s(t,x) = \left(\frac{x(t)}{a}\right)^n$$

The reliability function of the unit under a single stress is given by:

$$R(t,x(t)) = e^{-(I(t,x))^\beta}$$

where:

$$I(t,x) = \int_0^t \left(\frac{x(u)}{a}\right)^n du$$

Therefore, the *pdf* is:

$$f(t,x) = \beta s(t,x)(I(t,x))^{\beta-1} e^{-(I(t,x))^\beta}$$

Parameter estimation can be accomplished via maximum likelihood estimation methods, and confidence intervals can be approximated using the Fisher matrix approach. Once the parameters are determined, all other characteristics of interest can be obtained utilizing the statistical properties definitions (e.g., mean life, failure rate, etc.) presented in previous chapters. The log-likelihood equation is as follows:

$$\ln(L) = \Lambda = \sum_{i=1}^{Fe} N_i \ln[\beta s(T_i,x_i)(I(T_i,x_i))^{\beta-1}] - \sum_{i=1}^{Fe} N_i (I(T_i,x_i))^\beta - \sum_{i=1}^{S} N_i' \left(I(T_i',x_i')\right)^\beta + \sum_{i=1}^{FI} N_i'' \ln[R_{Li}'' - R_{Ri}'']$$

where:

$$\begin{aligned} R_{Li}''(T_{Li}'',x_i'') = &\quad e^{-(I(T_{Li}'',x_i''))^\beta} \\ R_{Ri}''(T_{Ri}'',x_i'') = &\quad e^{-(I(T_{Ri}'',x_i''))^\beta} \end{aligned}$$

and:

- F_e is the number of groups of exact times-to-failure data points.
- N_i is the number of times-to-failure in the i^{th} time-to-failure data group.
- T_i is the exact failure time of the i^{th} group.

- S is the number of groups of suspension data points.
- N_i' is the number of suspensions in the i^{th} group of suspension data points.
- T_i' is the running time of the i^{th} suspension data group.
- FI is the number of interval data groups.
- N_i'' is the number of intervals in the i^{th} group of data intervals.
- T_{Li}'' is the beginning of the i^{th} interval.
- T_{Ri}'' is the ending of the i^{th} interval.

Cumulative Damage-Power-Weibull Example

Using the simple step-stress data given here, one would define $x(t)$ as:

$$\begin{aligned} x(t) = &\ 2,\ 0 < t \le 250 \\ = &\ 3,\ 250 < t \le 350 \\ = &\ 4,\ 350 < t \le 370 \\ = &\ 5,\ 370 < t \le 380 \\ = &\ 6,\ 380 < t \le 390 \\ = &\ 7,\ 390 < t \le +\infty \end{aligned}$$

Assuming a power relation as the underlying life-stress relationship and the Weibull distribution as the underlying life distribution, one can then formulate the log-likelihood function for the above data set as,

$$\ln(L) = \Lambda = \sum_{i=1}^{F} \ln\left\{\beta\left[\frac{x(t)}{a}\right]^n \left[\int_0^{t_i} \left[\frac{[x(u)]}{a}\right]^n du\right]^{\beta-1}\right\} - \sum_{i=1}^{F} \left\{\left[\int_0^{t_i} \left[\frac{[x(u)]}{a}\right]^n du\right]^{\beta}\right\}$$

where:

- F is the number of exact time-to-failure data points.
- β is the Weibull shape parameter.

- a and n are the IPL parameters.
- $x(t)$ is the stress profile function.
- t_i is the i^{th} time to failure.

The parameter estimates for $\hat{\beta}$, $\hat{a}$ and $\hat{n}$ can be obtained by simultaneously solving, $\frac{\partial \Lambda}{\partial a}=0$ and $\frac{\partial \Lambda}{\partial n}=0$. Using Weibull++, the parameter estimates for this data set are:

$$\begin{aligned}\widehat{\beta} &= 2.67829\\ \widehat{\alpha} &= 11.72208\\ \widehat{n} &= 3.998466\end{aligned}$$

Once the parameters are obtained, one can now determine the reliability for these units at any time t and stress $x(t)$ from:

$$R\left(t,x\left(t\right)\right)=e^{-\left[\int_0^t\left[\frac{x(u)}{a}\right]^n du\right]^\beta}$$

or at a fixed stress level $x(t)=2\text{ V}$ and $t=300\text{ hours}$,

$$R\left(t=300,x(t)=2\right)=e^{-\left[\int_0^t\left[\frac{x(u)}{a}\right]^n du\right]^\beta}=97.5\%$$

The mean time to failure $(MTTF)$ at any stress $x(t)$ can be determined by:

$$MTTF\left(x\left(t\right)\right)=\int_0^\infty t\left[\left\{\beta\left[\frac{x\left(t\right)}{a}\right]^n\left[\int_0^t\left[\frac{x\left(u\right)}{a}\right]^n du\right]^{\beta-1}\right\}e^{-\left[\int_0^t\left[\frac{x(u)}{a}\right]^n du\right]^\beta}\right]dt$$

or at a fixed stress level $x\left(t\right)=2\text{ V}$,

$$MTTF\left(x\left(t\right)\right)=1046.3\text{ hours}$$

Any other metric of interest (e.g., failure rate, conditional reliability etc.) can also be determined using the basic definitions given in Appendix A and calculated automatically with Weibull++.

Cumulative Damage Power - Lognormal

Given a time-varying stress $x(t)$ and assuming the power law relationship, the median life is given by:

$$\frac{1}{\breve{T}(t,x)} = s(t,x) = \left(\frac{x(t)}{a}\right)^{n}$$

The reliability function of the unit under a single stress is given by:

$$R(t,x(t)) = 1 - \Phi(z)$$

where:

$$z(t,x) = \frac{\ln I(t,x)}{\sigma'_T}$$

and:

$$I(t,x) = \int_{0}^{t} \left(\frac{x(u)}{a}\right)^{n} du$$

Therefore, the *pdf* is:

$$f(t,x) = \frac{s(t,x)\varphi(z(t,x))}{\sigma'_T I(t,x)}$$

Parameter estimation can be accomplished via maximum likelihood estimation methods, and confidence intervals can be approximated using the Fisher matrix approach. Once the parameters are determined, all other characteristics of interest can be obtained utilizing the statistical properties definitions (e.g., mean life, failure rate, etc.) presented in previous chapters. The log-likelihood equation is as follows:

$$\ln(L) = \quad \Lambda = \sum_{i=1}^{Fe} N_i \ln[\frac{s(T_i,x_i)\varphi(z(T_i,x_i))}{\sigma'_T I(T_i,x_i)}] + \sum_{i=1}^{S} N'_i \ln(1 - \Phi(z(T'_i,x'_i))) + \sum_{i=1}^{FI} N''_i \ln[\Phi(z''_{Ri}) - \Phi(z''_{Li})]$$

where:

$$z''_{Ri} = \frac{\ln I(T''_{Ri}, x''_i)}{\sigma'_T}$$
$$z''_{Li} = \frac{\ln I(T''_{Li}, x''_i)}{\sigma'_T}$$

and:

- F_e is the number of groups of exact time-to-failure data points.
- N_i is the number of times-to-failure in the i^{th} time-to-failure data group.
- T_i is the exact failure time of the i^{th} group.
- S is the number of groups of suspension data points.
- N'_i is the number of suspensions in the i^{th} group of suspension data points.
- T'_i is the running time of the i^{th} suspension data group.
- FI is the number of interval data groups.
- N''_i is the number of intervals in the i^{th} group of data intervals.
- T''_{Li} is the beginning of the interval.
- T''_{Ri} is the ending of the i^{th} interval.

Cumulative Damage Arrhenius Relationship

This section presents a generalized formulation of the cumulative damage model where stress can be any function of time and the life-stress relationship is based on the Arrhenius life-stress relationship. Given a time-varying stress $x(t)$ and assuming the Arrhenius relationship, the life-stress relationship is given by:

$$L(x(t)) = Ce^{\frac{b}{x(t)}}$$

In Weibull++, the above relationship is actually presented in a format consistent with the general log-linear (GLL) relationship for the Arrhenius relationship:

$$L(x(t)) = e^{\alpha_0 + \alpha_1 \frac{1}{x(t)}}$$

Therefore, instead of displaying C and b as the calculated parameters, the following reparameterization is used:

$$\begin{aligned} \alpha_0 &= \ln(C) \\ \alpha_1 &= b \end{aligned}$$

Cumulative Damage Arrhenius - Exponential

Given a time-varying stress $x(t)$ and assuming the Arrhenius relationship, the mean life is:

$$\frac{1}{m(t,x)} = s(t,x) = \frac{e^{\frac{-b}{x(t)}}}{C}$$

The reliability function of the unit under a single stress is given by:

$$R(t, x(t)) = e^{-I(t,x)}$$

where:

$$I(t,x) = \int_0^t \frac{e^{\frac{-b}{x(u)}}}{C} du$$

Therefore, the *pdf* is:

$$f(t,x) = s(t,x) e^{-I(t,x)}$$

Parameter estimation can be accomplished via maximum likelihood estimation methods, and confidence intervals can be approximated using the Fisher matrix approach. Once the parameters are determined, all other characteristics of interest can be obtained utilizing the statistical properties definitions (e.g., mean life, failure rate, etc.) presented in previous chapters. The log-likelihood equation is as follows:

$$\ln(L) = \quad \Lambda = \sum_{i=1}^{Fe} N_i \ln[s(T_i, x_i)] - \sum_{i=1}^{Fe} N_i \left(I(T_i, x_i)\right) - \sum_{i=1}^{S} N_i' \left(I(T_i', x_i')\right) + \sum_{i=1}^{FI} N_i'' \ln[R_{Li}'' - R_{Ri}'']$$

where:

$$R''_{Li}(T''_{Li}, x''_i) = \qquad e^{-I(T''_{Li}, x''_i)}$$
$$R''_{Ri}(T''_{Ri}, x''_i) = \qquad e^{-I(T''_{Ri}, x''_i)}$$

and:

- F_e is the number of groups of exact time-to-failure data points.
- N_i is the number of times-to-failure in the i^{th} time-to-failure data group.
- T_i is the exact failure time of the i^{th} group.
- S is the number of groups of suspension data points.
- N'_i is the number of suspensions in the i^{th} group of suspension data points.
- T'_i is the running time of the i^{th} suspension data group.
- FI is the number of interval data groups.
- N''_i is the number of intervals in the i^{th} group of data intervals.
- T''_{Li} is the beginning of the i^{th} interval.
- T''_{Ri} is the ending of the i^{th} interval.

Cumulative Damage Arrhenius - Weibull

Given a time-varying stress $x(t)$ and assuming the Arrhenius relationship, the characteristic life is given by:

$$\frac{1}{\eta(t,x)} = s(t,x) = \frac{e^{\frac{-b}{x(t)}}}{C}$$

The reliability function of the unit under a single stress is given by:

$$R(t, x(t)) = e^{-(I(t,x))^{\beta}}$$

where:

$$I(t,x)=\int_{0}^{t}\frac{e^{\frac{-b}{x(u)}}}{C}du$$

Therefore, the *pdf* is:

$$f(t,x)=\beta s(t,x)(I(t,x))^{\beta-1}e^{-(I(t,x))^{\beta}}$$

Parameter estimation can be accomplished via maximum likelihood estimation methods, and confidence intervals can be approximated using the Fisher matrix approach. Once the parameters are determined, all other characteristics of interest can be obtained utilizing the statistical properties definitions (e.g., mean life, failure rate, etc.) presented in previous chapters. The log-likelihood equation is as follows:

$$\ln(L)= \qquad \Lambda=\sum_{i=1}^{Fe}N_i\ln[\beta s(T_i,x_i)(I(T_i,x_i))^{\beta-1}]-\sum_{i=1}^{Fe}N_i(I(T_i,x_i))^{\beta}-\sum_{i=1}^{S}N_i'\left(I(T_i',x_i')\right)^{\beta}+\sum_{i=1}^{FI}N_i''\ln[R_{Li}''-R_{Ri}'']$$

where:

$$R_{Li}''(T_{Li}'',x_i'')= \qquad e^{-(I(T_{Li}'',x_i''))^{\beta}}$$
$$R_{Ri}''(T_{Ri}'',x_i'')= \qquad e^{-(I(T_{Ri}'',x_i''))^{\beta}}$$

and:

- F_e is the number of groups of exact time-to-failure data points.
- N_i is the number of times-to-failure in the i^{th} time-to-failure data group.
- T_i is the exact failure time of the i^{th} group.
- S is the number of groups of suspension data points.
- N_i' is the number of suspensions in the i^{th} group of suspension data points.
- T_i' is the running time of the i^{th} suspension data group.
- FI is the number of interval data groups.
- N_i'' is the number of intervals in the i^{th} group of data intervals.

- T''_{Li} is the beginning of the i^{th} interval.
- T''_{Ri} is the ending of the i^{th} interval.

Cumulative Damage Arrhenius - Lognormal

Given a time-varying stress $x(t)$ and assuming the Arrhenius relationship, the median life is given by:

$$\frac{1}{\breve{T}(t,x)} = s(t,x) = \frac{e^{\frac{-b}{x(t)}}}{C}$$

The reliability function of the unit under a single stress is given by:

$$R(t,x(t)) = 1 - \Phi(z)$$

where:

$$z(t,x) = \frac{\ln I(t,x)}{\sigma'_T}$$

and:

$$I(t,\, x) = \int_0^t \frac{e^{\frac{-b}{x(u)}}}{C} du$$

Therefore, the *pdf* is:

$$f(t,x) = \frac{s(t,x)\varphi(z(t,x))}{\sigma'_T I(t,x)}$$

Parameter estimation can be accomplished via maximum likelihood estimation methods, and confidence intervals can be approximated using the Fisher matrix approach. Once the parameters are determined, all other characteristics of interest can be obtained utilizing the statistical properties definitions (e.g., mean life, failure rate, etc.) presented in previous chapters. The log-likelihood equation is as follows,

$$\ln(L) = \quad \Lambda = \sum_{i=1}^{Fe} N_i \ln[\frac{s(T_i, x_i)\varphi(z(T_i, x_i))}{\sigma_T' I(T_i, x_i)}] + \sum_{i=1}^{S} N_i' \ln\left(1 - \Phi(z(T_i', x_i'))\right) + \sum_{i=1}^{FI} N_i'' \ln[\Phi(z_{Ri}'') - \Phi(z_{Li}'')]$$

where:

$$z_{Ri}'' = \quad \frac{\ln I(T_{Ri}'', x_i'')}{\sigma_T'}$$
$$z_{Li}'' = \quad \frac{\ln I(T_{Li}'', x_i'')}{\sigma_T'}$$

and:

- F_e is the number of groups of exact times-to-failure data points.
- N_i is the number of times-to-failure in the i^{th} time-to-failure data group.
- T_i is the exact failure time of the i^{th} group.
- S is the number of groups of suspension data points.
- N_i' is the number of suspensions in the i^{th} group of suspension data points.
- T_i' is the running time of the i^{th} suspension data group.
- FI is the number of interval data groups.
- N_i'' is the number of intervals in the i^{th} group of data intervals.
- T_{Li}'' is the beginning of the i^{th} interval.
- T_{Ri}'' is the ending of the i^{th} interval.

Cumulative Damage Exponential Relationship

This section presents a generalized formulation of the cumulative damage model where stress can be any function of time and the life-stress relationship is based on the exponential relationship. Given a time-varying stress $x(t)$ and assuming the exponential relationship, the life-stress relationship is given by:

$$L(x(t)) = Ce^{bx(t)}$$

In Weibull++, the above relationship is actually presented in a format consistent with the general log-linear (GLL) relationship for the exponential relationship:

Therefore, instead of displaying C and b as the calculated parameters, the following reparameterization is used:

$$\begin{aligned} \alpha_0 &= \ln(C) \\ \alpha_1 &= b \end{aligned}$$

Cumulative Damage Exponential - Exponential

Given a time-varying stress $x(t)$ and assuming the exponential life-stress relationship, the mean life is given by:

$$\frac{1}{m(t,x)} = s(t,x) = \frac{e^{-bx(t)}}{C}$$

The reliability function of the unit under a single stress is given by:

$$R(t,x(t)) = e^{-I(t,x)}$$

where:

$$I(t,x) = \int_0^t \frac{e^{-bx(u)}}{C} du$$

Therefore, the *pdf* is:

$$f(t,x) = s(t,x)e^{-I(t,x)}$$

Parameter estimation can be accomplished via maximum likelihood estimation methods, and confidence intervals can be approximated using the Fisher matrix approach. Once the parameters are determined, all other characteristics of interest can be obtained utilizing the statistical properties definitions (e.g., mean life, failure rate, etc.) presented in previous chapters. The log-likelihood equation is as follows:

$$\ln(L) = \quad \Lambda = \sum_{i=1}^{Fe} N_i \ln[s(T_i, x_i)] - \sum_{i=1}^{Fe} N_i \left(I(T_i, x_i)\right) - \sum_{i=1}^{S} N'_i \left(I(T'_i, x'_i)\right) + \sum_{i=1}^{FI} N''_i \ln[R''_{Li} - R''_{Ri}]$$

where:

$$R''_{Li}(T''_{Li}, x''_i) = \quad e^{-I(T''_{Li}, x''_i)}$$
$$R''_{Ri}(T''_{Ri}, x''_i) = \quad e^{-I(T''_{Ri}, x''_i)}$$

and:

- F_e is the number of groups of exact time-to-failure data points.
- N_i is the number of times-to-failure in the i^{th} time-to-failure data group.
- T_i is the exact failure time of the i^{th} group.
- S is the number of groups of suspension data points.
- N'_i is the number of suspensions in the i^{th} group of suspension data points.
- T'_i is the running time of the i^{th} suspension data group.
- FI is the number of interval data groups.
- N''_i is the number of intervals in the i^{th} group of data intervals.
- T''_{Li} is the beginning of the i^{th} interval.
- T''_{Ri} is the ending of the i^{th} interval.

Cumulative Damage Exponential - Weibull

Given a time-varying stress $x(t)$ and assuming the exponential life-stress relationship, the characteristic life is given by:

$$\frac{1}{\eta(t, x)} = s(t, x) = \frac{e^{-b \cdot x(t)}}{C}$$

The reliability function of the unit under a single stress is given by:

$$R(t,x(t))=e^{-(I(t,x))^{\beta}}$$

where:

$$I(t,x)=\int_{0}^{t}\frac{e^{-bx(u)}}{C}du$$

Therefore, the *pdf* is:

$$f(t,x)=\beta s(t,x)(I(t,x))^{\beta-1}e^{-(I(t,x))^{\beta}}$$

Parameter estimation can be accomplished via maximum likelihood estimation methods, and confidence intervals can be approximated using the Fisher matrix approach. Once the parameters are determined, all other characteristics of interest can be obtained utilizing the statistical properties definitions (e.g., mean life, failure rate, etc.) presented in previous chapters. The log-likelihood equation is as follows:

$$\ln(L)=\quad \Lambda=\sum_{i=1}^{Fe}N_i\ln[\beta s(T_i,x_i)(I(T_i,x_i))^{\beta-1}]-\sum_{i=1}^{Fe}N_i(I(T_i,x_i))^{\beta}-\sum_{i=1}^{S}N_i'\left(I(T_i',x_i')\right)^{\beta}+\sum_{i=1}^{FI}N_i''\ln[R_{Li}''-R_{Ri}'']$$

where:

$$R_{Li}''(T_{Li}'',x_i'')=\quad e^{-(I(T_{Li}'',x_i''))^{\beta}}$$
$$R_{Ri}''(T_{Ri}'',x_i'')=\quad e^{-(I(T_{Ri}'',x_i''))^{\beta}}$$

and:

- F_e is the number of groups of exact time-to-failure data points.
- N_i is the number of times-to-failure in the i^{th} time-to-failure data group.
- T_i is the exact failure time of the i^{th} group.
- S is the number of groups of suspension data points.
- N_i' is the number of suspensions in the i^{th} group of suspension data points.
- T_i' is the running time of the i^{th} suspension data group.

- FI is the number of interval data groups.
- N_i'' is the number of intervals in the i^{th} group of data intervals.
- T_{Li}'' is the beginning of the i^{th} interval.
- T_{Ri}'' is the ending of the i^{th} interval.

Cumulative Damage Exponential - Lognormal

Given a time-varying stress $x(t)$ and assuming the exponential life-stress relationship, the median life is:

$$\frac{1}{\breve{T}(t,x)}=s(t,x)=\frac{e^{-bx(t)}}{C}$$

The reliability function of the unit under a single stress is given by:

$$R(t,x(t))=1-\Phi(z)$$

where:

$$z(t,x)=\frac{\ln I(t,x)}{\sigma_T'}$$

and:

$$I(t,x)=\int_0^t \frac{e^{-bx(u)}}{C}du$$

Therefore, the *pdf* is:

$$f(t,x)=\frac{s(t,x)\varphi(z(t,x))}{\sigma_T' I(t,x)}$$

Parameter estimation can be accomplished via maximum likelihood estimation methods, and confidence intervals can be approximated using the Fisher matrix approach. Once the parameters are determined, all other characteristics of interest can be obtained utilizing the statistical properties definitions (e.g., mean life, failure rate, etc.) presented in previous chapters. The log-likelihood equation is as follows:

$$\ln(L) = \quad \Lambda = \sum_{i=1}^{Fe} N_i \ln[\frac{s(T_i, x_i)\varphi(z(T_i, x_i))}{\sigma'_T I(T_i, x_i)}] + \sum_{i=1}^{S} N'_i \ln\big(1 - \Phi(z(T'_i, x'_i))\big) + \sum_{i=1}^{FI} N''_i \ln[\Phi(z''_{Ri}) - \Phi(z''_{Li})]$$

where:

$$z''_{Ri} = \quad \frac{\ln I(T''_{Ri}, x''_i)}{\sigma'_T}$$

$$z''_{Li} = \quad \frac{\ln I(T''_{Li}, x''_i)}{\sigma'_T}$$

and:

- F_e is the number of groups of exact times-to-failure data points.
- N_i is the number of times-to-failure in the i^{th} time-to-failure data group.
- T_i is the exact failure time of the i^{th} group.S
- S is the number of groups of suspension data points.
- N'_i is the number of suspensions in the i^{th} group of suspension data points.
- T'_i is the running time of the i^{th} suspension data group.
- FI is the number of interval data groups.
- N''_i is the number of intervals in the i^{th} group of data intervals.
- T''_{Li} is the beginning of the i^{th} interval.
- T''_{Ri} is the ending of the i^{th} interval.

Cumulative Damage General Loglinear Relationship

This section presents a generalized formulation of the cumulative damage model where multiple stress types are used in the analysis and where the stresses can be any function of time.

Cumulative Damage General Log-Linear - Exponential

Given n time-varying stresses $\underline{X} = (X_1(t), X_2(t) \ldots X_n(t))$, the life-stress relationship is:

$$\frac{1}{m(t, \bar{x})} = s(t, \bar{x}) = e^{-a_0 - \sum^n{}_{j=1} a_j x_j(t)}$$

where α_0 and α_j are model parameters. This relationship can be further modified through the use of transformations and can be reduced to the relationships discussed previously (power, Arrhenius and exponential), if so desired. The exponential reliability function of the unit under multiple stresses is given by:

$$R(t, \bar{x}) = e^{-I(t, \bar{x})}$$

where:

$$I(t, \bar{x}) = \int_0^t \frac{du}{e^{\alpha_0 + \sum_{j=1}{}^n \alpha_j x_j(t)}}$$

Therefore, the *pdf* is:

$$f(t, \bar{x}) = s(t, \bar{x}) e^{-I(t, \bar{x})}$$

Parameter estimation can be accomplished via maximum likelihood estimation methods, and confidence intervals can be approximated using the Fisher matrix approach. Once the parameters are determined, all other characteristics of interest can be obtained utilizing the statistical properties definitions (e.g., mean life, failure rate, etc.) presented in previous chapters. The log-likelihood equation is as follows:

$$\ln(L) = \quad \Lambda = \sum_{i=1}^{Fe} N_i \ln[s(T_i, \bar{x}_i)] - \sum_{i=1}^{Fe} N_i \left(I(T_i, \bar{x}_i)\right) - \sum_{i=1}^{S} N_i' \left(I(T_i', \bar{x}_i')\right) + \sum_{i=1}^{FI} N_i'' \ln[R_{Li}'' - R_{Ri}'']$$

where:

$$\begin{aligned} R_{Li}''(T_{Li}'', \bar{x}_i'') = & \quad e^{-I(T_{Li}'', \bar{x}_i'')} \\ R_{Ri}''(T_{Ri}'', \bar{x}_i'') = & \quad e^{-I(T_{Ri}'', \bar{x}_i'')} \end{aligned}$$

and:

- F_e is the number of groups of exact time-to-failure data points.
- N_i is the number of times-to-failure in the i^{th} time-to-failure data group.
- T_i is the exact failure time of the i^{th} group.
- S is the number of groups of suspension data points.
- N_i' is the number of suspensions in the i^{th} group of suspension data points.
- T_i' is the running time of the i^{th} suspension data group.
- FI is the number of interval data groups.
- N_i'' is the number of intervals in the i^{th} group of data intervals.
- T_{Li}'' is the beginning of the i^{th} interval.
- T_{Ri}'' is the ending of the i^{th} interval.

Cumulative Damage General Log-Linear - Weibull

Given n time-varying stresses $\underline{X} = (X_1(t), X_2(t)\ldots X_n(t))$, the life-stress relationship is given by:

$$\frac{1}{\eta(t,\bar{x})} = s(t,\bar{x}) = e^{-a_0 - \sum_{j=1}{}^{n} \alpha_j x_j(t)}$$

where α_j are model parameters.

The Weibull reliability function of the unit under multiple stresses is given by:

$$R(t,\bar{x}) = e^{-(I(t,\bar{x}))^{\beta}}$$

where:

$$I(t,\bar{x}) = \int_0^t \frac{du}{e^{a_0 + \sum^{n}{}_{j=1} \alpha_j x_j(u)}}$$

Therefore, the *pdf* is:

$$f(t,\bar{x}) = \beta s(t,\bar{x})(I(t,\bar{x}))^{\beta-1} e^{-(I(t,\bar{x}))^{\beta}}$$

Parameter estimation can be accomplished via maximum likelihood estimation methods, and confidence intervals can be approximated using the Fisher matrix approach. Once the parameters are determined, all other characteristics of interest can be obtained utilizing the statistical properties definitions (e.g., mean life, failure rate, etc.) presented in previous chapters. The log-likelihood equation is as follows:

$$\ln(L) = \quad \Lambda = \sum_{i=1}^{Fe} N_i \ln[\beta s(T_i,\bar{x}_i)(I(T_i,\bar{x}_i))^{\beta-1}] - \sum_{i=1}^{Fe} N_i (I(T_i,\bar{x}_i))^{\beta} - \sum_{i=1}^{S} N_i' \left(I(T_i',\bar{x}_i')\right)^{\beta} + \sum_{i=1}^{FI} N_i'' \ln[R_{Li}'' - R_{Ri}'']$$

where:

$$R_{Li}''(T_{Li}'',\bar{x}_i'') = \quad e^{-(I(T_{Li}'',\bar{x}_i''))^{\beta}}$$
$$R_{Ri}''(T_{Ri}'',\bar{x}_i'') = \quad e^{-(I(T_{Ri}'',\bar{x}_i''))^{\beta}}$$

and:

- F_e is the number of groups of exact time-to-failure data points.
- N_i is the number of times-to-failure in the i^{th} time-to-failure data group.
- T_i is the exact failure time of the i^{th} group.
- S is the number of groups of suspension data points.
- N_i' is the number of suspensions in the i^{th} group of suspension data points.
- T_i' is the running time of the i^{th} suspension data group.
- FI is the number of interval data groups.
- N_i'' is the number of intervals in the i^{th} group of data intervals.
- T_{Li}'' is the beginning of the i^{th} interval.
- T_{Ri}'' is the ending of the i^{th} interval.

Cumulative Damage General Log-Linear - Lognormal

Given n time-varying stresses $\underline{X} = (X_1(t), X_2(t)\ldots X_n(t))$, the life-stress relationship is given by:

$$\frac{1}{\breve{T}(t,\bar{x})} = s(t,\bar{x}\,) = e^{-a_0 - \sum_{j=1}{}^{n} \alpha_j x_j(t)}$$

where α_j are model parameters.

The lognormal reliability function of the unit under multiple stresses is given by:

$$R(t,\bar{x}) = 1 - \Phi(z(t,\bar{x}))$$

where:

$$z(t,\bar{x}) = \frac{\ln I(t,\bar{x})}{\sigma'_T}$$

and:

$$I(t,\bar{x}) = \int_0^t \frac{du}{e^{\alpha_0 + \sum^n{}_{j=1} \alpha_j x_j(u)}}$$

Therefore, the *pdf* is:

$$f(t,\bar{x}) = \frac{s(t,\bar{x})\varphi(z(t,\bar{x}))}{\sigma'_T I(t,\bar{x})}$$

Parameter estimation can be accomplished via maximum likelihood estimation methods, and confidence intervals can be approximated using the Fisher matrix approach. Once the parameters are determined, all other characteristics of interest can be obtained utilizing the statistical properties definitions (e.g., mean life, failure rate, etc.) presented in previous chapters. The log-likelihood equation is as follows:

$$\ln(L) = \quad \Lambda = \sum_{i=1}^{Fe} N_i \ln[\frac{s(T_i,\bar{x}_i)\varphi(z(T_i,\bar{x}_i))}{\sigma'_T I(T_i,\bar{x}_i)}] + \sum_{i=1}^{S} N'_i \ln\big(1 - \Phi(z(T'_i,\bar{x}'_i))\big) + \sum_{i=1}^{FI} N''_i \ln[\Phi(z''_{Ri}) - \Phi(z''_{Li})]$$

where:

$$z''_{Ri} = \frac{\ln I(T''_{Ri}, \bar{x}''_i)}{\sigma'_T}$$

$$z''_{Li} = \frac{\ln I(T''_{Li}, \bar{x}''_i)}{\sigma'_T}$$

and:

- F_e is the number of groups of exact time-to-failure data points.
- N_i is the number of times-to-failure in the i^{th} time-to-failure data group.
- T_i is the exact failure time of the i^{th} group.
- S is the number of groups of suspension data points.
- N'_i is the number of suspensions in the i^{th} group of suspension data points.
- T'_i is the running time of the i^{th} suspension data group.
- FI is the number of interval data groups.
- N''_i is the number of intervals in the i^{th} group of data intervals.
- T''_{Li} is the beginning of the i^{th} interval.
- T''_{Ri} is the ending of the i^{th} interval.

Additional Tools

IN THIS CHAPTER

Additional Tools

Common Shape Parameter Likelihood Ratio Test

In order to assess the assumption of a common shape parameter among the data obtained at various stress levels, the likelihood ratio (LR) test can be utilized, as described in Nelson [28]. This test applies to any distribution with a shape parameter. In the case of Weibull++, it applies to the Weibull and lognormal distributions. When Weibull is used as the underlying life distribution, the shape parameter, $\beta,$ is assumed to be constant across the different stress levels (i.e., stress independent). Similarly, $\sigma_{T'}$, the parameter of the lognormal distribution is assumed to be constant across the different stress levels.

The likelihood ratio test is performed by first obtaining the LR test statistic, T . If the true shape parameters are equal, then the distribution of T is approximately chi-square with $n-1$ degrees of freedom, where n is the number of test stress levels with two or more exact failure points. The LR test statistic, T , is calculated as follows:

$$T = 2(\hat{\Lambda}_1 + \ldots + \hat{\Lambda}_n - \hat{\Lambda}_0)$$

where $\hat{\Lambda}_1, \ldots, \hat{\Lambda}_n$ are the likelihood values obtained by fitting a separate distribution to the data from each of the n test stress levels (with two or more exact failure times). The likelihood value, $\hat{\Lambda}_0$, is obtained by fitting a model with a common shape parameter and a separate scale parameter for each of the n stress levels, using indicator variables.

Once the LR statistic has been calculated, then:

- If $T \leq \chi^2(1-\alpha; n-1)$, the n shape parameter estimates do not differ statistically significantly at the 100 $\alpha\%$ level.
- If $T > \chi^2(1-\alpha; n-1)$, the n shape parameter estimates differ statistically significantly at the 100 $\alpha\%$ level.

$\chi^2(1-\alpha; n-1)$ is the 100(1- α) percentile of the chi-square distribution with $n-1$ degrees of freedom.

Example: Likelihood Ratio Test Example

Consider the following times-to-failure data at three different stress levels.

Stress	406 K	416 K	426 K
Time Failed (hrs)	248	164	92
	456	176	105
	528	289	155
	731	319	184
	813	340	219
		543	235

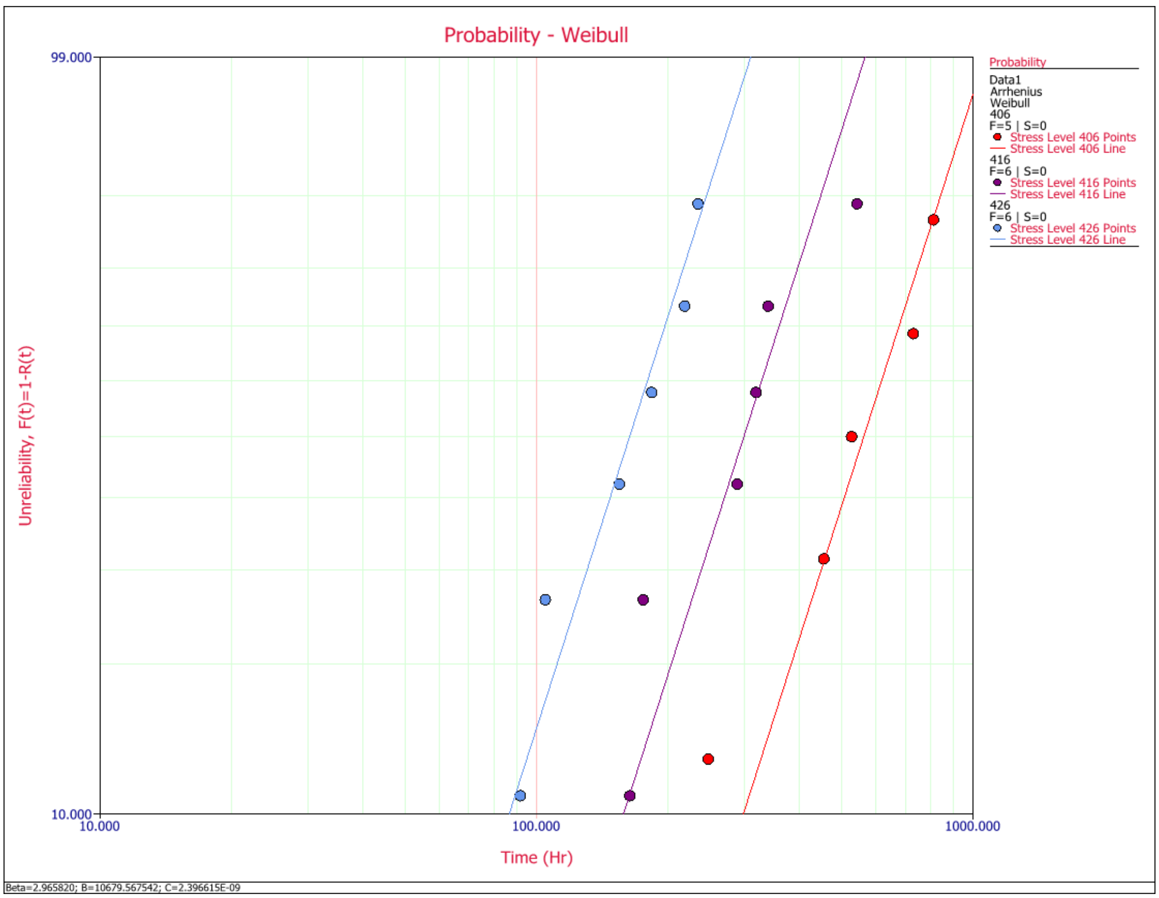
The data set was analyzed using an Arrhenius-Weibull model. The analysis yields:

$$\widehat{\beta} = 2.965820$$

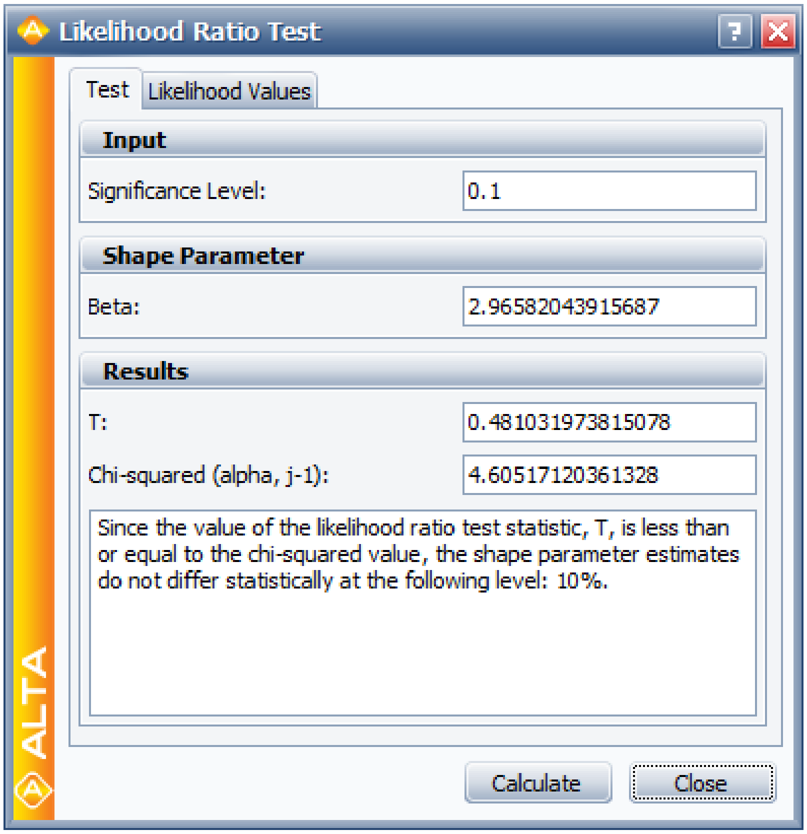
$$\widehat{B} = 10,679.567542$$

$$\widehat{C} = 2.396615 \cdot 10^{-9}$$

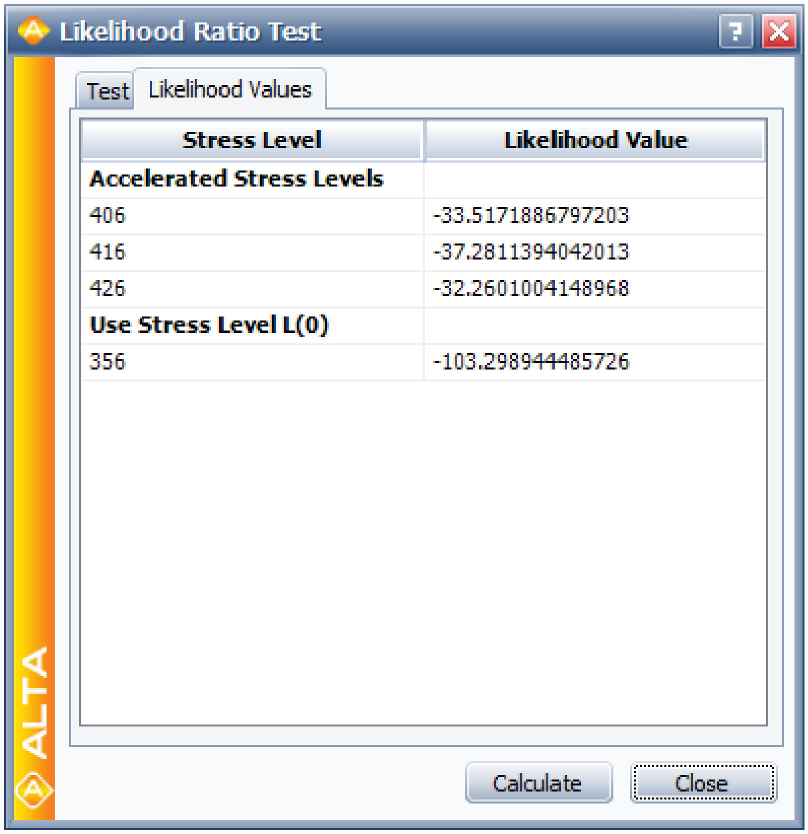
The assumption of a common β across the different stress levels can be visually assessed by using a probability plot. As you can see in the following plot, the plotted data from the different stress levels seem to be fairly parallel.



A better assessment can be made with the LR test, which can be performed using the Likelihood Ratio Test tool in Weibull++. For example, in the following figure, the βs are compared for equality at the 10% level.



The LR test statistic, T, is calculated to be 0.481. Therefore, $T = 0.481 \leq 4.605 = \chi^2(0.9; 2)$, the β' s do not differ significantly at the 10% level. The individual likelihood values for each of the test stresses are shown next.



Tests of Comparison

It is often desirable to be able to compare two sets of accelerated life data in order to determine which of the data sets has a more favorable life distribution. The units from which the data are obtained could either be from two alternate designs, alternate manufacturers or alternate lots or assembly lines. Many methods are available in statistical literature for doing this when the units come from a complete sample, (i.e., a sample with no censoring). This process becomes a little more difficult when dealing with data sets that have censoring, or when trying to compare two data sets that have different distributions. In general, the problem boils down to that of being able to determine any statistically significant difference between the two samples of potentially censored data from two possibly different populations. This section discusses some of the methods that are applicable to censored data, and are available in Weibull++.

Simple Plotting

One popular graphical method for making this determination involves plotting the data at a given stress level with confidence bounds and seeing whether the bounds overlap or separate at the point of interest. This can be effective for comparisons at a given point in time or a given reliability level, but it is difficult to assess the overall behavior of the two distributions, as the confidence bounds may overlap at some points and be far apart at others. This can be easily done using the overlay plot feature in Weibull++.

Using the Life Comparison Wizard

Another methodology, suggested by Gerald G. Brown and Herbert C. Rutemiller, is to estimate the probability of whether the times-to-failure of one population are better or worse than the times-to-failure of the second. The equation used to estimate this probability is given by:

$$P\left[t_2 \geq t_1\right] = \int_0^{\infty} f_1(t) \cdot R_2(t) \cdot dt$$

where $f_1(t)$ is the *pdf* of the first distribution and $R_2(t)$ is the reliability function of the second distribution. The evaluation of the superior data set is based on whether this probability is smaller or greater than 0.50. If the probability is equal to 0.50, then is equivalent to saying that the two distributions are identical.

For example, consider two alternate designs, where X and Y represent the life test data from each design. If we simply wanted to choose the component with the higher reliability, we could simply select the component with the higher reliability at time t. However, if we wanted to design the product to be as long-lived as possible, we would want to calculate the probability that the entire distribution of one design is better than the other. The statement "the probability that X is greater than or equal to Y" can be interpreted as follows:

- If $P = 0.50$, then the statement is equivalent to saying that both X and Y are equal.
- If $P < 0.50$ or, for example, $P = 0.10$, then the statement is equivalent to saying that $P = 1 - 0.10 = 0.90$, or Y is better than X with a 90% probability.

Weibull++'s Comparison Wizard allows you to perform such calculations. The comparison is performed at the given use stress levels of each data set, using the equation:

$$P\left[t_2 \geq t_1\right] = \int_0^{\infty} f_1(t, V_{Use,1}) \cdot R_2(t, V_{Use,2}) \cdot dt$$

Degradation Analysis

Given that products are frequently being designed with higher reliabilities and developed in shorter amounts of time, even accelerated life testing is often not sufficient to yield reliability results in the desired timeframe. In some cases, it is possible to infer the reliability behavior of unfailed test samples with only the accumulated test time information and assumptions about the distribution. However, this generally leads to a great deal of uncertainty in the results. Another option in this situation is the use of degradation analysis. Degradation analysis involves the measurement and extrapolation of degradation or performance data that can be directly related to the presumed failure of the product in question. Many failure mechanisms can be directly linked to the degradation of part of the product, and degradation analysis allows the user to extrapolate to an assumed failure time based on the measurements of degradation or performance over time. To reduce testing time even further, tests can be performed at elevated stresses and the degradation at these elevated stresses can be measured resulting in a type of analysis known as accelerated degradation. In some cases, it is possible to directly measure the degradation over time, as with the wear of brake pads or with the propagation of crack size. In other cases, direct measurement of degradation might not be possible without invasive or destructive measurement techniques that would directly affect the subsequent performance of the product. In such cases, the degradation of the product can be estimated through the measurement of certain performance characteristics, such as using resistance to gauge the degradation of a dielectric material. In either case, however, it is necessary to be able to define a level of degradation or performance at which a failure is said to have occurred. With this failure level of performance defined, it is a relatively simple matter to use basic mathematical models to extrapolate the performance measurements over time to the point where the failure is said to occur. This is done at different stress levels, and therefore each time-to-failure is also associated with a corresponding stress level. Once the times-to-failure at the corresponding stress levels have been determined, it is merely a matter of analyzing the extrapolated failure times in the same manner as you would conventional accelerated time-to-failure data.

Once the level of failure (or the degradation level that would constitute a failure) is defined, the degradation for multiple units over time needs to be measured (with different groups of units being at different stress levels). As with conventional accelerated data, the amount of certainty in the results is directly related to the number of units being tested, the number of units at each stress level, as well as in the amount of overstressing with respect to the normal operating conditions. The performance or degradation of these units needs to be measured over time, either continuously or at predetermined intervals. Once this information has been recorded, the next task is to extrapolate the performance measurements to the defined failure level in order to estimate the failure time. Weibull++ allows the user to perform such analysis using a linear, exponential, power, logarithmic, Gompertz or Lloyd-Lipow model to perform this extrapolation. These models have the following forms:

$$\begin{aligned} Linear &: \quad y = a \cdot x + b \\ Exponential &: \quad y = b \cdot e^{a \cdot x} \\ Power &: \quad y = b \cdot x^{a} \\ Logarithmic &: \quad y = a \cdot ln(x) + b \\ Gompertz &: \quad y = a \cdot b^{cx} \\ Lloyd - Lipow &: \quad y = a - b/x \end{aligned}$$

where y represents the performance, x represents time, and a and b are model parameters to be solved for. Once the model parameters a_i and b_i (and c_i for Lloyd-Lipow) are estimated for each sample i , a time, x_i, can be extrapolated that corresponds to the defined level of failure y . The computed x_i can now be used as our times-to-failure for subsequent accelerated life data analysis. As with any sort of extrapolation, one must be careful not to extrapolate too far beyond the actual range of data in order to avoid large inaccuracies (modeling errors).

One may also define a censoring time past which no failure times are extrapolated. In practice, there is usually a rather narrow band in which this censoring time has any practical meaning. With a relatively low censoring time, no failure times will be extrapolated, which defeats the purpose of degradation analysis. A relatively high censoring time would occur after all of the theoretical failure times, thus being rendered meaningless. Nevertheless, certain situations may arise in which it is beneficial to be able to censor the accelerated degradation data.

Accelerated Life Test Plans

Poor accelerated test plans waste time, effort and money and may not even yield the desired information. Before starting an accelerated test (which is sometimes an expensive and difficult endeavor), it is advisable to have a plan that helps in accurately estimating reliability at operating conditions while minimizing test time and costs. A test plan should be used to decide on the appropriate stress levels that should be used (for each stress type) and the amount of the test units that need to be allocated to the different stress levels (for each combination of the different stress types' levels). This section presents some common test plans for one-stress and two-stress accelerated tests.

General Assumptions

Most accelerated life testing plans use the following model and testing assumptions that correspond to many practical quantitative accelerated life testing problems.

1. The log-time-to-failure for each unit follows a location-scale distribution such that:

$$\Pr\ (Y \leq y) = \Phi\left(\frac{y-\mu}{\sigma}\right)$$

where μ and σ are the location and scale parameters respectively and $\Phi\ (\cdot)$ is the standard form of the location-scale distribution.

2. Failure times for all test units, at all stress levels, are statistically independent.

3. The location parameter μ is a linear function of stress. Specifically, it is assumed that:

$$\mu = \mu(z_1) = \gamma_0 + \gamma_1 x$$

4. The scale parameter, $\sigma,$ does not depend on the stress levels. All units are tested until a pre-specified test time.

5. Two of the most common models used in quantitative accelerated life testing are the linear Weibull and lognormal models. The Weibull model is given by:

$$Y \sim SEV\left[\mu(z) = \gamma_0 + \gamma_1 x, \sigma\right]$$

where SEV denotes the smallest extreme value distribution. The lognormal model is given by:

$$Y \sim Normal\left[\mu(z) = \gamma_0 + \gamma_1 z, \sigma\right]$$

That is, log life Y is assumed to have either an SEV or a normal distribution with location parameter $\mu(z)$, expressed as a linear function of z and constant scale parameter σ .

Planning Criteria and Problem Formulation

Without loss of generality, a stress can be standardized as follows:

$$\xi = \frac{x - x_D}{x_H - x_D}$$

where:

- x_D is the use stress or design stress at which product life is of primary interest.
- x_H is the highest test stress level.

The values of x , x_D and x_H refer to the actual values of stress or to the transformed values in case a transformation (e.g., the reciprocal transformation to obtain the Arrhenius relationship or the log transformation to obtain the power relationship) is used.

Typically, there will be a limit on the highest level of stress for testing because the distribution and life-stress relationship assumptions hold only for a limited range of the stress. The highest test level of stress, x_H, can be determined based on engineering knowledge, preliminary tests or experience with similar products. Higher stresses will help end the test faster, but might violate your distribution and life-stress relationship assumptions.

Therefore, $\xi = 0$ at the design stress and $\xi = 1$ at the highest test stress.

A common purpose of an accelerated life test experiment is to estimate a particular percentile (unreliability value of p), T_p , in the lower tail of the failure distribution at use stress. Thus a natural criterion is to minimize $Var(\hat{T}_p)$ or $Var(\hat{Y}_p)$ such that $Y_p = \ln(T_p)$. $Var(\hat{Y}_p)$ measures the precision of the p quantile estimator; smaller values mean less variation in the value of $\hat{Y}_p$ in repeated samplings. Hence a good test plan should yield a relatively small, if not the minimum, $Var(\hat{Y}_p)$ value. For the minimization problem, the decision variables are ξ_i (the standardized stress level used in the test) and π_i (the percentage of the total test units allocated at that level). The optimization problem can be formulized as follows.

Minimize:

$$Var(\hat{Y}_p) = f(\xi_i, \pi_i)$$

Subject to:

$$0 \leq \pi_i \leq 1,\ i = 1, 2, \ldots n$$

where:

$$\sum_{i=1}^{n} \pi_i = 1$$

An optimum accelerated test plan requires algorithms to minimize $Var(\hat{Y}_p)$.

Planning tests may involve compromise between efficiency and extrapolation. More failures correspond to better estimation efficiency, requiring higher stress levels but more extrapolation to the use condition. Choosing the best plan to consider must take into account the trade-offs between efficiency and extrapolation. Test plans with more stress levels are more robust than plans with fewer stress levels because they rely less on the validity of the life-stress relationship assumption. However, test plans with fewer stress levels can be more convenient.

Test Plans for a Single Stress Type

This section presents a discussion of some of the most popular test plans used when only one stress factor is applied in the test. In order to design a test, the following information needs to be determined beforehand:

1. The design stress, x_D, and the highest test stress, x_H .

2. The test duration (or censoring time), Υ .

3. The probability of failure at x_D $(\xi = 0)$ by Υ , denoted as P_D, and at x_H $(\xi = 1)$ by Υ , denoted as P_H .

Two Level Statistically Optimum Plan

The Two Level Statistically Optimum Plan is the most important plan, as almost all other plans are derived from it. For this plan, the highest stress, x_H , and the design stress, x_D , are pre-determined. The test is conducted at two levels. The high test level is fixed at x_H . The low test stress, x_L , together with the proportion of the test units allocated to the low level, π_L , are calculated such that $Var(\hat{Y}_p)$ is minimized. Meeker and Hahn [36] present more details about this test plan.

Three Level Best Standard Plan

In this plan, three stress levels are used. Let us use ξ_L, ξ_M and ξ_H to denote the three standardized stress levels from lowest to highest with:

$$\xi_M = \frac{\xi_L + \xi_H}{2} = \frac{\xi_L + 1}{2}$$

An equal number of units is tested at each level, $\pi_L = \pi_M = \pi_H = 1/3$. Therefore, the test plan is $(\xi_L, \xi_M, \xi_H, \pi_L, \pi_M, \pi_H) = (\xi_L, \frac{\xi_L+1}{2}, 1, 1/3, 1/3, 1/3)$ with ξ_L being the only decision variable. ξ_L is determined such that $Var(\hat{Y}_p)$ is minimized. Escobar and Meeker [37] present more details about this test plan.

Three Level Best Compromise Plan

In this plan, three stress levels are used $(\xi_L, \frac{\xi_L+1}{2}, 1)$. π_M, which is a value between 0 and 1, is pre-determined. $\pi_M = 0.1$ and $\pi_M = 0.2$ are commonly used; values less than or equal to 0.2 can give good results. The test plan is $(\xi_L, \xi_M, \xi_H, \pi_L, \pi_M, \pi_H) = (\xi_L, \frac{\xi_L+1}{2}, 1, \pi_L, \pi_M, 1 - \pi_L - \pi_M)$ with ξ_L and π_L being the decision variables determined such that $Var(\hat{Y}_p)$ is minimized. Meeker [38] presents more details about this test plan.

Three Level Best Equal Expected Number Failing Plan

In this plan, three stress levels are used $(\xi_L, \frac{\xi_L+1}{2}, 1)$ and there is a constraint that an equal number of failures at each stress level is expected. The constraint can be written as:

$$\pi_L P_L = \pi_M P_M = \pi_H P_H$$

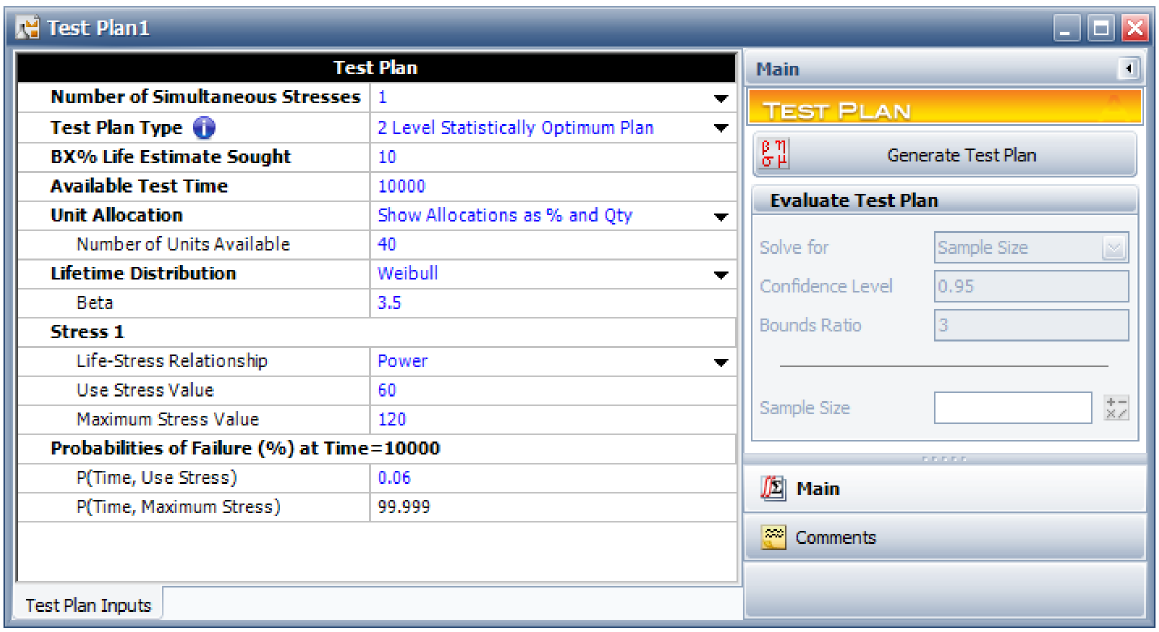
where P_L, P_M and P_H are the failure probability at the low, middle and high test level, respectively. P_L and P_M can be expressed in terms of ξ_L and ξ_M. Therefore, all variables can be expressed in terms of ξ_L, which is chosen such that $Var(\hat{Y}_p)$ is minimized. Meeker [38] presents more details about this test plan.

Three Level 4:2:1 Allocation Plan

In this plan, three stress levels are used $(\xi_L, \frac{\xi_L+1}{2}, 1)$. The allocation of units at each level is pre-given as $\pi_L : \pi_M : \pi_H = 4 : 2 : 1$. Therefore $\pi_L = 4/7, \pi_M = 2/7$ and $\pi_H = 1/7$. ξ_L is the only decision variable that is chosen such that $Var(\hat{Y}_p)$ is minimized. The optimum ξ_L can also be multiplied by a constant k (defined by the user) to make the low stress level closer to the use stress than to the optimized plan, in order to make a better extrapolation at the use stress. Meeker and Hahn [40] present more details about this test plan.

Example of a Single Stress Test Plan

A reliability engineer is planning an accelerated test for a mechanical component. Torque is the only factor in the test. The purpose of the experiment is to estimate the B10 life (time equivalent to unreliability = 0.1) of the diodes. The reliability engineer wants to use a 2 Level Statistically Optimum Plan because it would require fewer test chambers than a 3 level test plan. 40 units are available for the test. The mechanical component is assumed to follow a Weibull distribution with beta = 3.5, and a power model is assumed for the life-stress relationship. The test is planned to last for 10,000 cycles. The engineer has estimated that there is a 0.06% probability that a unit will fail by 10,000 cycles at the use stress level of 60 N • m. The highest level allowed in the test is 120 N • m and a unit is estimated to fail with a probability of 99.999% at 120 N • m. The following setup shows the test plan in Weibull++.



The Two Level Statistically Optimum Plan is shown next.

Test Plan Inputs				
Number of Simultaneous Stresses	1			
Test Plan Type	2 Level Statistically Optimum Plan			
BX% Life Estimate Sought	10			
Available Test Time	10000			
Number of Units Available	40			
Lifetime Distribution	Weibull			
Beta	3.5			
Stress1				
Life-Stress Relationship	Power			
Use Stress Value	60			
Maximum Stress Value	120			
Probabilities of Failure (%) at Time=10000				
P(Time, Use Stress)	0.06			
P(Time, Maximum Stress)	99.999			
Recommended Test Plan				
Stress Level	**Stress Value**	**Unit Allocation (%)**	**Unit Allocation (Qty)**	**Probability of Failure**
Low Stress Level	95.39823	70.6	28.24	0.356023
High Stress Level	120	29.4	11.76	0.99999
BX% Life Estimate				
Time at Which Unreliability (Tp)=10%	43778.01748			
Standard Deviation of Tp	14379.70074			

The Two Level Statistically Optimum Plan is to test 28.24 units at 95.39 N • m and 11.76 units at 120 N • m. The variance of the test at B10 is

$$Var(T_p = B10) = StdDev(T_p = B10)^2 = 14380^2 .$$

Test Plan Evaluation

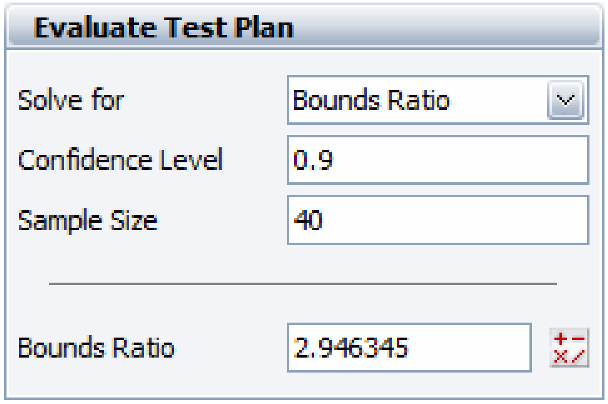
In addition to assessing $Var(\hat{T}_p)$, the test plan can also be evaluated based on three different criteria: confidence level, bounds ratio or sample size. These criteria can be assessed before conducting the recommended test to decide whether the test plan is satisfactory or whether some modifications would be beneficial. We can solve for any one of three criteria, given the two other criteria.

The bounds ratio is defined as follows:

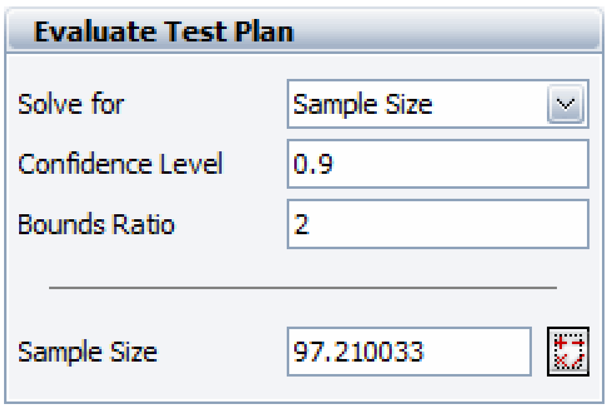
$$\text{Bounds Ratio} = \frac{\text{Two Sided Upper Bound on } T_p}{\text{Two Sided Lower Bound on } T_p}$$

This ratio is analogous to the ratio that can be calculated if a test is conducted and life data are obtained and used to calculate the ratio of the confidence bounds based on the results.

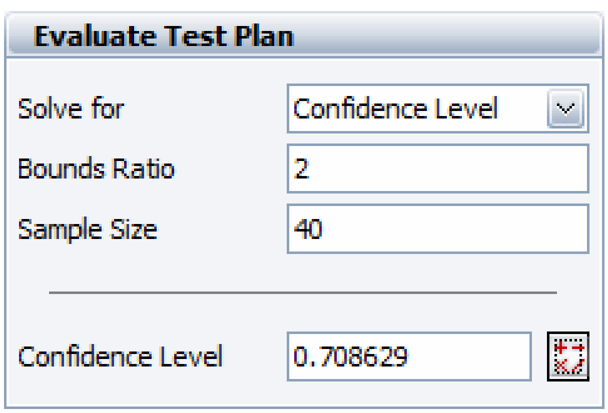
For this example, assume that a 90% confidence is desired and 40 units are to be used in the test. The bounds ratio is calculated as 2.946345, as shown next.



If this calculated bounds ratio is unsatisfactory, we can calculate the required number of units that would meet a certain bounds ratio criterion. For example, if a bounds ratio of 2 is desired, the required sample size is calculated as 97.210033, as shown next.



If the sample size is kept at 40 units and a bounds ratio of 2 is desired, the equivalent confidence level we have in the test drops to 70.8629%, as shown next.



Test Plans for Two Stress Types

This section presents a discussion of some of the most popular test plans used when two stress factors are applied in the test and interactions are assumed not to exists between the factors. The location parameter μ can be expressed as function of stresses x_1 and x_2 or as a function of their normalized stress levels as follows:

$$\mu = \gamma_0 + \gamma_1 \xi_1 + \gamma_2 \xi_2$$

In order to design a test, the following information needs to be determined beforehand:

1. The stress limits (the design stress, x_D, and the highest test stress, x_H) of each stress type.

2. The test time (or censoring time), Υ . .

3. The probability of failure at Υ at three stress combinations. Usually P_{DD} , P_{HD} and P_{DH} are used (P indicates probability and the subscript D represents the design stress, while H represents the highest stress level in the test).

For two-stress test planning, two methods are available: the Three Level Optimum Plan and the Five Level Best Compromise Plan.

Three Level Optimum Plan

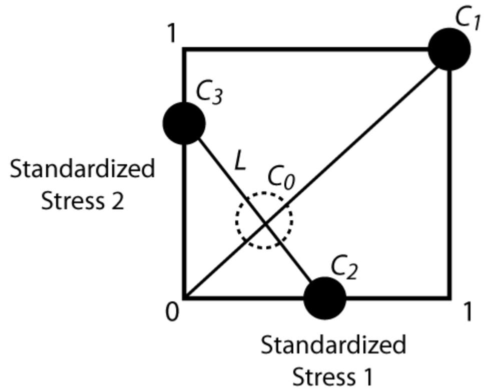
The Three Level Optimum Plan is obtained by first finding a one-stress degenerate Two Level Statistically Optimum Plan and splitting this degenerate plan into an appropriate two-stress plan. In a degenerate test plan, the test is conducted at any two (or more) stress level combinations on a line with slope s that passes through the design $\xi_D = (\xi_{1D}, \xi_{2D})$. Therefore, in the case of a degenerate design, we have:

$$\mu = \gamma_0 + (\gamma_1 + \gamma_2 s)\, \xi_1$$

Degenerate plans help reducing the two-stress problem into a one-stress problem. Although these degenerate plans do not allow the estimation of all the model parameters and would be an unlikely choice in practice, they are used as a starting point for developing more reasonable optimum and compromise test plans. After finding the one stress degenerate Two Level Statistically Optimum Plan using the methodology explained in 13.4.3.1, the plan is split into an appropriate Three Level Optimum Plan.

The next figure illustrates the concept of the Three Level Optimum Plan for a two-stress test. ξ_D is the (0,0) point. C_O and C_1 are the one-stress degenerate Two Level Statistically

Optimum Plan. C_1 , which corresponds to ($\xi_1 = 1, \xi_2 = 1$), is always used for this type of test and is the high stress level of the degenerate plan. C_O corresponds to the low stress level of the degenerate plan. A line, L , is drawn passing through C_O such that all the points along the line have the same probability of failures at the end of the test with the stress levels of the C_O plan. C_2 and C_3 are then determined by obtaining the intersections of L with the boundaries of the square.

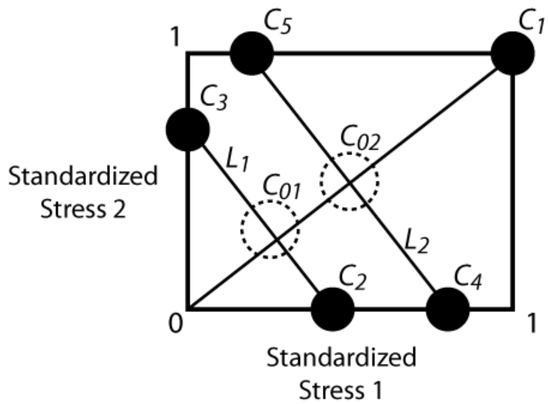


C_1 , C_2 and C_3 represent the the Three Level Optimum Plan. Readers are encouraged to review Escobar and Meeker [37] for more details about this test plan.

Five Level Best Compromise Plan

The Five Level Best Compromise Plan is obtained by first finding a degenerate one-stress Three Level Best Compromise Plan, using the methodology explained in the Three Level Best Compromise Plan (with $\pi_M = 0.2$) , and splitting this degenerate plan into an appropriate two-stress plan.

In the next figure, ξ_D is the (0,0) point. C_{O1}, C_{O2} and C_1 are the degenerate one-stress Three Level Best Compromise Plan. Points along the L_1 line have the same probability of failure at the end of the C_{O1} test plan, while points on L_2 have the same probability of failure at the end of the C_{O2} test plan. C_2, C_3 , C_4 and C_5 are then determined by obtaining the intersections of L_1 and L_2 with the boundaries of the square.



C_1 , C_2, C_3 , C_4 and C_5 represent the the Five Level Best Compromise Plan. Readers are encouraged to review Escobar and Meeker [37] for more details about this test plan.

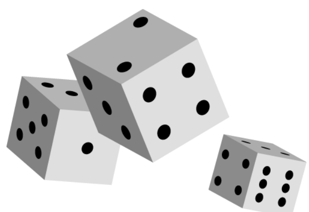
Appendices

IN THIS CHAPTER

Appendix A: Brief Statistical Background

In this appendix, we attempt to provide a brief elementary introduction to the most common and fundamental statistical equations and definitions used in reliability engineering and life data analysis. The equations and concepts presented in this appendix are used extensively throughout this reference.

Random Variables



In general, most problems in reliability engineering deal with quantitative measures, such as the time-to-failure of a component, or qualitative measures, such as whether a component is defective or non-defective. We can then use a random variable X to denote these possible measures.

In the case of times-to-failure, our random variable X is the time-to-failure of the component and can take on an infinite number of possible values in a range from 0 to infinity (since we do not know the exact time *a priori*). Our component can be found failed at any time after time 0 (e.g., at 12 hours or at 100 hours and so forth), thus X can take on any value in this range. In this case, our random variable X is said to be a *continuous random variable*. In this reference, we will deal almost exclusively with continuous random variables.

In judging a component to be defective or non-defective, only two outcomes are possible. That is, X is a random variable that can take on one of only two values (let's say defective = 0 and non-defective = 1). In this case, the variable is said to be a discrete random variable.

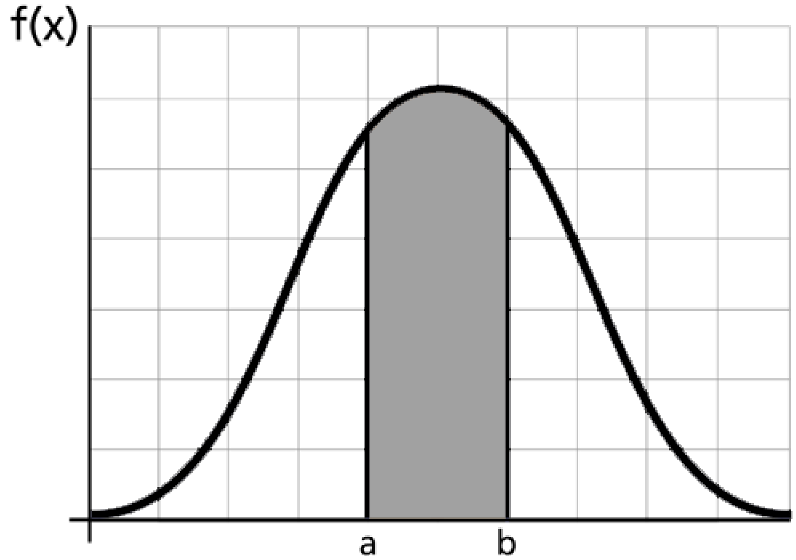
The Probability Density Function and the Cumulative Distribution Function

The probability density function (*pdf*) and cumulative distribution function (*cdf*) are two of the most important statistical functions in reliability and are very closely related. When these functions are known, almost any other reliability measure of interest can be derived or obtained. We will now take a closer look at these functions and how they relate to other reliability measures, such as the reliability function and failure rate.

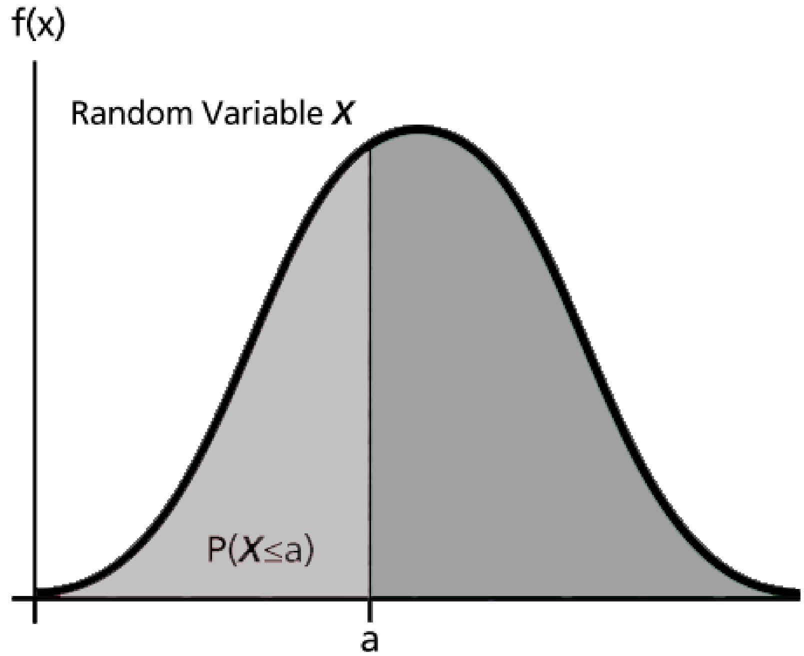
From probability and statistics, given a continuous random variable $X,$ we denote:

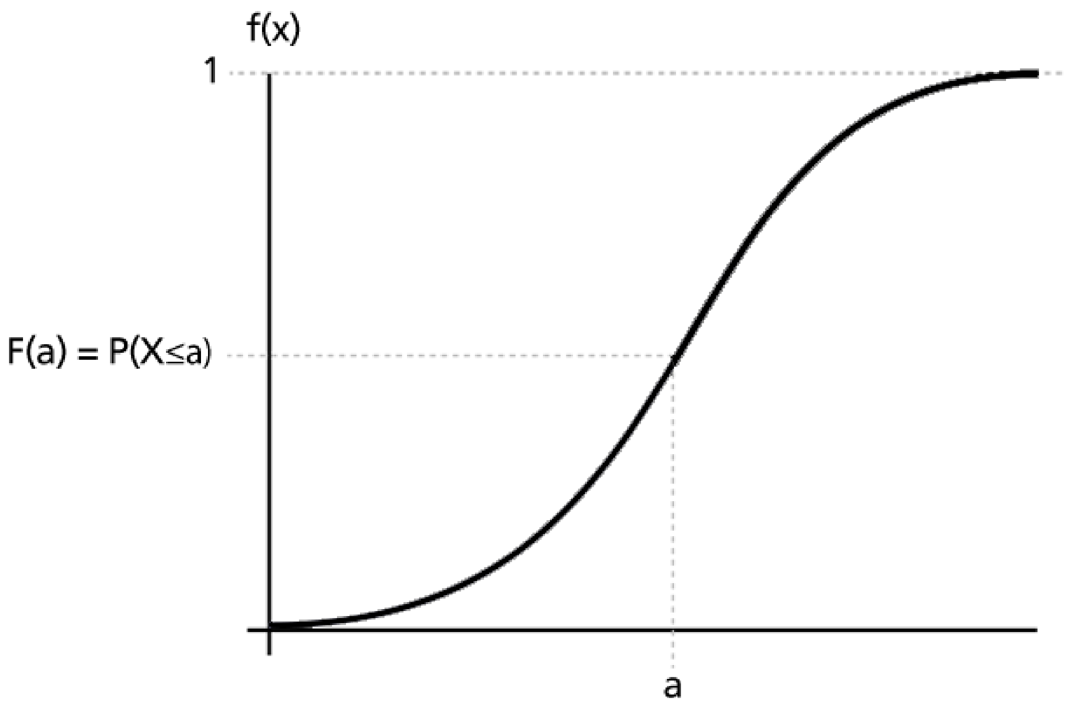
- The probability density function, *pdf*, as $f(x)$.
- The cumulative distribution function, *cdf*, as $F(x)$.

The *pdf* and *cdf* give a complete description of the probability distribution of a random variable. The following figure illustrates a *pdf*.



The next figures illustrate the *pdf* - *cdf* relationship.





If X is a continuous random variable, then the *pdf* of X is a function, $f(x)$, such that for any two numbers, a and b with $a \leq b$:

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

That is, the probability that X takes on a value in the interval $[a, b]$ is the area under the density function from a to $b,$ as shown above. The *pdf* represents the relative frequency of failure times as a function of time.

The *cdf* is a function, $F(x)$, of a random variable X, and is defined for a number x by:

$$F(x) = P(X \leq x) = \int_0^x f(s)ds$$

That is, for a number x, $F(x)$ is the probability that the observed value of X will be at most x. The *cdf* represents the cumulative values of the *pdf*. That is, the value of a point on the curve of the *cdf* represents the area under the curve to the left of that point on the *pdf*. In reliability, the *cdf* is used to measure the probability that the item in question will fail before the associated time value, t, and is also called *unreliability*.

Note that depending on the density function, denoted by $f(x)$, the limits will vary based on the region over which the distribution is defined. For example, for the life distributions considered in this reference, with the exception of the normal distribution, this range would be $[0, +\infty].$

Mathematical Relationship: *pdf* and *cdf*

The mathematical relationship between the *pdf* and *cdf* is given by:

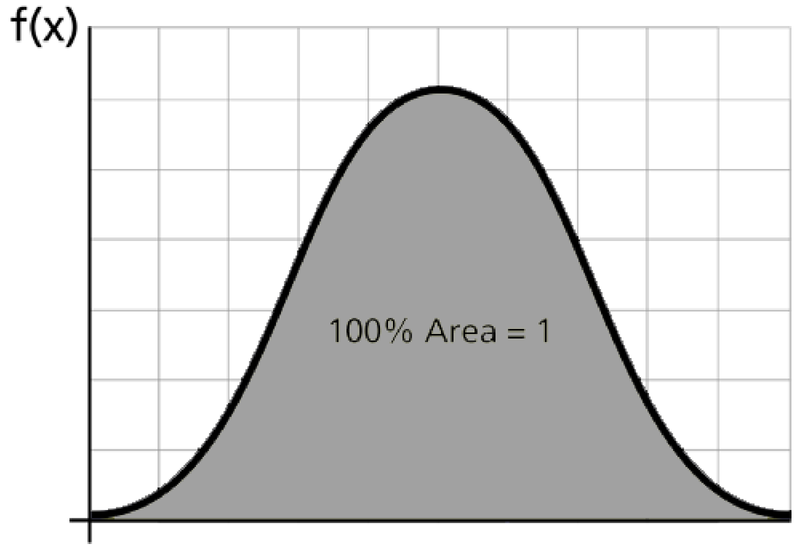
$$F(x) = \int_0^x f(s)ds$$

where s is a dummy integration variable.

Conversely:

$$f(x) = \frac{d(F(x))}{dx}$$

The *cdf* is the area under the probability density function up to a value of x. The total area under the *pdf* is always equal to 1, or mathematically:



$$\int_{-\infty}^{+\infty} f(x)dx = 1$$

The well-known normal (or Gaussian) distribution is an example of a probability density function. The *pdf* for this distribution is given by:

$$f(t) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2}$$

where μ is the mean and σ is the standard deviation. The normal distribution has two parameters, μ and σ.

Another is the lognormal distribution, whose *pdf* is given by:

$$f(t) = \frac{1}{t \cdot \sigma' \sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{t'-\mu'}{\sigma'}\right)^2}$$

where μ' is the mean of the natural logarithms of the times-to-failure and σ' is the standard deviation of the natural logarithms of the times-to-failure. Again, this is a 2-parameter distribution.

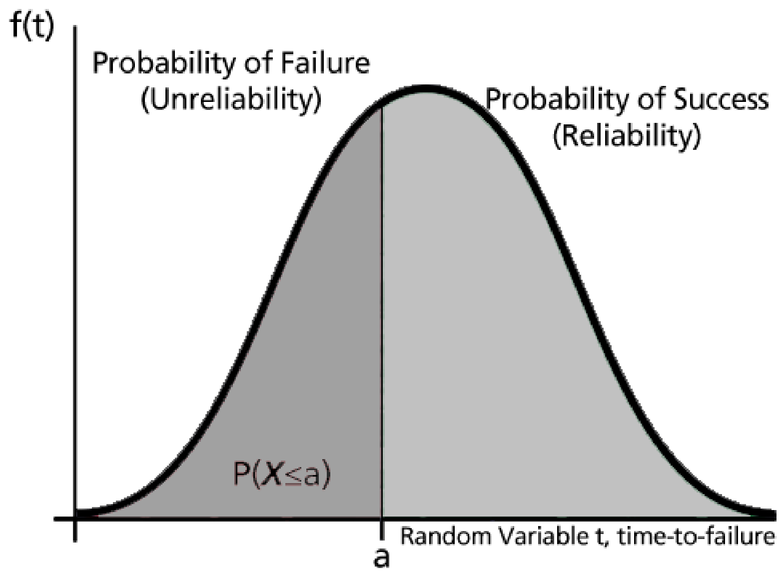
Reliability Function

The reliability function can be derived using the previous definition of the cumulative distribution function, $F(x)=\int_0^x f(s)ds$. From our definition of the *cdf*, the probability of an event occurring by time t is given by:

$$F(t)=\int_0^t f(s)ds$$

Or, one could equate this event to the probability of a unit failing by time t.

Since this function defines the probability of failure by a certain time, we could consider this the unreliability function. Subtracting this probability from 1 will give us the reliability function, one of the most important functions in life data analysis. The reliability function gives the probability of success of a unit undertaking a mission of a given time duration. The following figure illustrates this.



To show this mathematically, we first define the unreliability function, $Q(t)$, which is the probability of failure, or the probability that our time-to-failure is in the region of 0 and t. This is the same as the *cdf*. So from $F(t)=\int_0^t f(s)ds$:

$$Q(t) = F(t) = \int_0^t f(s)ds$$

Reliability and unreliability are the only two events being considered and they are mutually exclusive; hence, the sum of these probabilities is equal to unity.

Then:

$$\begin{aligned} Q(t) + R(t) &= 1 \\ R(t) &= 1 - Q(t) \\ R(t) &= 1 - \int_0^t f(s)ds \\ R(t) &= \int_t^\infty f(s)ds \end{aligned}$$

Conversely:

$$f(t) = -\frac{d(R(t))}{dt}$$

Conditional Reliability Function

Conditional reliability is the probability of successfully completing another mission following the successful completion of a previous mission. The time of the previous mission and the time for the mission to be undertaken must be taken into account for conditional reliability calculations. The conditional reliability function is given by:

$$R(t|T) = \frac{R(T+t)}{R(T)}$$

Failure Rate Function

The failure rate function enables the determination of the number of failures occurring per unit time. Omitting the derivation, the failure rate is mathematically given as:

$$\lambda(t) = \frac{f(t)}{R(t)}$$

This gives the instantaneous failure rate, also known as the hazard function. It is useful in characterizing the failure behavior of a component, determining maintenance crew allocation, planning for spares provisioning, etc. Failure rate is denoted as failures per unit time.

Mean Life (MTTF)

The mean life function, which provides a measure of the average time of operation to failure, is given by:

$$\overline{T} = m = \int_0^{\infty} t \cdot f(t)dt$$

This is the expected or average time-to-failure and is denoted as the MTTF (Mean Time To Failure).

The MTTF, even though an index of reliability performance, does not give any information on the failure distribution of the component in question when dealing with most lifetime distributions. Because vastly different distributions can have identical means, it is unwise to use the MTTF as the sole measure of the reliability of a component.

Median Life

Median life, $\tilde{T}$, is the value of the random variable that has exactly one-half of the area under the *pdf* to its left and one-half to its right. It represents the centroid of the distribution. The median is obtained by solving the following equation for $\breve{T}$. (For individual data, the median is the midpoint value.)

$$\int_{-\infty}^{\breve{T}} f(t)dt = 0.5$$

Modal Life (or Mode)

The modal life (or mode), $\tilde{T}$, is the value of T that satisfies:

$$\frac{d\left[f(t)\right]}{dt} = 0$$

For a continuous distribution, the mode is that value of t that corresponds to the maximum probability density (the value at which the *pdf* has its maximum value, or the peak of the curve).

Lifetime Distributions

A statistical distribution is fully described by its *pdf*. In the previous sections, we used the definition of the *pdf* to show how all other functions most commonly used in reliability engineering and life data analysis can be derived. The reliability function, failure rate function, mean time function, and median life function can be determined directly from the *pdf* definition, or $f(t)$. Different distributions exist, such as the normal (Gaussian), exponential, Weibull, etc., and each has a predefined form of $f(t)$ that can be found in many references. In fact, there are certain references that are devoted exclusively to different types of statistical distributions. These distributions were formulated by statisticians, mathematicians and engineers to mathematically model or represent certain behavior. For example, the Weibull distribution was formulated by Waloddi Weibull and thus it bears his name. Some distributions tend to better represent life data and are most commonly called "lifetime distributions".

A more detailed introduction to this topic is presented in Life Distributions.

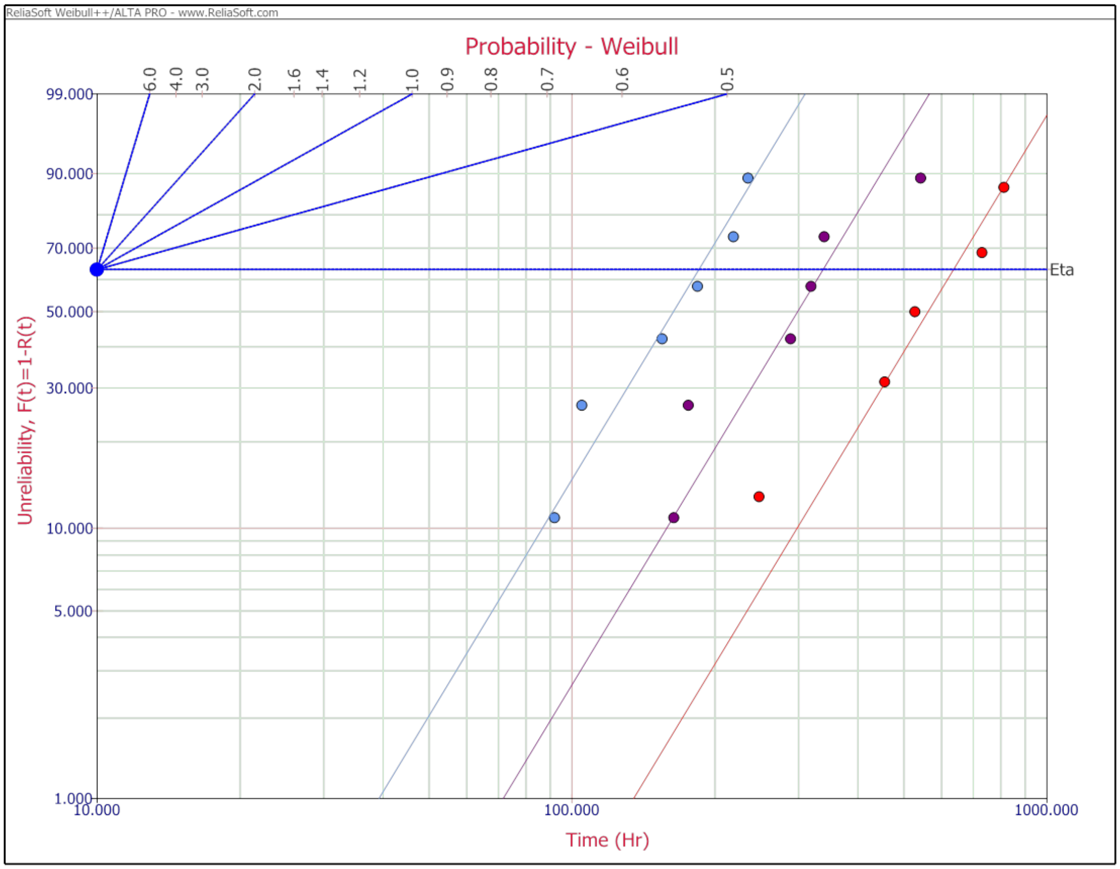
Appendix B: Parameter Estimation

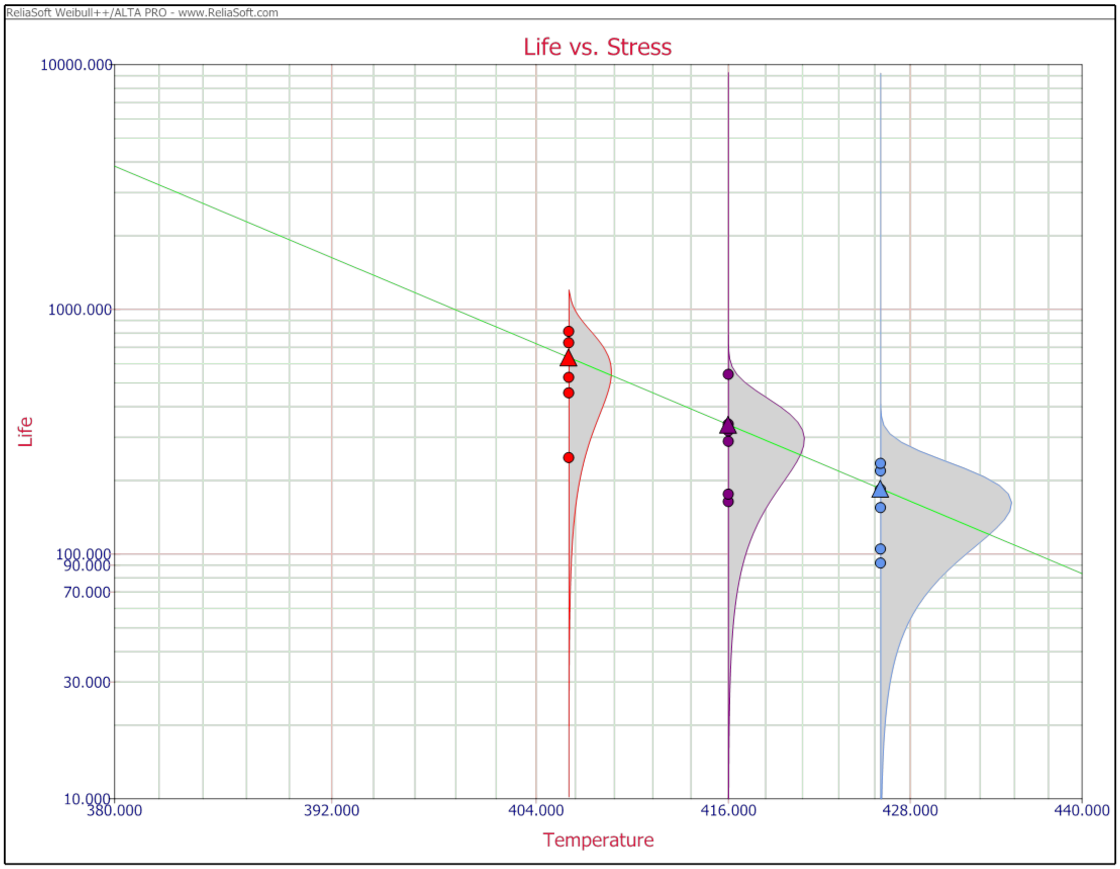
This appendix presents two methods for estimating the parameters of accelerated life test data analysis models (ALTA models). The graphical method, which is based on probability plotting or least squares (Rank Regression on X or Rank Regression on Y), has some limitations. Therefore, the Maximum Likelihood Estimation (MLE) method is used for all parameter estimation in Weibull++.

Graphical Method

The graphical method for estimating the parameters of accelerated life data involves generating two types of plots. First, the life data at each individual stress level are plotted on a probability paper appropriate to the assumed life distribution (i.e., Weibull, exponential, or lognormal). This can be done using either Probability Plotting or Least Squares (Rank Regression).

The parameters of the distribution at each stress level are then estimated from the plot. Once these parameters have been estimated at each stress level, the second plot is created on a paper that linearizes the assumed life-stress relationship (e.g., Arrhenius, inverse power law, etc.). To do this, a life characteristic must be chosen to be plotted. The life characteristic can be any percentile, such as BX% life, the scale parameter, mean life, etc. The plotting paper used is a special type of paper that linearizes the life-stress relationship. For example, a log-log paper linearizes the inverse power law relationship, and a log-reciprocal paper linearizes the Arrhenius relationship. The parameters of the model are then estimated by solving for the slope and the intercept of the line.





Example of Graphical Method for Accelerated Life Data

Consider the following times-to-failure data at three different stress levels.

Stress	393 psi	408 psi	423 psi

Time Failed (hrs)	3450	3300	2645
	4340	3720	3100
	4760	4180	3400
	5320	4560	3800
	5740	4920	4100
	6160	5280	4400
	6580	5640	4700
	7140	6233	5100
	8101	6840	5700
	8960	7380	6400

Estimate the parameters for a Weibull assumed life distribution and for the inverse power law life-stress relationship.

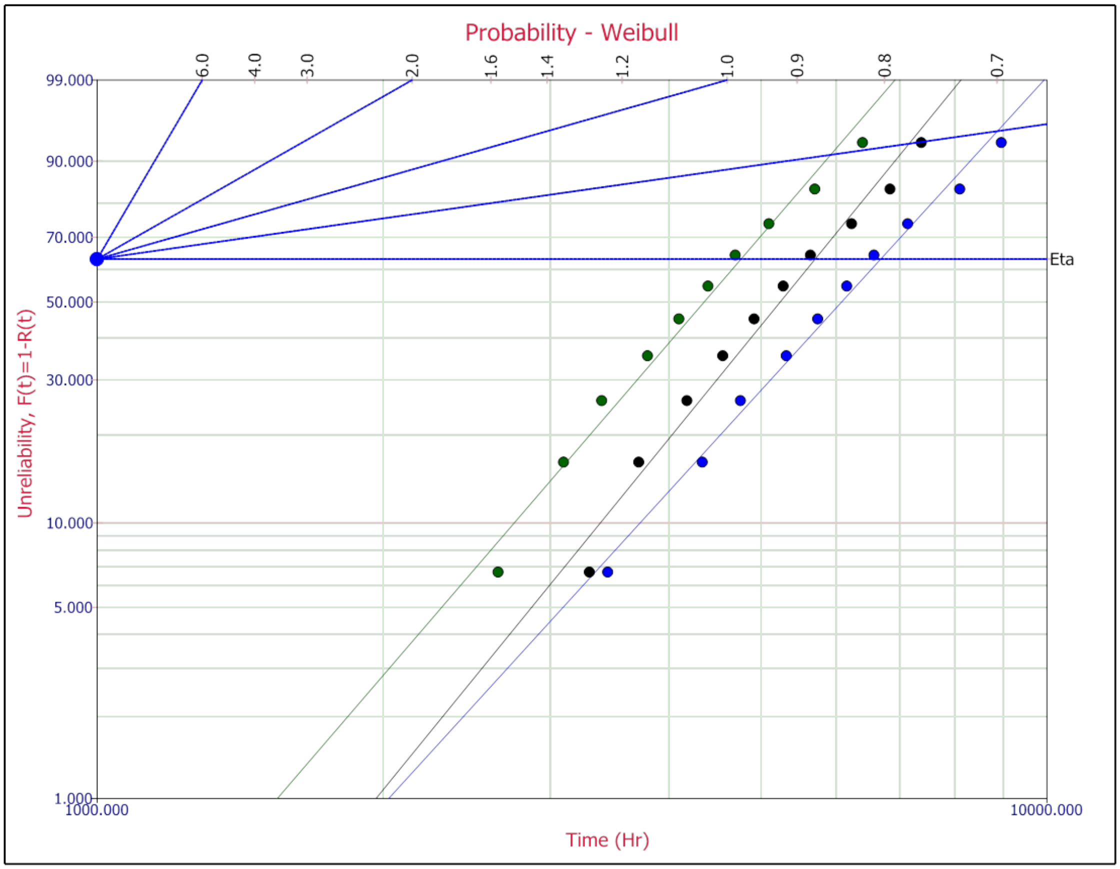
Solution

First the parameters of the Weibull distribution need to be determined. The data are individually analyzed (for each stress level) using the probability plotting method, or software such as ReliaSoft's Weibull++, with the following results:

$$[\widehat{\beta}_1 = \quad 3.8,\ \hat{\eta}_1 = 6692]$$
$$[\widehat{\beta}_2 = \quad 4.2,\ \hat{\eta}_2 = 5716]$$
$$[\widehat{\beta}_3 = \quad 4.0,\ \hat{\eta}_3 = 4774]$$

where:

- $\widehat{\beta}_1, \hat{\eta}_1$ are the parameters of the 393 psi data.
- $\widehat{\beta}_2, \hat{\eta}_2$ are the parameters of the 408 psi data.
- $\widehat{\beta}_3, \hat{\eta}_3$ are the parameters of the 423 psi data.



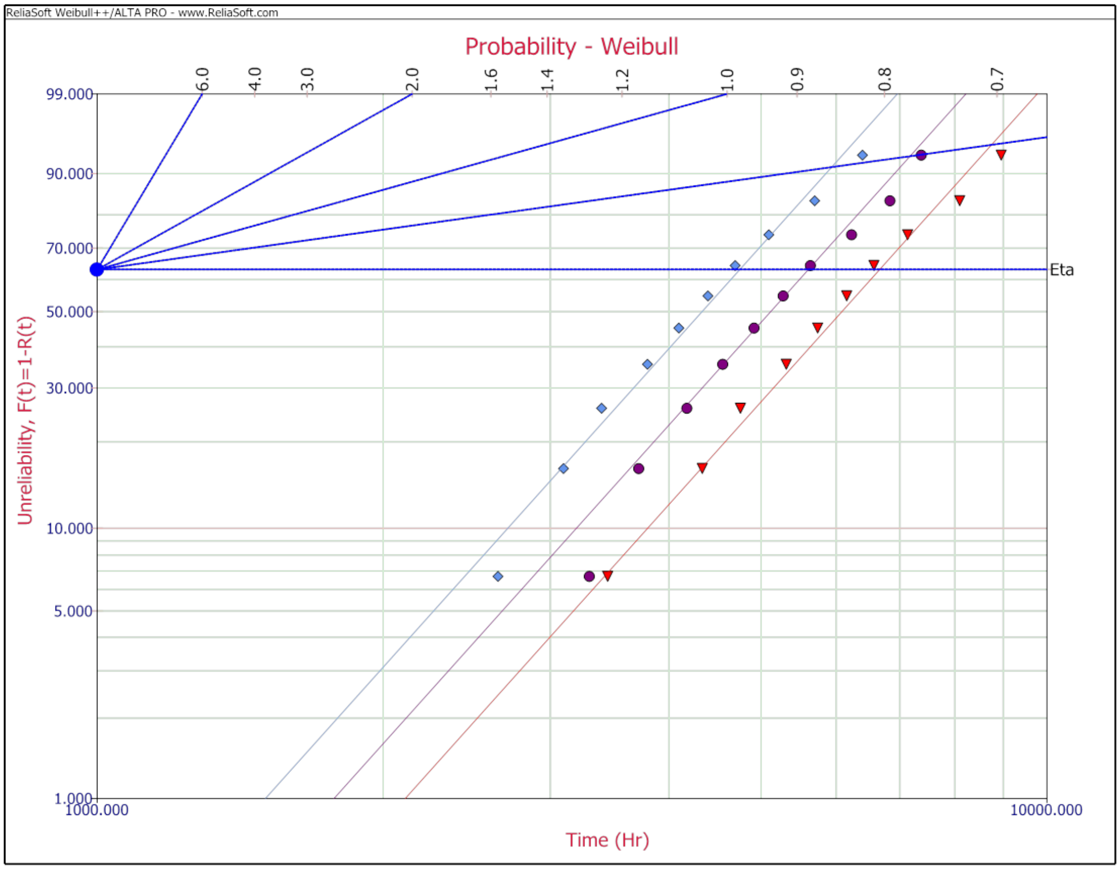
Since the shape parameter, β, is not common for the three stress levels, the average value is estimated.

$$\widehat{\beta}_{common} = 4$$

Averaging the betas is one of many simple approaches available. One can also use a weighted average, since the uncertainty on beta is greater for smaller sample sizes. In most practical applications the value of $\widehat{\beta}$ will vary (even though it is assumed constant) due to sampling error, etc. The variability in the value of $\widehat{\beta}$ is a source of error when performing analysis by averaging the betas. MLE analysis, which uses a common $\widehat{\beta}$, is not susceptible to this error. MLE analysis is the method of parameter estimation used in Weibull++ and it is explained in the next section.

Redraw each line with a $\widehat{\beta} = 4$, and estimate the new etas, as follows:

$$\hat{\eta}_1 = 6650$$
$$\hat{\eta}_2 = 5745$$
$$\hat{\eta}_3 = 4774$$



The IPL relationship is given by:

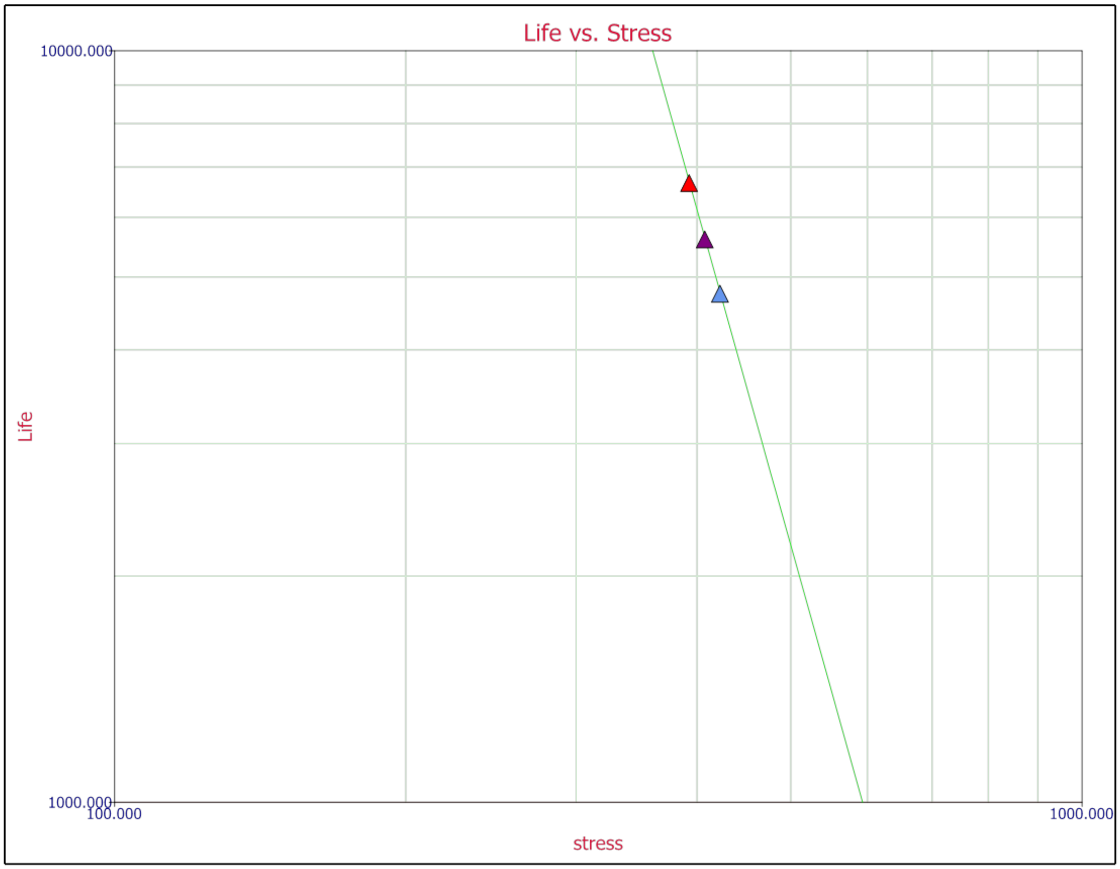
$$L(V) = \frac{1}{KV^n}$$

where L represents a quantifiable life measure (eta in the Weibull case), V represents the stress level, K is one of the parameters, and n is another model parameter. The relationship is linearized by taking the logarithm of both sides which yields:

$$\ln(L) = -\ln K - n \ln V$$

where $L = \eta$, $(-\ln K)$ is the intercept, and $(-n)$ is the slope of the line.

The values of eta obtained previously are now plotted on a log-linear scale yielding the following plot:



The slope of the line is the n parameter, which is obtained from the plot:

$$Slope = \frac{\ln(T_2) - \ln(T_1)}{\ln(V_2) - \ln(V_1)} = \frac{\ln(10,000) - \ln(6,000)}{\ln(360) - \ln(403)} = -4.5272$$

Thus:

$$\widehat{n} = 4.5272$$

Solving the inverse power law equation with respect to K yields:

$$\widehat{K} = \frac{1}{LV^n}$$

Substituting V=403, the corresponding L (from the plot), L=6,000 and the previously estimated n :

$$\widehat{K} = \frac{1}{6000 \cdot 403^{4.5272}} = 2.67 \cdot 10^{-16}$$

Comments on the Graphical Method

Although the graphical method is simple, it is quite laborious. Furthermore, many issues surrounding its use require careful consideration. Some of these issues are presented next:

- What happens when no failures are observed at one or more stress level? In this case, plotting methods cannot be employed. Discarding the data would be a mistake since every piece of life data information is important.
- In the step at which the life-stress relationship is linearized and plotted to obtain its parameters, you must be able to linearize the function, which is not always possible.
- In real accelerated tests the data sets are small. Separating them and individually plotting them, and then subsequently replotting the results, increases the underlying error.
- During initial parameter estimation, the parameter that is assumed constant will more than likely vary. What value do you use?
- Confidence intervals on all of the results cannot be ascertained using graphical methods.

The maximum likelihood estimation parameter estimation method described next overcomes these shortfalls, and is the method utilized in Weibull++.

Maximum Likelihood Estimation (MLE) Method

The idea behind maximum likelihood parameter estimation is to determine the parameters that maximize the probability (likelihood) of the sample data. From a statistical point of view, the method of maximum likelihood is considered to be more robust (with some exceptions) and yields estimators with good statistical properties. In other words, MLE methods are versatile and apply to most models and to different types of data. In addition, they provide efficient methods for quantifying uncertainty through confidence bounds. For a detailed discussion of this analysis method for a single life distribution, see Maximum Likelihood Estimation.

The maximum likelihood solution for accelerated life test data is formulated in the same way as described in Maximum Likelihood Estimation for a single life distribution. However, in this case, the stress level of each individual observation is included in the likelihood function. Consider a continuous random variable $x(v),$ where v is the stress. The *pdf* of the random variable now becomes a function of both x and v :

$$f(x, v; \theta_1, \theta_2, \ldots, \theta_k)$$

where $\theta_1, \theta_2, \ldots, \theta_k$ are k unknown constant parameters which need to be estimated. Conduct an experiment and obtain N independent observations, $x_1, x_2, \ldots, x_N$ each at a corresponding stress, $v_1, v_2, \ldots, v_N$. Then the likelihood function for complete data is given by:

$$L((x_1,\ v_1),(x_2,\ v_2),\ldots,(x_N,\ v_N)|\theta_1,\theta_2,\ldots,\theta_k) = \prod_{i=1}^{N} f(x_i, v_i; \theta_1, \theta_2, \ldots, \theta_k)$$

$$i = 1, 2, \ldots, N$$

The logarithmic likelihood function is given by:

$$\Lambda = \ln L = \sum_{i=1}^{N} \ln f(x_i, v_i; \theta_1, \theta_2, \ldots, \theta_k)$$

The maximum likelihood estimators (MLE) of $\theta_1, \theta_2, \ldots, \theta_k,$ are obtained by maximizing L or Λ.

In this case, $\theta_1, \theta_2, \ldots, \theta_k$ are the parameters of the combined model which includes the parameters of the life distribution and the parameters of the life-stress relationship. Note that in the above equations, N is the total number of observations. This means that the sample size is no longer broken into the number of observations at each stress level. In the graphical method example, the sample size at the stress level of 20V was 4, and 15 at 36V. Using the above equations, however, the test's sample size is 19.

Once the parameters are estimated, they can be substituted back into the life distribution and the life-stress relationship.

Example of MLE for Accelerated Life Data

The following example illustrates the use of the MLE method on accelerated life test data. Consider the inverse power law relationship, given by:

$$L(V) = \frac{1}{KV^n}$$

where L represents a quantifiable life measure, V represents the stress level, K is one of the parameters, and n is another model parameter.

Assume that the life at each stress follows a Weibull distribution, with a *pdf* given by:

$$f(t) = \frac{\beta}{\eta}\left(\frac{T}{\eta}\right)^{\beta-1} e^{-\left(\frac{T}{\eta}\right)^{\beta}}$$

where the time-to-failure, t , is a function of stress, V .

A common life measure needs to determined so that it can be easily included in the Weibull *pdf*. In this case, setting $\eta = L(V)$ (which is the life at 63.2%) and substituting in the Weibull *pdf*, yields the following IPL-Weibull *pdf* :

$$f(t, V) = \beta K V^n (K V^n T)^{\beta-1} e^{-(K V^n T)^{\beta}}$$

The log-likelihood function for the complete data is given by:

$$\Lambda = \ln L = \sum_{i=1}^{N} \ln\left(\beta K V^n (K V^n T_i)^{\beta-1} e^{-(K V^n T_i)^{\beta}}\right)$$

Note that β is now the common shape parameter to solve for, along with K and n.

Conclusions

In this appendix, two methods for estimating the parameters of accelerated life testing models were presented. First, the graphical method was illustrated using a probability plotting method for obtaining the parameters of the life distribution. The parameters of the life-stress relationship were then estimated graphically by linearizing the model. However, not all life-stress relationships can be linearized. In addition, estimating the parameters of each individual distribution leads to an accumulation of uncertainties, depending on the number of failures and suspensions observed at each stress level. Furthermore, the slopes (shape parameters) of each individual distribution are rarely equal (common). Using the graphical method, one must estimate a common shape parameter (usually the average) and repeat the analysis. By doing so, further uncertainties are introduced on the estimates, and these are uncertainties that cannot be quantified. The second method, the Maximum Likelihood Estimation, treated both the life distribution and the life-stress relationship as one model, the parameters of that model can be estimated using the complete likelihood function. Doing so, a common shape parameter is estimated for the model, thus eliminating the uncertainties of averaging the individual shape parameters. All uncertainties are accounted for in the form of confidence bounds (presented in detail in Appendix D), which are quantifiable because they are obtained based on the overall model.

Appendix C: Benchmark Examples

In this section, five published examples are presented for comparison purposes. ReliaSoft's R&D validated the Weibull++ software with hundreds of data sets and methods. Weibull++ also cross-validates each provided solution by independently re-evaluating the second partial derivatives based on the estimated parameters each time a calculation is performed. These partials will be equal to zero when a solution is reached. Double precision is used throughout Weibull++.

Example 1

From Wayne Nelson [28, p. 135].

Published Results for Example 1

- Published Results:

$$\begin{aligned}\widehat{\sigma}_{T'} &= 0.59673\\ \widehat{B} &= 9920.195\\ \widehat{C} &= 9.69517\cdot 10^{-7}\end{aligned}$$

Computed Results for Example 1

This same data set can be entered into Weibull++ by selecting the data sheet for grouped times-to-failure data with suspensions and using the Arrhenius model, the lognormal distribution, and MLE. Weibull++ computed parameters for maximum likelihood are:

$$\begin{aligned}\widehat{\sigma}_{T'} &= 0.59678\\ \widehat{B} &= 9924.804\\ \widehat{C} &= 9.58978\cdot 10^{-7}\end{aligned}$$

Example 2

From Wayne Nelson [28, p. 453], time to breakdown of a transformer oil, tested at 26kV, 28kV, 30kV, 32kV, 34kV, 36kV and 38kV.

Published Results for Example 2

- Published Results:

$$\begin{aligned}\widehat{\beta} &= 0.777\\ \widehat{K} &= 6.8742\cdot 10^{-29}\\ \widehat{n} &= 17.72958\end{aligned}$$

- Published 95% confidence limits on β :

$$\{0.653, 0.923\}$$

Computed Results for Example 2

Use the inverse power law model and Weibull as the underlying life distribution. Weibull++ computed parameters are:

$$\begin{aligned}\widehat{\beta} &= 0.7765,\\ \widehat{K} &= 6.8741\cdot 10^{-29}\\ \widehat{n} &= 17.7296\end{aligned}$$

- Weibull++ computed 95% confidence limits on the parameters:

$$\{0.6535, 0.9228\}\ \text{for}\ \widehat{\beta}$$

Example 3

From Wayne Nelson [28, p. 157], forty bearings were tested to failure at four different test loads. The data were analyzed using the inverse power law Weibull model.

Published Results for Example 3

Nelson's [28, p. 306] IPL-Weibull parameter estimates:

$$\begin{aligned}\widehat{\beta} &= 1.243396\\ \widehat{K} &= 0.4350735\\ \widehat{n} &= 13.8528\end{aligned}$$

- The 95% 2-sided confidence bounds on the parameters:

$$\{0.9746493, 1.586245\} \text{ for } \widehat{\beta}$$
$$\{0.332906, 0.568596\} \text{ for } \widehat{K}$$
$$\{11.43569, 16.26991\} \text{ for } \widehat{n}$$

- Percentile estimates at a stress of 0.87, with 95% 2-sided confidence bounds:

Percentile	Life Estimate	95% Lower	95% Upper
1%	0.3913096	0.1251383	1.223632
10%	2.589731	1.230454	5.450596
90%	30.94404	19.41020	49.33149
99%	54.03563	33.02691	88.40821

Computed Results for Example 3

Use the inverse power law model and Weibull as the underlying life distribution.

- Weibull++ computed parameters are:

$$\widehat{\beta} = 1.243375$$
$$\widehat{K} = 0.4350548$$
$$\widehat{n} = 13.8529$$

- The 95% 2-sided confidence bounds on the parameters:

$$\{0.9745811, 1.586303\} \text{ for } \widehat{\beta}$$
$$\{0.330007, 0.573542\} \text{ for } \widehat{K}$$
$$\{11.43510, 16.27079\} \text{ for } \widehat{n}$$

- Percentile estimates at a stress of 0.87, with 95% 2-sided confidence bounds:

Percentile	Life Estimate	95% Lower	95% Upper
1%	0.3913095	0.1251097	1.223911
10%	2.589814	1.230384	5.451588
90%	30.94632	19.40876	49.34240
99%	54.04012	33.02411	88.43039

Example 4

From Meeker and Escobar [26, p. 504], Mylar-Polyurethane Insulating Structure data using the inverse power law lognormal model.

Published Results for Example 4

- Published Results:

$$\begin{aligned} \widehat{\sigma}_{T'} &= 1.05, \\ \widehat{K} &= 1.14 \cdot 10^{-12}, \\ \widehat{n} &= 4.28. \end{aligned}$$

- The 95% 2-sided confidence bounds on the parameters:

$$\begin{aligned} &\{0.83, 1.32\} \text{ for } \widehat{\sigma}_{T'} \\ &\left\{3.123 \cdot 10^{-15}, 4.16 \cdot 10^{-10}\right\} \text{ for } \widehat{K} \\ &\{3.11, 5.46\} \text{ for } \widehat{n} \end{aligned}$$

Computed Results for Example 4

Use the inverse power law lognormal.

- Weibull++ computed parameters are:

$$\begin{aligned} \widehat{\sigma}_{T'} &= 1.04979 \\ \widehat{K} &= 1.15 \cdot 10^{-12} \\ \widehat{n} &= 4.289 \end{aligned}$$

- Weibull++ computed 95% confidence limits on the parameters:

$$\begin{aligned} &\{0.833, 1.323\} \text{ for } \widehat{\sigma}_{T'} \\ &\left\{3.227 \cdot 10^{-15}, 4.095 \cdot 10^{-10}\right\} \text{ for } \widehat{K} \\ &\{3.115, 5.464\} \text{ for } \widehat{n} \end{aligned}$$

Example 5

From Meeker and Escobar [26, p. 515], Tantalum capacitor data using the combination (Temperature-NonThermal) Weibull model.

Published Results for Example 5

- Published Results:

$$\begin{aligned}\widehat{\beta} &= 0.4292 \\ \widehat{B} &= 3829.468 \\ \widehat{C} &= 4.513 \cdot 10^{36} \\ \widehat{n} &= 20.1\end{aligned}$$

- The 95% 2-sided confidence bounds on the parameters:

$$\begin{aligned}&\{0.3165, 0.58\} \text{ for } \widehat{\beta} \\ &\{-464.177, 8007.069\} \text{ for } \widehat{B} \\ &\left\{1.265 \cdot 10^{25}, 1.609 \cdot 10^{48}\right\} \text{ for } \widehat{C} \\ &\{11.4, 28.8\} \text{ for } \widehat{n}\end{aligned}$$

Computed Results for Example 5

Use the Temperature-NonThermal model and Weibull as the underlying life distribution.

- Weibull++ computed parameters are:

$$\begin{aligned}\widehat{\beta} &= 0.4287 \\ \widehat{B} &= 3780.298 \\ \widehat{C} &= 4.772 \cdot 10^{36} \\ \widehat{n} &= 20.09\end{aligned}$$

- Weibull++ computed 95% confidence limits on the parameters:

$$\begin{aligned}&\{0.3169, 0.5799\} \text{ for } \widehat{\beta} \\ &\{-483.83, 8044.426\} \text{ for } \widehat{B} \\ &\left\{1.268 \cdot 10^{25}, 1.796 \cdot 10^{48}\right\} \text{ for } \widehat{C} \\ &\{11.37, 28.8\} \text{ for } \widehat{n}\end{aligned}$$

Appendix D: Confidence Bounds

What Are Confidence Bounds?

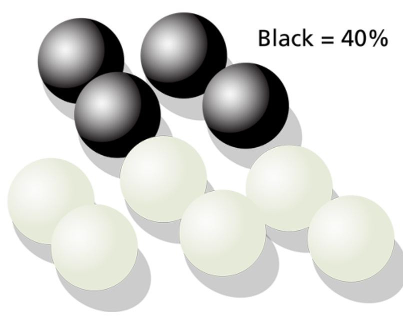
One of the most confusing concepts to a novice reliability engineer is estimating the precision of an estimate. This is an important concept in the field of reliability engineering, leading to the use of confidence intervals (or bounds). In this section, we will try to briefly present the concept in relatively simple terms but based on solid common sense.

The Black and White Marbles

To illustrate, consider the case where there are millions of perfectly mixed black and white marbles in a rather large swimming pool and our job is to estimate the percentage of black marbles. The only way to be absolutely certain about the exact percentage of marbles in the pool is to accurately count every last marble and calculate the percentage. However, this is too time- and resource-intensive to be a viable option, so we need to come up with a way of estimating the percentage of black marbles in the pool. In order to do this, we would take a relatively small sample of marbles from the pool and then count how many black marbles are in the sample.

Taking a Small Sample of Marbles

First, pick out a small sample of marbles and count the black ones. Say you picked out ten marbles and counted four black marbles. Based on this, your estimate would be that 40% of the marbles are black.



If you put the ten marbles back in the pool and repeat this example again, you might get six black marbles, changing your estimate to 60% black marbles. Which of the two is correct? Both estimates are correct! As you repeat this experiment over and over again, you might find out that this estimate is usually between $X_1\%$ and $X_2\%$, and you can assign a percentage to the number of times your estimate falls between these limits. For example, you notice that 90% of the time this estimate is between $X_1\%$ and $X_2\%$.

Taking a Larger Sample of Marbles

If you now repeat the experiment and pick out 1,000 marbles, you might get results for the number of black marbles such as 545, 570, 530, etc., for each trial. The range of the estimates in this case will be much narrower than before. For example, you observe that 90% of the time, the number of black marbles will now be from $Y_1\%$ to $Y_2\%$, where $X_1\% < Y_1\%$ and $X_2\% > Y_2\%$, thus giving you a more narrow estimate interval. The same principle is true for confidence intervals; the larger the sample size, the more narrow the confidence intervals.

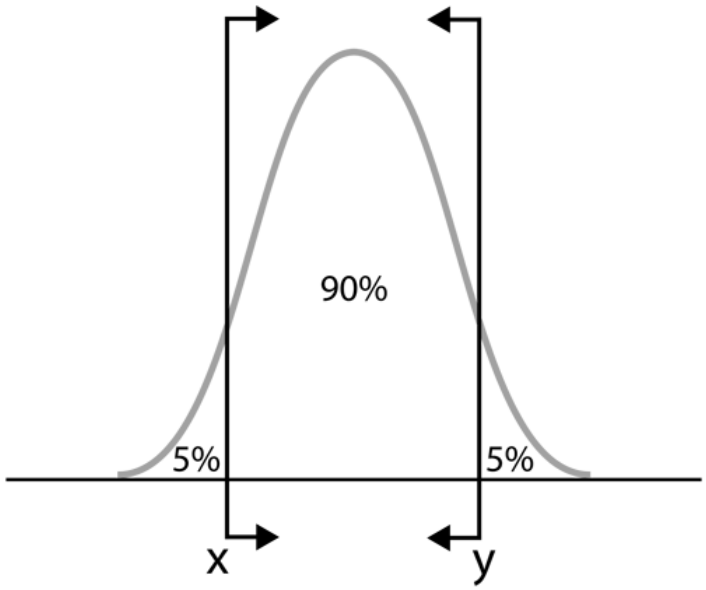
Back to Reliability

We will now look at how this phenomenon relates to reliability. Overall, the reliability engineer's task is to determine the probability of failure, or reliability, of the population of units in question. However, one will never know the exact reliability value of the population unless one is able to obtain and analyze the failure data for every single unit in the population. Since this usually is not a realistic situation, the task then is to estimate the reliability based on a sample, much like estimating the number of black marbles in the pool. If we perform ten different reliability tests for our units, and analyze the results, we will obtain slightly different parameters for the distribution each time, and thus slightly different reliability results. However, by employing confidence bounds, we obtain a range within which these reliability values are likely to occur a certain percentage of the time. This helps us gauge the utility of the data and the accuracy of the resulting estimates. Plus, it is always useful to remember that each parameter is an estimate of the true parameter, one that is unknown to us. This range of plausible values is called a confidence interval.

One-Sided and Two-Sided Confidence Bounds

Confidence bounds are generally described as being one-sided or two-sided.

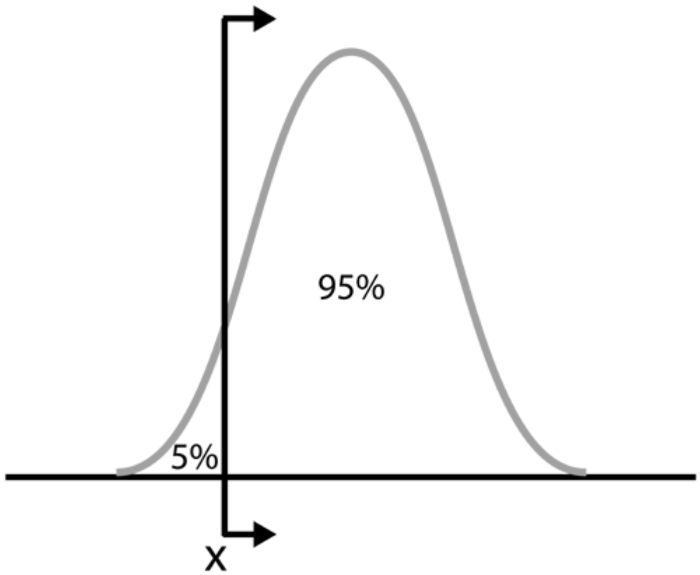
Two-Sided Bounds

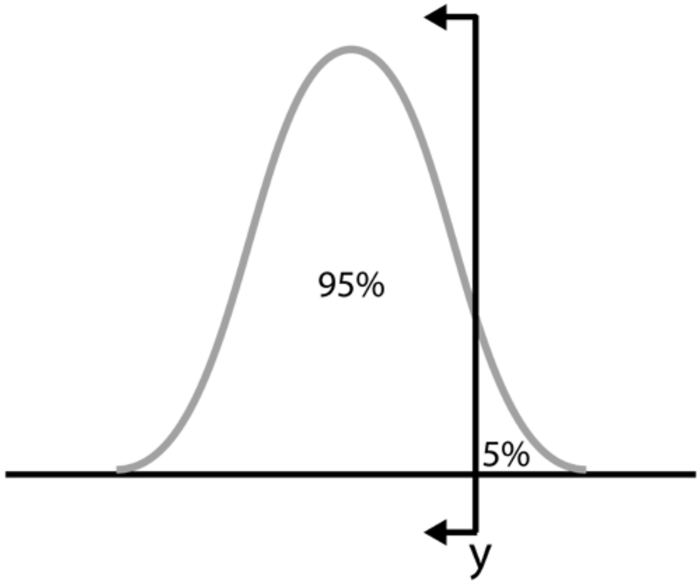


When we use two-sided confidence bounds (or intervals), we are looking at a closed interval where a certain percentage of the population is likely to lie. That is, we determine the values, or bounds, between which lies a specified percentage of the population. For example, when dealing with 90% two-sided confidence bounds of (X,Y), we are saying that 90% of the population lies between X and Y with 5% less than X and 5% greater than Y.

One-Sided Bounds

One-sided confidence bounds are essentially an open-ended version of two-sided bounds. A one-sided bound defines the point where a certain percentage of the population is either higher or lower than the defined point. This means that there are two types of one-sided bounds: upper and lower. An upper one-sided bound defines a point that a certain percentage of the population is less than. Conversely, a lower one-sided bound defines a point that a specified percentage of the population is greater than.





For example, if X is a 95% upper one-sided bound, this would imply that 95% of the population is less than X. If X is a 95% lower one-sided bound, this would indicate that 95% of the population is greater than X. Care must be taken to differentiate between one- and two-sided confidence bounds, as these bounds can take on identical values at different percentage levels. For example, in the figures above, we see bounds on a hypothetical distribution. Assuming that this is the same distribution in all of the figures, we see that X marks the spot below which 5% of the distribution's population lies. Similarly, Y represents the point above which 5% of the population lies. Therefore, X and Y represent the 90% two-sided bounds, since 90% of the population lies between the two points. However, X also represents the lower one-sided 95% confidence bound, since 95% of the population lies above that point; and Y

represents the upper one-sided 95% confidence bound, since 95% of the population is below Y. It is important to be sure of the type of bounds you are dealing with, particularly as both one-sided bounds can be displayed simultaneously in Weibull++. In Weibull++, we use upper to represent the higher limit and lower to represent the lower limit, regardless of their position, but based on the value of the results. So if obtaining the confidence bounds on the reliability, we would identify the lower value of reliability as the lower limit and the higher value of reliability as the higher limit. If obtaining the confidence bounds on probability of failure we will again identify the lower numeric value for the probability of failure as the lower limit and the higher value as the higher limit.

Fisher Matrix Confidence Bounds

This section presents an overview of the theory on obtaining approximate confidence bounds on suspended (multiple censored) data. The methodology used is the so-called Fisher matrix bounds (FM), described in Nelson [30] and Lloyd and Lipow [24]. These bounds are employed in most other commercial statistical applications. In general, these bounds tend to be more optimistic than the non-parametric rank based bounds. This may be a concern, particularly when dealing with small sample sizes. Some statisticians feel that the Fisher matrix bounds are too optimistic when dealing with small sample sizes and prefer to use other techniques for calculating confidence bounds, such as the likelihood ratio bounds.

Approximate Estimates of the Mean and Variance of a Function

In utilizing FM bounds for functions, one must first determine the mean and variance of the function in question (i.e., reliability function, failure rate function, etc.). An example of the methodology and assumptions for an arbitrary function G is presented next.

Single Parameter Case

For simplicity, consider a one-parameter distribution represented by a general function G, which is a function of one parameter estimator, say $G(\hat{\theta})$. For example, the mean of the exponential distribution is a function of the parameter $\lambda: G(\lambda) = 1/\lambda = \mu$. Then, in general, the expected value of $G\left(\hat{\theta}\right)$ can be found by:

$$E\left(G\left(\hat{\theta}\right)\right) = G(\theta) + O\left(\frac{1}{n}\right)$$

where $G(\theta)$ is some function of θ, such as the reliability function, and θ is the population parameter where $E\left(\hat{\theta}\right)=\theta$ as $n\to\infty$. The term $O\left(\frac{1}{n}\right)$ is a function of n, the sample size, and tends to zero, as fast as $\frac{1}{n}$, as $n\to\infty$. For example, in the case of $\hat{\theta}=1/\overline{x}$ and $G(x)=1/x$, then $E(G(\hat{\theta}))=\overline{x}+O\left(\frac{1}{n}\right)$ where $O\left(\frac{1}{n}\right)=\frac{\sigma^2}{n}$. Thus as $n\to\infty$, $E(G(\hat{\theta}))=\mu$ where μ and σ are the mean and standard deviation, respectively. Using the same one-parameter distribution, the variance of the function $G\left(\hat{\theta}\right)$ can then be estimated by:

$$Var\left(G\left(\hat{\theta}\right)\right)=\left(\frac{\partial G}{\partial\hat{\theta}}\right)^2_{\hat{\theta}=\theta}Var\left(\hat{\theta}\right)+O\left(\frac{1}{n^{\frac{3}{2}}}\right)$$

Two-Parameter Case

Consider a Weibull distribution with two parameters β and η. For a given value of t, $R(t)=G(\beta,\eta)=e^{-\left(\frac{t}{\eta}\right)^{\beta}}$. Repeating the previous method for the case of a two-parameter distribution, it is generally true that for a function G, which is a function of two parameter estimators, say $G\left(\hat{\theta}_1,\hat{\theta}_2\right)$, that:

$$E\left(G\left(\hat{\theta}_1,\hat{\theta}_2\right)\right)=G\left(\theta_1,\theta_2\right)+O\left(\frac{1}{n}\right)$$

and:

$$Var(G(\hat{\theta}_1,\hat{\theta}_2)) = \left(\frac{\partial G}{\partial \hat{\theta}_1}\right)^2_{\widehat{\theta_1}=\theta_1} Var(\widehat{\theta_1}) + \left(\frac{\partial G}{\partial \hat{\theta}_2}\right)^2_{\widehat{\theta_2}=\theta_2} Var(\widehat{\theta_2})$$
$$+ 2\left(\frac{\partial G}{\partial \hat{\theta}_1}\right)_{\widehat{\theta_1}=\theta_1}\left(\frac{\partial G}{\partial \hat{\theta}_2}\right)_{\widehat{\theta_2}=\theta_2} Cov(\widehat{\theta_1},\widehat{\theta_2})$$
$$+ O\left(\frac{1}{n^{\frac{3}{2}}}\right)$$

Note that the derivatives of the above equation are evaluated at $\hat{\theta}_1 = \theta_1$ and $\hat{\theta}_2 = \theta_2$, where E $\left(\hat{\theta}_1\right) \simeq \theta_1$ and E $\left(\hat{\theta}_2\right) \simeq \theta_2$.

Parameter Variance and Covariance Determination

The determination of the variance and covariance of the parameters is accomplished via the use of the Fisher information matrix. For a two-parameter distribution, and using maximum likelihood estimates (MLE), the log-likelihood function for censored data is given by:

$$\ln[L] = \Lambda = \sum_{i=1}^{R} \ln[f(T_i;\theta_1,\theta_2)]$$
$$+ \sum_{j=1}^{M} \ln[1 - F(S_j;\theta_1,\theta_2)]$$
$$+ \sum_{l=1}^{P} \ln\{F(I_{l_U};\theta_1,\theta_2) - F(I_{l_L};\theta_1,\theta_2)\}$$

In the equation above, the first summation is for *complete data*, the second summation is for *right censored data* and the third summation is for *interval or left censored data*.

Then the Fisher information matrix is given by:

$$F_0 = \begin{bmatrix} E_0\left[-\frac{\partial^2\Lambda}{\partial\theta_1^2}\right]_0 & E_0\left[-\frac{\partial^2\Lambda}{\partial\theta_1\partial\theta_2}\right]_0 \\ E_0\left[-\frac{\partial^2\Lambda}{\partial\theta_2\partial\theta_1}\right]_0 & E_0\left[-\frac{\partial^2\Lambda}{\partial\theta_2^2}\right]_0 \end{bmatrix}$$

The subscript 0 indicates that the quantity is evaluated at $\theta_1 = \theta_{1_0}$ and $\theta_2 = \theta_{2_0}$, the true values of the parameters.

So for a sample of N units where R units have failed, S have been suspended, and P have failed within a time interval, and $N = R + M + P,$ one could obtain the sample local information matrix by:

$$F = \begin{bmatrix} -\frac{\partial^2 \Lambda}{\partial \theta_1^2} & -\frac{\partial^2 \Lambda}{\partial \theta_1 \partial \theta_2} \\ \\ -\frac{\partial^2 \Lambda}{\partial \theta_2 \partial \theta_1} & -\frac{\partial^2 \Lambda}{\partial \theta_2^2} \end{bmatrix}$$

Substituting the values of the estimated parameters, in this case $\hat{\theta}_1$ and $\hat{\theta}_2$, and then inverting the matrix, one can then obtain the local estimate of the covariance matrix or:

$$\begin{bmatrix} \widehat{Var}\left(\hat{\theta}_1\right) & \widehat{Cov}\left(\hat{\theta}_1, \hat{\theta}_2\right) \\ \\ \widehat{Cov}\left(\hat{\theta}_1, \hat{\theta}_2\right) & \widehat{Var}\left(\hat{\theta}_2\right) \end{bmatrix} = \begin{bmatrix} -\frac{\partial^2 \Lambda}{\partial \theta_1^2} & -\frac{\partial^2 \Lambda}{\partial \theta_1 \partial \theta_2} \\ \\ -\frac{\partial^2 \Lambda}{\partial \theta_2 \partial \theta_1} & -\frac{\partial^2 \Lambda}{\partial \theta_2^2} \end{bmatrix}^{-1}$$

Then the variance of a function ($Var(G)$) can be estimated using equation for the variance. Values for the variance and covariance of the parameters are obtained from Fisher Matrix equation. Once they have been obtained, the approximate confidence bounds on the function are given as:

$$CB_R = E(G) \pm z_\alpha \sqrt{Var(G)}$$

which is the estimated value plus or minus a certain number of standard deviations. We address finding z_α next.

Approximate Confidence Intervals on the Parameters

In general, MLE estimates of the parameters are asymptotically normal, meaning that for large sample sizes, a distribution of parameter estimates from the same population would be very close to the normal distribution. Thus if $\hat{\theta}$ is the MLE estimator for θ, in the case of a single parameter distribution estimated from a large sample of n units, then:

$$z \equiv \frac{\hat{\theta} - \theta}{\sqrt{Var\left(\hat{\theta}\right)}}$$

follows an approximating normal distribution. That is

$$P\left(x \leq z\right) \rightarrow \Phi\left(z\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{t^2}{2}} dt$$

for large n. We now place confidence bounds on $\theta,$ at some confidence level δ, bounded by the two end points C_1 and C_2 where:

$$P\left(C_1 < \theta < C_2\right) = \delta$$

From the above equation:

$$P\left(-K_{\frac{1-\delta}{2}} < \frac{\hat{\theta} - \theta}{\sqrt{Var\left(\hat{\theta}\right)}} < K_{\frac{1-\delta}{2}}\right) \simeq \delta$$

where K_α is defined by:

$$\alpha = \frac{1}{\sqrt{2\pi}} \int_{K_\alpha}^{\infty} e^{-\frac{t^2}{2}} dt = 1 - \Phi\left(K_\alpha\right)$$

Now by simplifying the equation for the confidence level, one can obtain the approximate two-sided confidence bounds on the parameter $\theta,$ at a confidence level $\delta,$ or:

$$\left(\hat{\theta} - K_{\frac{1-\delta}{2}} \cdot \sqrt{Var\left(\hat{\theta}\right)} < \theta < \hat{\theta} + K_{\frac{1-\delta}{2}} \cdot \sqrt{Var\left(\hat{\theta}\right)}\right)$$

The upper one-sided bounds are given by:

$$\theta < \hat{\theta} + K_{1-\delta}\sqrt{Var(\hat{\theta})}$$

while the lower one-sided bounds are given by:

$$\theta > \hat{\theta} - K_{1-\delta}\sqrt{Var(\hat{\theta})}$$

If $\hat{\theta}$ must be positive, then $\ln\hat{\theta}$ is treated as normally distributed. The two-sided approximate confidence bounds on the parameter θ, at confidence level δ, then become:

$$\theta_U = \hat{\theta} \cdot e^{\frac{K_{\frac{1-\delta}{2}}\sqrt{Var(\hat{\theta})}}{\hat{\theta}}} \text{ (Two-sided upper)}$$

$$\theta_L = \frac{\hat{\theta}}{e^{\frac{K_{\frac{1-\delta}{2}}\sqrt{Var(\hat{\theta})}}{\hat{\theta}}}} \text{ (Two-sided lower)}$$

The one-sided approximate confidence bounds on the parameter θ, at confidence level δ, can be found from:

$$\theta_U = \hat{\theta} \cdot e^{\frac{K_{1-\delta}\sqrt{Var(\hat{\theta})}}{\hat{\theta}}} \text{ (One-sided upper)}$$

$$\theta_L = \frac{\hat{\theta}}{e^{\frac{K_{1-\delta}\sqrt{Var(\hat{\theta})}}{\hat{\theta}}}} \text{ (One-sided lower)}$$

The same procedure can be extended for the case of a two or more parameter distribution. Lloyd and Lipow [24] further elaborate on this procedure.

Confidence Bounds on Time (Type 1)

Type 1 confidence bounds are confidence bounds around time for a given reliability. For example, when using the one-parameter exponential distribution, the corresponding time for a given exponential percentile (i.e., y-ordinate or unreliability, $Q = 1 - R$) is determined by solving the unreliability function for the time, T, or:

$$\widehat{T}(Q) = -\frac{1}{\widehat{\lambda}}\ln(1-Q) = -\frac{1}{\widehat{\lambda}}\ln(R)$$

Bounds on time (Type 1) return the confidence bounds around this time value by determining the confidence intervals around $\widehat{\lambda}$ and substituting these values into the above equation. The bounds on $\widehat{\lambda}$ are determined using the method for the bounds on parameters, with its variance obtained from the Fisher Matrix. Note that the procedure is slightly more complicated for distributions with more than one parameter.

Confidence Bounds on Reliability (Type 2)

Type 2 confidence bounds are confidence bounds around reliability. For example, when using the two-parameter exponential distribution, the reliability function is:

$$\widehat{R}(T)=e^{-\widehat{\lambda}\cdot T}$$

Reliability bounds (Type 2) return the confidence bounds by determining the confidence intervals around $\widehat{\lambda}$ and substituting these values into the above equation. The bounds on $\widehat{\lambda}$ are determined using the method for the bounds on parameters, with its variance obtained from the Fisher Matrix. Once again, the procedure is more complicated for distributions with more than one parameter.

Beta Binomial Confidence Bounds

Another less mathematically intensive method of calculating confidence bounds involves a procedure similar to that used in calculating median ranks (see Parameter Estimation). This is a non-parametric approach to confidence interval calculations that involves the use of rank tables and is commonly known as beta-binomial bounds (BB). By non-parametric, we mean that no underlying distribution is assumed. (Parametric implies that an underlying distribution, with parameters, is assumed.) In other words, this method can be used for any distribution, without having to make adjustments in the underlying equations based on the assumed distribution. Recall from the discussion on the median ranks that we used the binomial equation to compute the ranks at the 50% confidence level (or median ranks) by solving the cumulative binomial distribution for Z (rank for the j^{th} failure):

$$P=\sum_{k=j}^{N}\binom{N}{k}Z^{k}(1-Z)^{N-k}$$

where N is the sample size and j is the order number.

The median rank was obtained by solving the following equation for Z :

$$0.50=\sum_{k=j}^{N}\binom{N}{k}Z^{k}(1-Z)^{N-k}$$

The same methodology can then be repeated by changing P for 0.50 (50%) to our desired confidence level. For $P=90\%$ one would formulate the equation as

$$0.90=\sum_{k=j}^{N}\binom{N}{k}Z^{k}(1-Z)^{N-k}$$

Keep in mind that one must be careful to select the appropriate values for P based on the type of confidence bounds desired. For example, if two-sided 80% confidence bounds are to be calculated, one must solve the equation twice (once with $P=0.1$ and once with $P=0.9$) in order to place the bounds around 80% of the population.

Using this methodology, the appropriate ranks are obtained and plotted based on the desired confidence level. These points are then joined by a smooth curve to obtain the corresponding confidence bound.

In Weibull++, this non-parametric methodology is used only when plotting bounds on the mixed Weibull distribution. Full details on this methodology can be found in Kececioglu [20]. These binomial equations can again be transformed using the beta and F distributions, thus the name beta binomial confidence bounds.

Likelihood Ratio Confidence Bounds

Another method for calculating confidence bounds is the likelihood ratio bounds (LRB) method. Conceptually, this method is a great deal simpler than that of the Fisher matrix, although that does not mean that the results are of any less value. In fact, the LRB method is often preferred over the FM method in situations where there are smaller sample sizes.

Likelihood ratio confidence bounds are based on the following likelihood ratio equation:

$$-2\cdot\ln\left(\frac{L(\theta)}{L(\hat{\theta})}\right)\geq\chi^{2}_{\alpha;k}$$

where:

- $L(\theta)$ is the likelihood function for the unknown parameter vector θ
- $L(\hat{\theta})$ is the likelihood function calculated at the estimated vector $\hat{\theta}$
- $\chi^2_{\alpha;k}$ is the chi-squared statistic with probability α and k degrees of freedom, where k is the number of quantities jointly estimated

If δ is the confidence level, then $\alpha=\delta$ for two-sided bounds and $\alpha=(2\delta-1)$ for one-sided. Recall from the Brief Statistical Background chapter that if x is a continuous random variable with *pdf*:

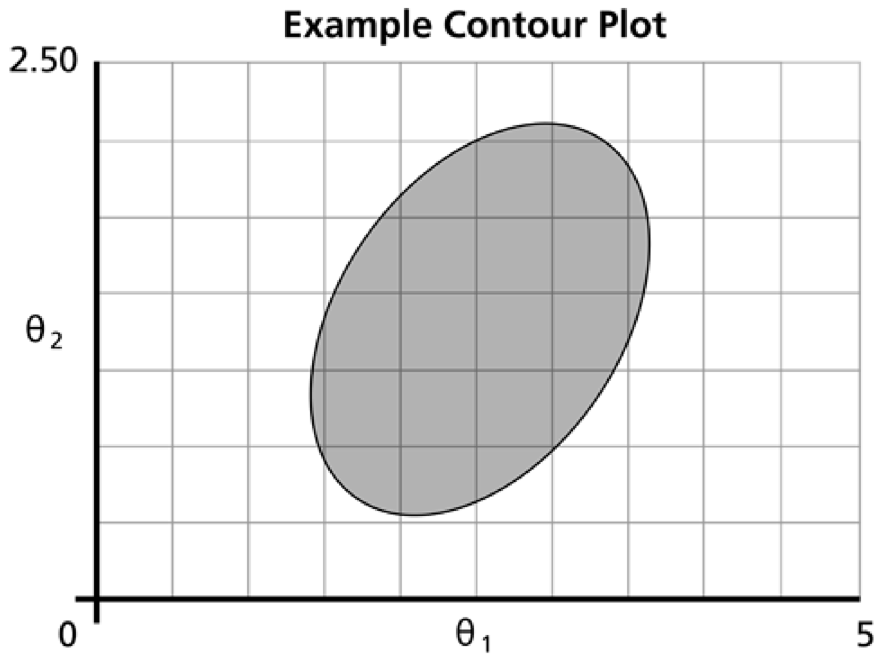
$$f(x;\theta_1,\theta_2,\ldots,\theta_k)$$

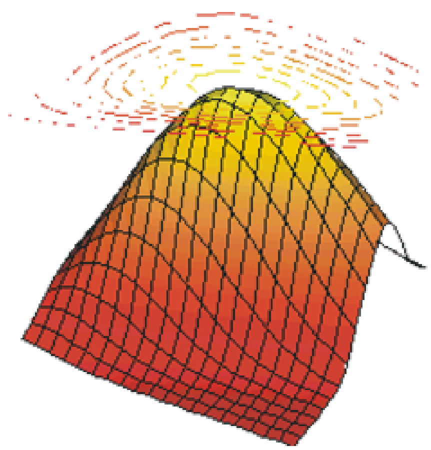
where $\theta_1,\theta_2,\ldots,\theta_k$ are k unknown constant parameters that need to be estimated, one can conduct an experiment and obtain R independent observations, $x_1,x_2,\ldots,x_R$, which correspond in the case of life data analysis to failure times. The likelihood function is given by:

$$L(x_1,x_2,\ldots,x_R|\theta_1,\theta_2,\ldots,\theta_k)=L=\prod_{i=1}^{R}f(x_i;\theta_1,\theta_2,\ldots,\theta_k)$$

$$i=1,2,\ldots,R$$

The maximum likelihood estimators (MLE) of $\theta_1,\theta_2,\ldots,\theta_k$ are k are obtained by maximizing $L.$ These are represented by the $L(\hat{\theta})$ term in the denominator of the ratio in the likelihood ratio equation. Since the values of the data points are known, and the values of the parameter estimates $\hat{\theta}$ have been calculated using MLE methods, the only unknown term in the likelihood ratio equation is the $L(\theta)$ term in the numerator of the ratio. It remains to find the values of the unknown parameter vector θ that satisfy the likelihood ratio equation. For distributions that have two parameters, the values of these two parameters can be varied in order to satisfy the likelihood ratio equation. The values of the parameters that satisfy this equation will change based on the desired confidence level $\delta;$ but at a given value of δ there is only a certain region of values for θ_1 and θ_2 for which the likelihood ratio equation holds true. This region can be represented graphically as a contour plot, an example of which is given in the following graphic.

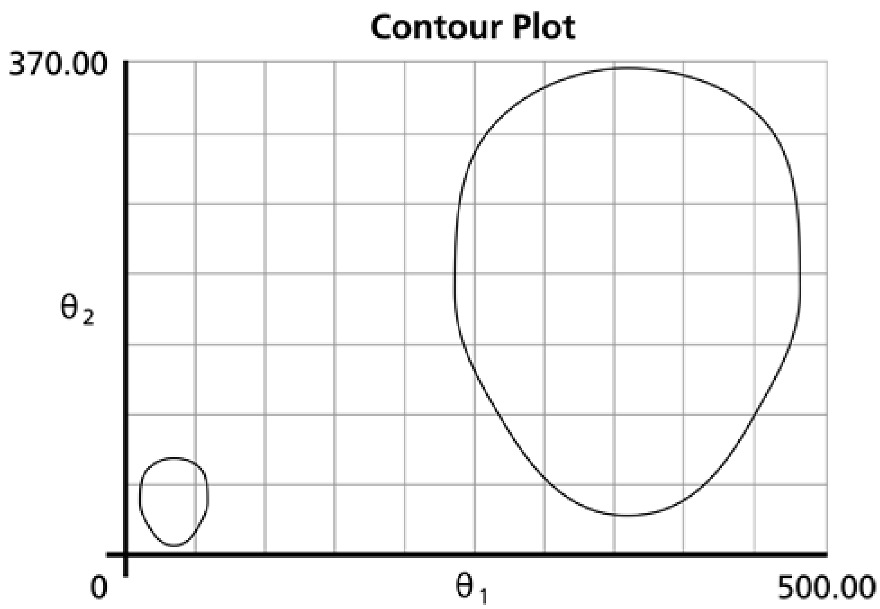




The region of the contour plot essentially represents a cross-section of the likelihood function surface that satisfies the conditions of the likelihood ratio equation.

Note on Contour Plots in Weibull++

Contour plots can be used for comparing data sets. Consider two data sets, one for an&.html#160;old product design and another for a new design. The engineer would like to determine if the two designs are significantly different and at what confidence. By plotting the contour plots of each data set in an overlay plot (the same distribution must be fitted to each data set), one can determine the confidence at which the two sets are significantly different. If, for example, there is no overlap (i.e., the two plots do not intersect) between the two 90% contours, then the two data sets are significantly different with a 90% confidence. If the two 95% contours overlap, then the two designs are NOT significantly different at the 95% confidence level. An example of non-intersecting contours is shown next. For details on comparing data sets, see Comparing Life Data Sets.



Confidence Bounds on the Parameters

The bounds on the parameters are calculated by finding the extreme values of the contour plot on each axis for a given confidence level. Since each axis represents the possible values of a given parameter, the boundaries of the contour plot represent the extreme values of the parameters that satisfy the following:

$$-2 \cdot \ln\left(\frac{L(\theta_1,\theta_2)}{L(\hat{\theta}_1,\hat{\theta}_2)}\right) = \chi^2_{\alpha;1}$$

This equation can be rewritten as:

$$L(\theta_1,\theta_2) = L(\hat{\theta}_1,\hat{\theta}_2) \cdot e^{\frac{-\chi^2_{\alpha;1}}{2}}$$

The task now is to find the values of the parameters θ_1 and θ_2 so that the equality in the likelihood ratio equation shown above is satisfied. Unfortunately, there is no closed-form solution; therefore, these values must be arrived at numerically. One way to do this is to hold one parameter constant and iterate on the other until an acceptable solution is reached. This can prove to be rather tricky, since there will be two solutions for one parameter if the other is held constant. In situations such as these, it is best to begin the iterative calculations with values close to those of the MLE values, so as to ensure that one is not attempting to perform calculations outside of the region of the contour plot where no solution exists.

Example 1:Likelihood Ratio Bounds on Parameters

Five units were put on a reliability test and experienced failures at 10, 20, 30, 40 and 50 hours. Assuming a Weibull distribution, the MLE parameter estimates are calculated to be $\widehat{\beta} = 2.2938$ and $\hat{\eta} = 33.9428.$ Calculate the 90% two-sided confidence bounds on these parameters using the likelihood ratio method.

Solution

The first step is to calculate the likelihood function for the parameter estimates:

$$L(\widehat{\beta}, \hat{\eta}) = \prod_{i=1}^{N} f(x_i; \widehat{\beta}, \hat{\eta}) = \prod_{i=1}^{5} \frac{\widehat{\beta}}{\hat{\eta}} \cdot \left(\frac{x_i}{\hat{\eta}}\right)^{\widehat{\beta}-1} \cdot e^{-\left(\frac{x_i}{\hat{\eta}}\right)^{\widehat{\beta}}}$$

$$L(\widehat{\beta}, \hat{\eta}) = \prod_{i=1}^{5} \frac{2.2938}{33.9428} \cdot \left(\frac{x_i}{33.9428}\right)^{1.2938} \cdot e^{-\left(\frac{x_i}{33.9428}\right)^{2.2938}}$$

$$L(\widehat{\beta}, \hat{\eta}) = 1.714714 \times 10^{-9}$$

where x_i are the original time-to-failure data points. We can now rearrange the likelihood ratio equation to the form:

$$L(\beta, \eta) - L(\widehat{\beta}, \hat{\eta}) \cdot e^{\frac{-\chi^2_{\alpha;1}}{2}} = 0$$

Since our specified confidence level, δ, is 90%, we can calculate the value of the chi-squared statistic, $\chi^2_{0.9;1} = 2.705543.$ We then substitute this information into the equation:

$$L(\beta,\eta)-L(\widehat{\beta},\hat{\eta})\cdot e^{\frac{-\chi^2_{\alpha;1}}{2}}=0$$

$$L(\beta,\eta)-1.714714\times 10^{-9}\cdot e^{\frac{-2.705543}{2}}=0$$

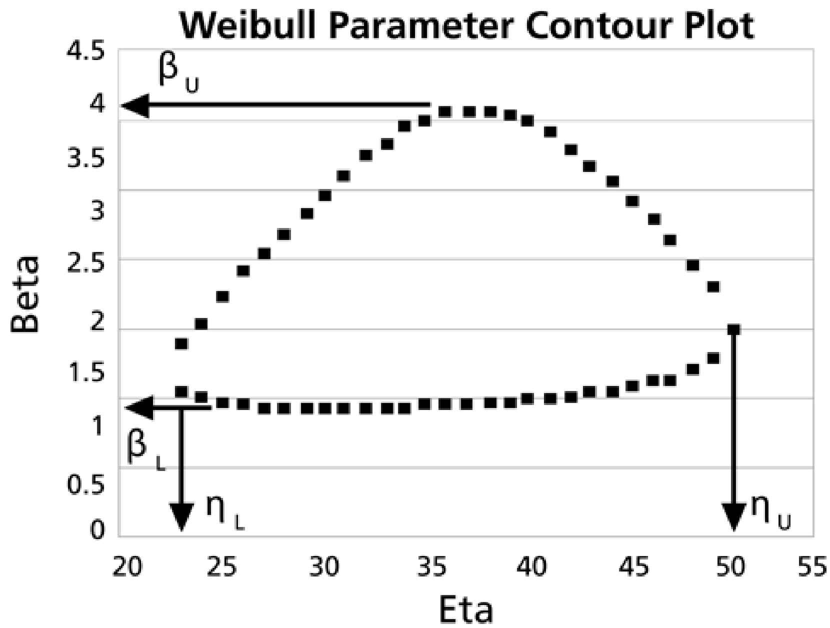
$$L(\beta,\eta)-4.432926\cdot 10^{-10}=0$$

The next step is to find the set of values of β and η that satisfy this equation, or find the values of β and η such that $L(\beta,\eta)=4.432926\cdot 10^{-10}$.

The solution is an iterative process that requires setting the value of β and finding the appropriate values of η, and vice versa. The following table gives values of β based on given values of η.

η	β_1	β_2	η	β_1	β_2
23	1.321	1.742	**37**	1.183	3.950
24	1.241	1.985	**38**	1.195	3.943
25	1.201	2.193	**39**	1.210	3.897
26	1.177	2.390	**40**	1.226	3.814
27	1.161	2.582	**41**	1.245	3.701
28	1.151	2.770	**42**	1.267	3.564
29	1.145	2.955	**43**	1.291	3.409
30	1.142	3.135	**44**	1.319	3.241
31	1.142	3.308	**45**	1.352	3.065
32	1.145	3.471	**46**	1.391	2.883
33	1.149	3.619	**47**	1.439	2.695
34	1.155	3.746	**48**	1.502	2.495
35	1.162	3.848	**49**	1.594	2.272
36	1.172	3.917	**50**	1.871	1.871

These data are represented graphically in the following contour plot:



(Note that this plot is generated with degrees of freedom $k = 1$, as we are only determining bounds on one parameter. The contour plots generated in Weibull++ are done with degrees of freedom $k = 2$, for use in comparing both parameters simultaneously.) As can be determined from the table, the lowest calculated value for β is 1.142, while the highest is 3.950. These represent the two-sided 90% confidence limits on this parameter. Since solutions for the equation do not exist for values of η below 23 or above 50, these can be considered the 90% confidence limits for this parameter. In order to obtain more accurate values for the confidence limits on η , we can perform the same procedure as before, but finding the two values of η that correspond with a given value of β Using this method, we find that the 90% confidence limits on η are 22.474 and 49.967, which are close to the initial estimates of 23 and 50.

Note that the points where β are maximized and minimized do not necessarily correspond with the points where η are maximized and minimized. This is due to the fact that the contour plot is not symmetrical, so that the parameters will have their extremes at different points.

Confidence Bounds on Time (Type 1)

The manner in which the bounds on the time estimate for a given reliability are calculated is much the same as the manner in which the bounds on the parameters are calculated. The difference lies in the form of the likelihood functions that comprise the likelihood ratio. In the preceding section, we used the standard form of the likelihood function, which was in terms of the parameters θ_1 and θ_2 . In order to calculate the bounds on a time estimate, the likelihood function needs to be rewritten in terms of one parameter and time, so that the maximum and minimum values of the time can be observed as the parameter is varied. This process is best illustrated with an example.

Example 2:Likelihood Ratio Bounds on Time (Type I)

For the data given in Example 1, determine the 90% two-sided confidence bounds on the time estimate for a reliability of 50%. The ML estimate for the time at which $R(t)=50\%$ is 28.930.

Solution

In this example, we are trying to determine the 90% two-sided confidence bounds on the time estimate of 28.930. As was mentioned, we need to rewrite the likelihood ratio equation so that it is in terms of t and $\beta.$ This is accomplished by using a form of the Weibull reliability equation, $R=e^{-\left(\frac{t}{\eta}\right)^{\beta}}$. This can be rearranged in terms of η, with R being considered a known variable or:

$$\eta=\frac{t}{(-\ln(R))^{\frac{1}{\beta}}}$$

This can then be substituted into the η term in the likelihood ratio equation to form a likelihood equation in terms of t and β or:

$$L(\beta,t)=\prod_{i=1}^{N}f(x_i;\beta,t,R)$$

$$=\prod_{i=1}^{5}\frac{\beta}{\left(\frac{t}{(-\ln(R))^{\frac{1}{\beta}}}\right)}\cdot\left(\frac{x_i}{\left(\frac{t}{(-\ln(R))^{\frac{1}{\beta}}}\right)}\right)^{\beta-1}\cdot\exp\left[-\left(\frac{x_i}{\left(\frac{t}{(-\ln(R))^{\frac{1}{\beta}}}\right)}\right)^{\beta}\right]$$

where x_i are the original time-to-failure data points. We can now rearrange the likelihood ratio equation to the form:

$$L(\beta,t)-L(\widehat{\beta},\hat{\eta})\cdot e^{\frac{-\chi^2_{\alpha;1}}{2}}=0$$

Since our specified confidence level, δ, is 90%, we can calculate the value of the chi-squared statistic, $\chi^2_{0.9;1}=2.705543.$ We can now substitute this information into the equation:

$$L(\beta,t)-L(\widehat{\beta},\hat{\eta})\cdot e^{\frac{-\chi^2_{\alpha;1}}{2}}=0$$

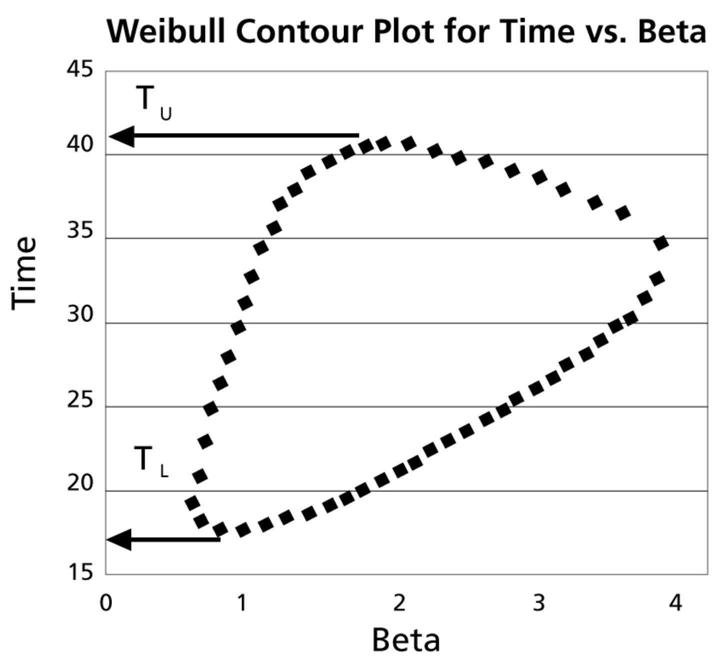
$$L(\beta,t)-1.714714\times 10^{-9}\cdot e^{\frac{-2.705543}{2}}=0$$

$$L(\beta,t)-4.432926\cdot 10^{-10}=0$$

Note that the likelihood value for $L(\widehat{\beta},\hat{\eta})$ is the same as it was for Example 1. This is because we are dealing with the same data and parameter estimates or, in other words, the maximum value of the likelihood function did not change. It now remains to find the values of β and t which satisfy this equation. This is an iterative process that requires setting the value of β and finding the appropriate values of t. The following table gives the values of t based on given values of β.

β	t_1	t_2	β	t_1	t_2
1.2	18.448	28.244	**2.6**	23.533	41.241
1.3	17.489	32.694	**2.7**	24.119	41.009
1.4	17.389	35.568	**2.8**	24.705	40.747
1.5	17.602	37.576	**2.9**	25.292	40.459
1.6	17.971	39.005	**3.0**	25.883	40.146
1.7	18.428	40.019	**3.1**	26.478	39.809
1.8	18.937	40.726	**3.2**	27.082	39.450
1.9	19.478	41.201	**3.3**	27.696	39.066
2.0	20.039	41.499	**3.4**	28.329	38.654
2.1	20.612	41.660	**3.5**	28.987	38.205
2.2	21.192	41.714	**3.6**	29.684	37.709
2.3	21.776	41.684	**3.7**	30.444	37.142
2.4	22.361	41.587	**3.8**	31.321	36.452
2.5	22.947	41.436	**3.9**	32.496	35.457

These points are represented graphically in the following contour plot:



As can be determined from the table, the lowest calculated value for t is 17.389, while the highest is 41.714. These represent the 90% two-sided confidence limits on the time at which reliability is equal to 50%.

Confidence Bounds on Reliability (Type 2)

The likelihood ratio bounds on a reliability estimate for a given time value are calculated in the same manner as were the bounds on time. The only difference is that the likelihood function must now be considered in terms of β and R. The likelihood function is once again altered in the same way as before, only now R is considered to be a parameter instead of t, since the value of t must be specified in advance. Once again, this process is best illustrated with an example.

Example 3:Likelihood Ratio Bounds on Reliability (Type 2)

For the data given in Example 1, determine the 90% two-sided confidence bounds on the reliability estimate for $t=45$. The ML estimate for the reliability at $t=45$ is 14.816%.

Solution

In this example, we are trying to determine the 90% two-sided confidence bounds on the reliability estimate of 14.816%. As was mentioned, we need to rewrite the likelihood ratio equation so that it is in terms of R and β. This is again accomplished by substituting the Weibull reliability equation into the η term in the likelihood ratio equation to form a likelihood equation in terms of R and β:

$$L(\beta,R)=\prod_{i=1}^{N}f(x_i;\beta,t,R)$$

$$=\prod_{i=1}^{5}\frac{\beta}{\left(\frac{t}{(-\ln(R))^{\frac{1}{\beta}}}\right)}\cdot\left(\frac{x_i}{\left(\frac{t}{(-\ln(R))^{\frac{1}{\beta}}}\right)}\right)^{\beta-1}\cdot\exp\left[-\left(\frac{x_i}{\left(\frac{t}{(-\ln(R))^{\frac{1}{\beta}}}\right)}\right)^{\beta}\right]$$

where x_i are the original time-to-failure data points. We can now rearrange the likelihood ratio equation to the form:

$$L(\beta, R) - L(\widehat{\beta}, \hat{\eta}) \cdot e^{\frac{-\chi^2_{\alpha;1}}{2}} = 0$$

Since our specified confidence level, δ, is 90%, we can calculate the value of the chi-squared statistic, $\chi^2_{0.9;1} = 2.705543.$ We can now substitute this information into the equation:

$$L(\beta, R) - L(\widehat{\beta}, \hat{\eta}) \cdot e^{\frac{-\chi^2_{\alpha;1}}{2}} = 0$$

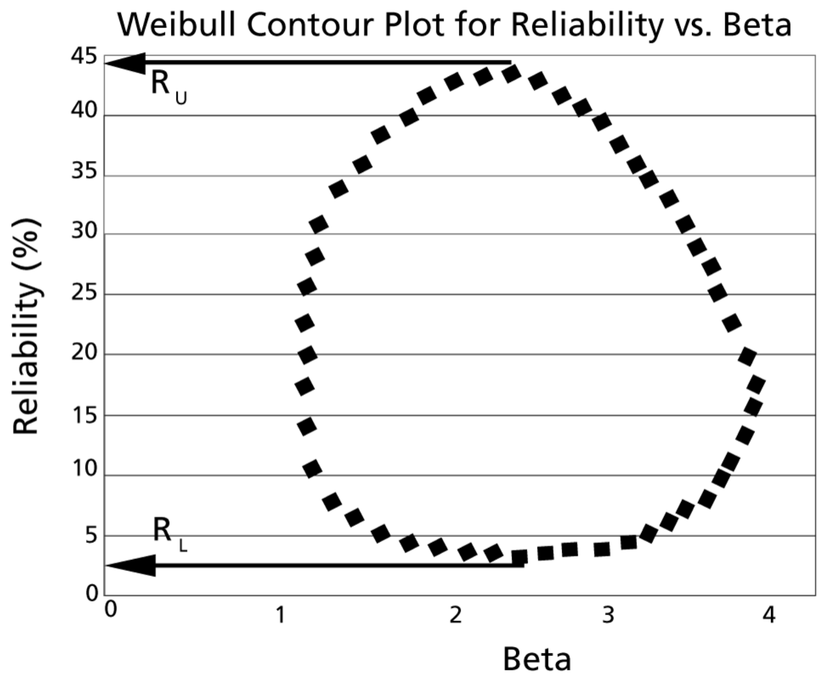
$$L(\beta, R) - 1.714714 \times 10^{-9} \cdot e^{\frac{-2.705543}{2}} = 0$$

$$L(\beta, R) - 4.432926 \cdot 10^{-10} = 0$$

It now remains to find the values of β and R that satisfy this equation. This is an iterative process that requires setting the value of β and finding the appropriate values of R. The following table gives the values of R based on given values of β.

β	R_1	R_2	β	R_1	R_2
1.2	0.1325	0.2975	**2.6**	0.0238	0.4191
1.3	0.0936	0.3499	**2.7**	0.0239	0.4104
1.4	0.0725	0.3816	**2.8**	0.0243	0.4004
1.5	0.0588	0.4032	**2.9**	0.0251	0.3892
1.6	0.0493	0.4184	**3.0**	0.0262	0.3767
1.7	0.0423	0.4291	**3.1**	0.0277	0.3629
1.8	0.0372	0.4362	**3.2**	0.0296	0.3478
1.9	0.0333	0.4406	**3.3**	0.0321	0.3311
2.0	0.0303	0.4426	**3.4**	0.0353	0.3128
2.1	0.0281	0.4426	**3.5**	0.0395	0.2925
2.2	0.0264	0.4409	**3.6**	0.0451	0.2699
2.3	0.0252	0.4375	**3.7**	0.0527	0.2442
2.4	0.0244	0.4327	**3.8**	0.0641	0.2136
2.5	0.0239	0.4266	**3.9**	0.0848	0.1727

These points are represented graphically in the following contour plot:



As can be determined from the table, the lowest calculated value for R is 2.38%, while the highest is 44.26%. These represent the 90% two-sided confidence limits on the reliability at $t = 45$.

Bayesian Confidence Bounds

A fourth method of estimating confidence bounds is based on the Bayes theorem. This type of confidence bounds relies on a different school of thought in statistical analysis, where prior information is combined with sample data in order to make inferences on model parameters and their functions. An introduction to Bayesian methods is given in the Parameter Estimation chapter. Bayesian confidence bounds are derived from Bayes's rule, which states that:

$$f(\theta|Data) = \frac{L(Data|\theta)\varphi(\theta)}{\int_{\varsigma} L(Data|\theta)\varphi(\theta)d\theta}$$

where:

- $f(\theta|Data$) is the *posterior pdf* of θ
- θ is the parameter vector of the chosen distribution (i.e., Weibull, lognormal, etc.)
- $L(\bullet)$ is the likelihood function

- $\varphi(\theta)$ is the *prior pdf* of the parameter vector θ
- ς is the range of θ.

In other words, the prior knowledge is provided in the form of the prior *pdf* of the parameters, which in turn is combined with the sample data in order to obtain the posterior *pdf*. Different forms of prior information exist, such as past data, expert opinion or non-informative (refer to Parameter Estimation). It can be seen from the above Bayes's rule formula that we are now dealing with distributions of parameters rather than single value parameters. For example, consider a one-parameter distribution with a positive parameter θ_1. Given a set of sample data, and a prior distribution for θ_1, $\varphi(\theta_1)$, the above Bayes's rule formula can be written as:

$$f(\theta_1|Data)=\frac{L(Data|\theta_1)\varphi(\theta_1)}{\int_0^{\infty}L(Data|\theta_1)\varphi(\theta_1)d\theta_1}$$

In other words, we now have the distribution of θ_1 and we can now make statistical inferences on this parameter, such as calculating probabilities. Specifically, the probability that θ_1 is less than or equal to a value x, $P(\theta_1\leq x)$ can be obtained by integrating the posterior probability density function (*pdf*), or:

$$P(\theta_1\leq x)=\int_0^{x}f(\theta_1|Data)d\theta_1$$

The above equation is the posterior *cdf*, which essentially calculates a confidence bound on the parameter, where $P(\theta_1\leq x)$ is the confidence level and x is the confidence bound. Substituting the posterior *pdf* into the above posterior *cdf* yields:

$$CL=\frac{\int_0^{x}L(Data|\theta_1)\varphi(\theta_1)d\theta_1}{\int_0^{\infty}L(Data|\theta_1)\varphi(\theta_1)d\theta_1}$$

The only question at this point is, what do we use as a prior distribution of θ_1? For the confidence bounds calculation application, non-informative prior distributions are utilized. Non-informative prior distributions are distributions that have no population basis and play a minimal role in the posterior distribution. The idea behind the use of non-informative prior distributions is to make inferences that are not affected by external information, or when external information is not available. In the general case of calculating confidence bounds using Bayesian methods, the method should be independent of external information and it should only rely on the current data. Therefore, non-informative priors are used. Specifically, the uniform distribution is used as a prior distribution for the different parameters of the selected fitted distribution. For example, if the Weibull distribution is fitted to the data, the prior distributions for beta and eta are assumed to be uniform. The above equation can be generalized for any distribution having a vector of parameters $\theta,$ yielding the general equation for calculating Bayesian confidence bounds:

$$CL=\frac{\int_{\xi} L(Data|\theta)\varphi(\theta)d\theta}{\int_{\varsigma} L(Data|\theta)\varphi(\theta)d\theta}$$

where:

- CL is the confidence level
- θ is the parameter vector
- $L(\bullet)$ is the likelihood function
- $\varphi(\theta)$ is the prior *pdf* of the parameter vector θ
- ς is the range of θ
- ξ is the range in which θ changes from $\Psi(T,R)$ till θ 's maximum value, or from θ 's minimum value till $\Psi(T,R)$
- $\Psi(T,R)$ is a function such that if T is given, then the bounds are calculated for R. If R is given, then the bounds are calculated for T.

If T is given, then from the above equation and Ψ and for a given CL, the bounds on R are calculated. If R is given, then from the above equation and Ψ and for a given C the bounds on T are calculated.

Confidence Bounds on Time (Type 1)

For a given failure time distribution and a given reliability R, $T(R)$ is a function of R and the distribution parameters. To illustrate the procedure for obtaining confidence bounds, the two-parameter Weibull distribution is used as an example. The bounds in other types of distributions can be obtained in similar fashion. For the two-parameter Weibull distribution:

$$T(R)=\eta \exp(\frac{\ln(-\ln R)}{\beta})$$

For a given reliability, the Bayesian one-sided upper bound estimate for $T(R)$ is:

$$CL=\Pr(T\le T_U)=\int_0^{T_U(R)} f(T|Data,R)dT$$

where $f(T|Data,R)$ is the posterior distribution of Time T. Using the above equation, we have the following:

$$CL=\Pr(T\le T_U)=\Pr(\eta \exp(\frac{\ln(-\ln R)}{\beta})\le T_U)$$

The above equation can be rewritten in terms of η as:

$$CL=\Pr(\eta\le T_U \exp(-\frac{\ln(-\ln R)}{\beta}))$$

Applying the Bayes's rule by assuming that the priors of β and η are independent, we then obtain the following relationship:

$$CL=\frac{\int_0^\infty \int_0^{T_U \exp(-\frac{\ln(-\ln R)}{\beta})} L(\beta,\eta)\varphi(\beta)\varphi(\eta)d\eta d\beta}{\int_0^\infty \int_0^\infty L(\beta,\eta)\varphi(\beta)\varphi(\eta)d\eta d\beta}$$

The above equation can be solved for $T_U(R)$, where:

- CL is the confidence level,
- $\varphi(\beta)$ is the prior *pdf* of the parameter β. For non-informative prior distribution, $\varphi(\beta)=\frac{1}{\beta}$.
- $\varphi(\eta)$ is the prior *pdf* of the parameter η. For non-informative prior distribution, $\varphi(\eta)=\frac{1}{\eta}$.
- $L(\bullet)$ is the likelihood function.

The same method can be used to get the one-sided lower bound of $T(R)$ from:

$$CL = \frac{\int_0^\infty \int_{T_L \exp(\frac{-\ln(-\ln R)}{\beta})}^\infty L(\beta,\eta)\varphi(\beta)\varphi(\eta)d\eta d\beta}{\int_0^\infty \int_0^\infty L(\beta,\eta)\varphi(\beta)\varphi(\eta)d\eta d\beta}$$

The above equation can be solved to get $T_L(R)$.

The Bayesian two-sided bounds estimate for $T(R)$ is:

$$CL = \int_{T_L(R)}^{T_U(R)} f(T|Data,R)dT$$

which is equivalent to:

$$(1+CL)/2 = \int_0^{T_U(R)} f(T|Data,R)dT$$

and:

$$(1-CL)/2 = \int_0^{T_L(R)} f(T|Data,R)dT$$

Using the same method for the one-sided bounds, $T_U(R)$ and $T_L(R)$ can be solved.

Confidence Bounds on Reliability (Type 2)

For a given failure time distribution and a given time T , $R(T)$ is a function of T and the distribution parameters. To illustrate the procedure for obtaining confidence bounds, the two-parameter Weibull distribution is used as an example. The bounds in other types of distributions can be obtained in similar fashion. For example, for two parameter Weibull distribution:

$$R = \exp(-(\frac{T}{\eta})^\beta)$$

The Bayesian one-sided upper bound estimate for $R(T)$ is:

$$CL=\int_{0}^{R_U(T)} f(R|Data,T)dR$$

Similar to the bounds on Time, the following is obtained:

$$CL=\frac{\int_0^\infty \int_0^{T\exp(-\frac{\ln(-\ln R_U)}{\beta})} L(\beta,\eta)\varphi(\beta)\varphi(\eta)d\eta d\beta}{\int_0^\infty \int_0^\infty L(\beta,\eta)\varphi(\beta)\varphi(\eta)d\eta d\beta}$$

The above equation can be solved to get $R_U(T)$.

The Bayesian one-sided lower bound estimate for R(T) is:

$$1-CL=\int_{0}^{R_L(T)} f(R|Data,T)dR$$

Using the posterior distribution, the following is obtained:

$$CL=\frac{\int_0^\infty \int_{T\exp(-\frac{\ln(-\ln R_L)}{\beta})}^{\infty} L(\beta,\eta)\varphi(\beta)\varphi(\eta)d\eta d\beta}{\int_0^\infty \int_0^\infty L(\beta,\eta)\varphi(\beta)\varphi(\eta)d\eta d\beta}$$

The above equation can be solved to get $R_L(T)$.

The Bayesian two-sided bounds estimate for $R(T)$ is:

$$CL=\int_{R_L(T)}^{R_U(T)} f(R|Data,T)dR$$

which is equivalent to:

$$\int_{0}^{R_U(T)} f(R|Data,T)dR=(1+CL)/2$$

and

$$\int_0^{R_L(T)} f(R|Data, T)dR = (1 - CL)/2$$

Using the same method for one-sided bounds, $R_U(T)$ and $R_L(T)$ can be solved.

Simulation Based Bounds

The SimuMatic tool in Weibull++ can be used to perform a large number of reliability analyses on data sets that have been created using Monte Carlo simulation. This utility can assist the analyst to a) better understand life data analysis concepts, b) experiment with the influences of sample sizes and censoring schemes on analysis methods, c) construct simulation-based confidence intervals, d) better understand the concepts behind confidence intervals and e) design reliability tests. This section describes how to use simulation for estimating confidence bounds.

SimuMatic generates confidence bounds and assists in visualizing and understanding them. In addition, it allows one to determine the adequacy of certain parameter estimation methods (such as rank regression on X, rank regression on Y and maximum likelihood estimation) and to visualize the effects of different data censoring schemes on the confidence bounds.

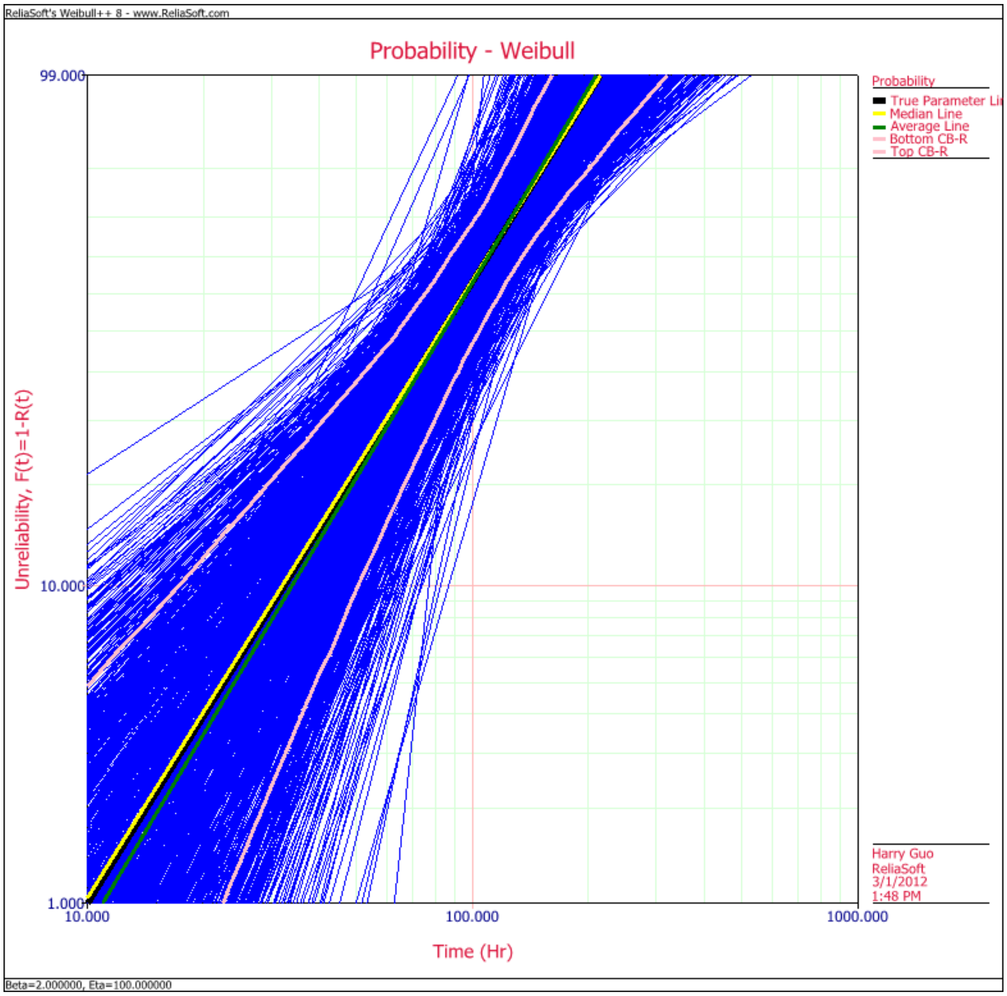
Example:Comparing Parameter Estimation Methods Using Simulation Based Bounds

The purpose of this example is to determine the best parameter estimation method for a sample of ten units with complete time-to-failure data for each unit (i.e., no censoring). The data set follows a Weibull distribution with $\beta = 2$ and $\eta = 100$ hours.

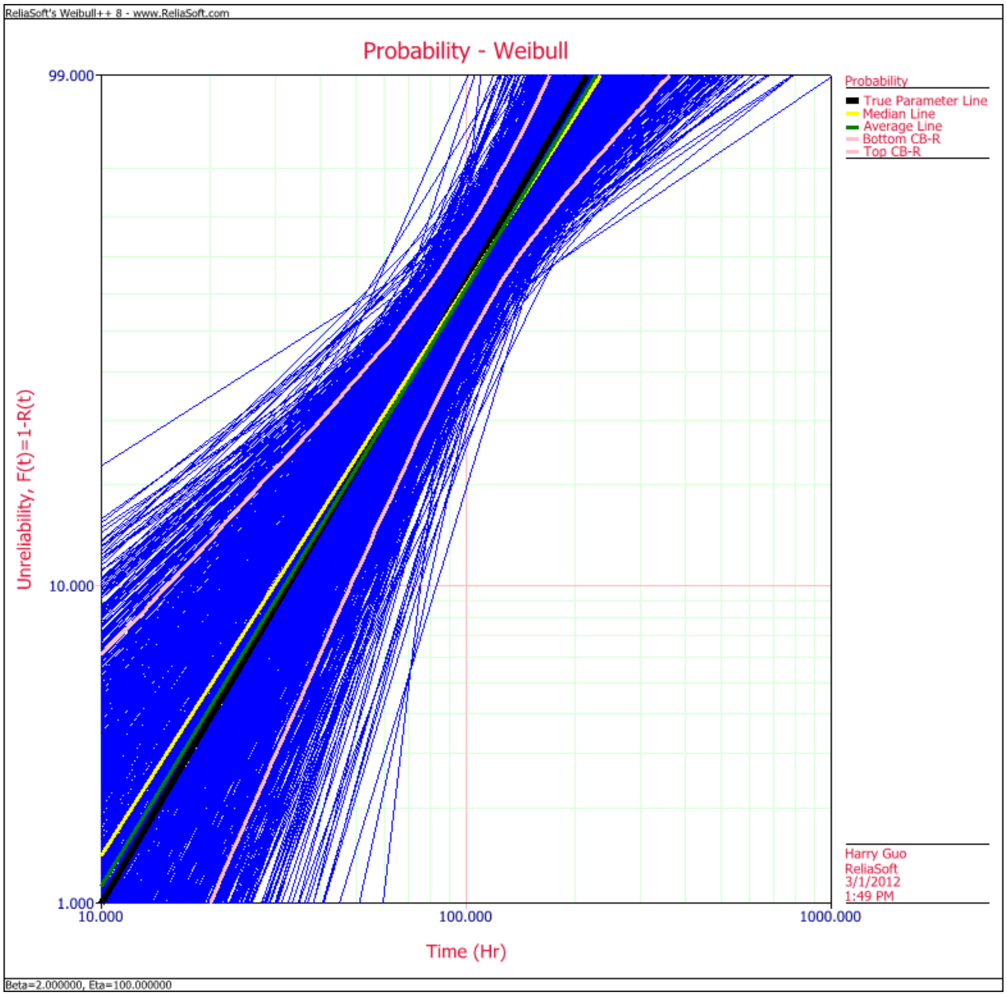
The confidence bounds for the data set could be obtained by using **Weibull++'s** SimuMatic utility. To obtain the results, use the following settings in SimuMatic.

1. On the Main tab, choose the **2P-Weibull** distribution and enter the given parameters (i.e., $\beta = 2$ and $\eta = 100$ hours)
2. On the Censoring tab, select the **No censoring** option.
3. On the Settings tab, set the number of data sets to **1,000** and the number of data points to **10**.
4. On the Analysis tab, choose the **RRX** analysis method and set the confidence bounds to **90**.

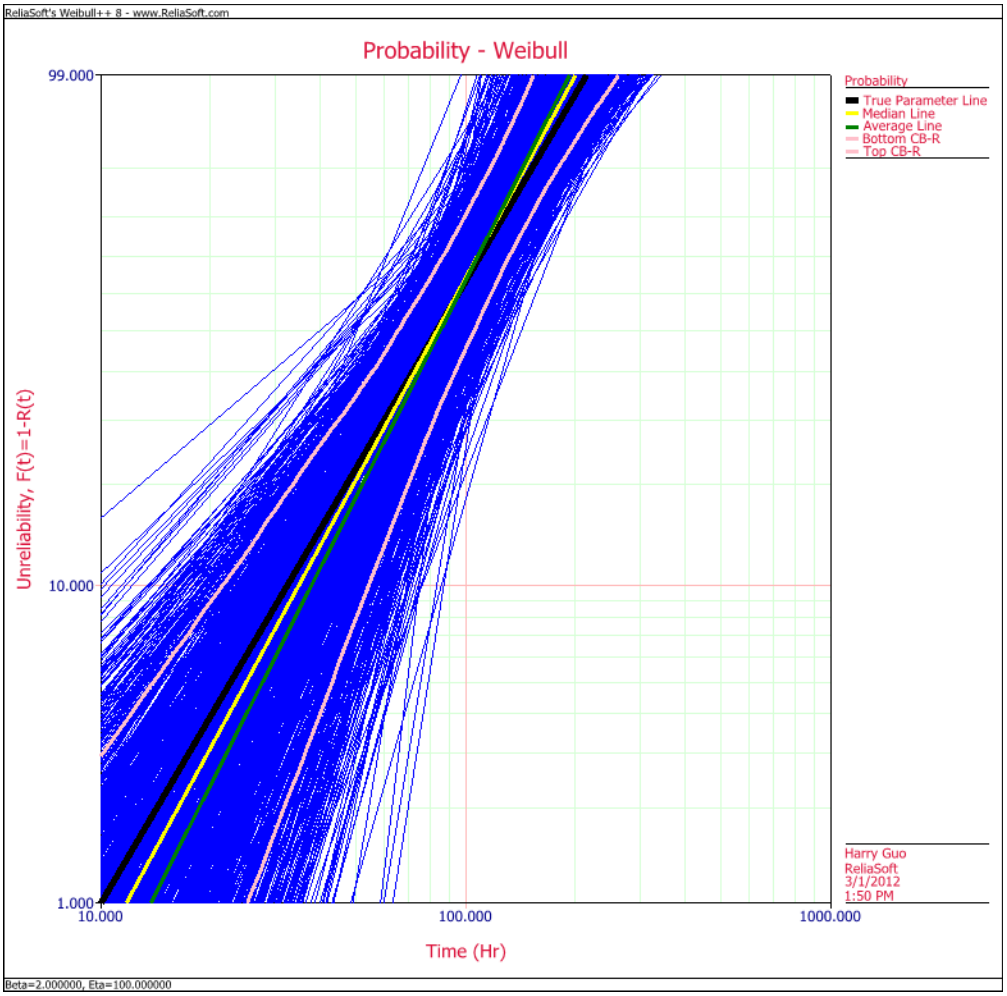
The following plot shows the simulation-based confidence bounds for the RRX parameter estimation method, as well as the expected variation due to sampling error.



Create another SimuMatic folio and generate a second data using the same settings, but this time, select the RRY analysis method on the Analysis tab. The following plot shows the result.



The following plot shows the results using the MLE analysis method.



The results clearly demonstrate that the median RRX estimate provides the least deviation from the truth for this sample size and data type. However, the MLE outputs are grouped more closely together, as evidenced by the bounds.

This experiment can be repeated in SimuMatic using multiple censoring schemes (including Type I and Type II right censoring as well as random censoring) with various distributions. Multiple experiments can be performed with this utility to evaluate assumptions about the appropriate parameter estimation method to use for data sets.

Appendix E: References

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